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by

Asher Auel

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Asher Auel

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
Germany

Department of Mathematics & CS
Emory University
400 Dowman Drive NE W401
Atlanta, GA 30322
USA

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CLIFFORD INVARIANTS OF LINE BUNDLE-VALUED QUADRATIC FORMS

ASHER AUEL

Abstract. We construct an extension of the Clifford (also known as the Hasse–Witt or 2nd Stiefel–Whitney) invariant to similarity classes of line bundle-valued quadratic forms of even rank $n = 2m$ and fixed discriminant δ on a scheme X (where 2 is invertible). This invariant resides in the étale cohomology group $H^2_{\text{ét}}(X, \boldsymbol{\kappa}_\delta^m)$ with coefficients in a twisted group scheme of order four and it "interpolates" between the classical étale cohomological Clifford invariant of ($\mathscr{O}_X$-valued) quadratic forms and the 1st Chern class (modulo 2) of the value line bundle. We further relate this invariant to natural classes in the "involutive" Brauer group arising from the even Clifford algebra and Clifford bimodule. In rank ≤ 6, we use the classification of line bundle-valued quadratic forms in terms of twisted norm and pfaffian forms to explicitly compute this invariant and give necessary and sufficient conditions for its vanishing.

Contents

Introduction

For a line bundle $\mathscr{L}$ on a space X, the notion of a quadratic (or symmetric bilinear) form on X with values in $\mathscr{L}$ dates back to the early 1970s. Geyer–Harder–Knebusch–Scharlau [44] introduced the notion of symmetric bilinear forms with values in the module of Kähler differentials over a global function field. This notion enabled a consistent choice of local traces in order to generalize residue theorems to nonrational function fields. For a smooth complete algebraic curve X, Mumford [77] introduced the notion of locally free $\mathscr{O}_X$-modules with pairings taking values in the sheaf of differentials ω_X to study theta characteristics. For a commutative ring R, Kanzaki [56] introduced the notion of a Witt group $W(I)$ of quadratic forms with values in an invertible R-module I.

Implicit in these early developments are generalizations to algebraic varieties of the classical transfer (or trace) maps from the theory of quadratic forms over fields. The general context in which such transfer maps exist has recently been established by the work of Gille [45], Nenashev [78], [79], Calmès–Hornbostel [25], [26], and Balmer–Calmès [10]. If $f : X \to Y$ is a proper morphism of relative dimension d between connected, noetherian, regular Spec $\mathbb{Z}[\frac{1}{2}]$-schemes of finite Krull dimension, then the total derived direct image functor gives rise to a transfer map,

$$f_* : GW^{i+d}(X, \omega_f) \to GW^i(Y, \mathscr{O}_Y),$$

between the shifted derived (or coherent) Grothendieck–Witt groups introduced by Balmer [6], [7], [8], and Walter [96]. Here, ω_f is the relative dualizing sheaf from Grothendieck duality theory. In particular, in order to define the transfer along a proper morphism $f : X \to Y$, one is forced to consider ω_f-valued forms on X. It's in this setting that Balmer [9, §1.4] and Walter [96, §10] considered the existence of cohomological invariants defined on general shifted and line bundle-valued Witt and Grothendieck-Witt groups.

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Line bundle-valued forms have also emerged in the context of involutions on Azumaya algebras, initially to generalize theorems of Albert [1, Thm. 10.19] on the existence of involutions on central simple algebras. Saltman [88, Thm. 4.2] and Knus–Parimala–Srinivas [67, §3] showed that involutions on endomorphism algebras are adjoint to symmetric bilinear forms with valued in line bundles. This paved the way for a version of Jacobson's [52] (see also Tits [93, §4]) even Clifford algebra of an Azumaya algebra with orthogonal involution. In his thesis, Bichsel [18] (reported in Bichsel–Knus [17]) constructs an even Clifford algebra of a line bundle-valued form over an affine scheme, then Parimala–Sridharan [84] use étale descent to construct an even Clifford algebra of an Azumaya algebra with orthogonal involution. These constructions enable much of the theory in [69] to be generalized (with some care) to arbitrary base schemes. In this context, line bundle-valued forms should be viewed as intermediary objects between quadratic forms and algebras with orthogonal involution: they are "unsplit" versions of quadratic forms, yet they still represent orthogonal involutions on "split" algebras.

Kapranov [57, §4.1] also considered the *homogeneous* Clifford algebra of a quadratic form—the same as the *generalized* Clifford algebra (in the sense Bichsel–Knus [17]) or the *graded* Clifford algebra (in the sense of Caenepeel–van Oystaeyen [23])—to study the derived category of projective quadrics and quadric fibrations. This theme was later taken up and developed by Bondal–Orlov [20, §2], [19], Kuznetsov [71], and Bolognesi–Bernardara [12]. These authors consider line bundle-valued quadratic forms with possible degeneration.

Over Dedekind domains, regular line bundle-valued bilinear forms (so called *modular lattices*) have been studied by Bushnell [21], [22], generalizing results of Fröhlich [38]. Recent number theoretic developments have also seen the appearance of line bundle-valued forms. Bhargava's [13], [14], [15] classification of "rings of low rank" over $\mathbb{Z}$ is used in his work [16] on the distribution of discriminants of number fields of fixed low degree. An approach to this classification over arbitrary base rings is initiated in Wood's thesis [97], where line bundle-valued forms (of higher degree) were essential to the generalization of Bhargava's results. Also see Ho's thesis [49]. In related developments, Venkata Balaji [5] and independently Voight [94], used Clifford algebras of (possibly degenerate) ternary line bundle-valued forms to classify "degenerate" quaternion algebras over arbitrary bases.

In a topological context, Holla–Nitsure [50], [51] study characteristic classes (i.e. cohomological invariants in singular cohomology) for possibly mildly degenerate line bundle-valued quadratic forms (on complex vector bundles) by computing the cohomology of the relevant classifying space.

While the cohomological invariant theory of quadratic forms is well developed, the theory for line bundle-valued forms is substantially less so. When $\mathscr{L}$ is the trivial line bundle, one can use the work of Delzant [32], Laborde [72], or Jardine [53] to define invariants in étale cohomology analogous to the Stiefel–Whitney (or higher Hasse–Witt) invariants for quadratic forms over fields (see [91], for example). These invariants have been intensively studied by many people in various contexts, including Serre [89], Knus–Ojanguren [70], Parimala–Srinivas [85], Ojanguren–Parimala–Sridharan [80], Knus–Parimala–Sridharan [66], Esnault–Kahn–Viehweg [34], Jardine [54], Cassou-Noguès–Erez–Taylor [27], and Esnault–Kahn–Levine–Viehweg [35]. In these approaches, invariants are constructed is several ways: directly, in the case of the classical invariants; à la Grothendieck, using the étale cohomology ring of a grassmannian bundle of anisotropic vectors; or as characteristic classes, using the étale cohomology groups of the simplicial classifying scheme (or topos) of a standard orthogonal group. The étale cohomology of the anisotropic grassmannian can be computed because it's an inner form of projective space with a standard subvariety of isotropic lines removed. The étale cohomology of the classifying space can be computed by analogy with the topological case because the standard orthogonal group is the base change of a smooth affine group scheme on $\operatorname{Spec}\mathbb{Z}[\frac{1}{2}]$ (even on $\operatorname{Spec}\mathbb{Z}$). When $\mathscr{L}$ is nontrivial, these approaches break down. When $\mathscr{L}$ is not a square in the Picard group, there are no known calculations of the étale cohomology rings of the required spaces; the calculations of Holla–Nitsure [50] in the topological case indicate that this might be a daunting task. The orthogonal group of such an $\mathscr{L}$-valued form is *not* a twist of the standard orthogonal group by a cocycle representing a class in the image of the natural map $H^1_{\text{ét}}(X, \mathbf{O}_n) \to H^1_{\text{ét}}(X, \mathbf{PO}_n)$. As Serre famously points out, the torsors for a group have in general no relation to the torsors for such a form of the group.[1]

In this work, we employ a different approach. While isometry classes of $\mathscr{L}$-valued forms are associated to torsors for a twisted form of the orthogonal group, their similarity classes are associated to torsors for a standard group $\mathbf{GO}_n$ of orthogonal similitudes. One complication is the nonexistence of a natural "pin" double covering of $\mathbf{GO}_{2m}$ (this is even true in the topological context). Our

[1] Serre [90, I, §5.5, Remarque] warns, "Par contre, $H^1(G, {}_aA)$ *n'a en général aucune relation avec* $H^1(G, A)$." Then later in [90, I, §5.7, Remarque 1], Serre feels the need to reiterate himself, "Ici encore, il est faux en général que $H^1(G, {}_cB)$ soit en correspondance bijective avec $H^1(G, B)$." This is indeed an important point.

essential contribution is the realization that, while there is no natural double covering, there is a natural four-fold covering of $\mathbf{GO}_{2m}$ by the Clifford group $\mathbf{\Gamma}_{2m}$ that becomes a central isogeny when restricted to the subgroup of proper similitudes, and which "interpolates" between the Kummer double covering of the multiplicative group and the classical pin covering of the orthogonal group (see §2.4). Such a central isogeny does not seem to exist in the literature (though see Joshi [55] for an ad hoc construction in rank 2). The kernel of this covering is the group scheme $\boldsymbol{\kappa}_\delta^m$ mentioned in the abstract. The appearance of invariants with values in cohomology with coefficients in a group scheme of order 4 is perhaps natural in light of Berhuy's theory [11, Prop. 9] of descent of cohomological invariants. An example of this is the cohomology class of the Tits algebra of a central simple algebra with orthogonal involution, see [69, VII §31.11].

Finally, since our methods are mostly torsorial in nature, we restrict ourselves to the case where 2 is invertible. This eliminates the consideration of nonsmooth algebraic group schemes and allows us to work purely in the étale site. The consideration of characteristic 2 phenomena will appear elsewhere.

Main results. Let X be a noetherian separated scheme on which 2 is invertible. A *line bundle-valued quadratic form* on X consists of a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank, an invertible $\mathscr{O}_X$-module $\mathscr{L}$, and a quadratic map $q : \mathscr{E} \to \mathscr{L}$. Since we assume that 2 is invertible on X, we will not concern ourselves with the difference between line-bundle valued quadratic and symmetric bilinear forms. Consult §1 for a review of these notions.

When $\mathscr{L} = \mathscr{O}_X$, the generalization to étale cohomology of the (signed) discriminant $d_\pm(\mathscr{E}, q)$ and classical Clifford invariant $c(\mathscr{E}, q)$ yield important invariants of quadratic forms $(\mathscr{E}, q)$ on schemes. When $\mathscr{L} \neq \mathscr{O}_X$, the discriminant immediately yields a generalization to line bundle-valued forms of even rank (see Parimala–Sridharan [84]), while the Clifford invariant—due to the nonexistence of a "full" Clifford algebra—does not. The main construction of this current work is an extension of the Clifford invariant to similarity classes of line bundle-valued quadratic forms.

For $n \geq 1$ and $f : Z \to X$ an étale quadratic morphism with isomorphism class $\delta \in H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \cong H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$, denote by $\mathrm{GQ}_n^\delta(X)$ the set of similarity classes of regular line bundle-valued quadratic forms of rank n and discriminant δ. In §2.3, we construct a group scheme $\boldsymbol{\kappa}_n^Z$ on X together with canonical isomorphisms

$$\boldsymbol{\kappa}_n^Z \cong \begin{cases} \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \text{if } n \equiv 0, 1 \bmod 4 \\ \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \text{if } n \equiv 2, 3 \bmod 4 \end{cases}$$

where $\mathbf{R}_{Z/X}$ and $\mathbf{R}^1_{Z/X}$ denote the Weil restriction functor and subfunctor of norm one elements (see §2.2), respectively. There is also a canonical exact sequence

$$(1) \qquad 1 \to \boldsymbol{\mu}_2 \xrightarrow{i} \boldsymbol{\kappa}_n^Z \xrightarrow{p} \boldsymbol{\mu}_2 \to 1$$

of group schemes on X. For the relationship between $\boldsymbol{\kappa}_n^Z$ and the center of the spin group, see §3.4.

Theorem 1 (Theorem 2.10)**.** *Let X be a scheme with 2 invertible. For each $n \geq 1$ and étale quadratic $f : Z \to X$ with class $\delta \in H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, there exists a map, called the* similarity Clifford invariant*,*

$$gc : \mathrm{GQ}_n^\delta(X) \to H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z)$$

which "interpolates" between the classical étale cohomological Clifford invariant and the 1st Chern class (modulo 2) of the value line bundle. More precisely, via the exact sequence of étale cohomology groups

$$\cdots \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \xrightarrow{i^2} H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \xrightarrow{p^2} H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \to \cdots$$

arising from (1), we have

$$i^2 c(\mathscr{E}, q) = gc(\mathscr{E}, q, \mathscr{O}_X), \qquad p^2 gc(\mathscr{E}, q, \mathscr{L}) = c_1(\mathscr{L}, \boldsymbol{\mu}_2),$$

for every isometry class of $\mathscr{O}_X$-valued quadratic form $(\mathscr{E}, q)$ and every similarity class of line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank n and discriminant δ.

The similarity Clifford invariant is constructed as the coboundary map in nonabelian étale cohomology associated to a short exact sequence of sheaves of groups

$$1 \to \boldsymbol{\kappa}_n^Z \to \mathbf{S\Gamma}_n^Z \to \mathbf{GSO}_n^Z \to 1$$

where $\mathbf{S\Gamma}_n^Z$ and $\mathbf{GSO}_n^Z$ are the even Clifford group and group of proper orthogonal similitudes of a fixed choice of $\mathscr{O}_X$-valued form of rank n and discriminant δ (see §2.4). This particular central isogeny does not seems to have a place in the existing literature, except in small ranks (see Joshi [55], for example). Here we use étale descent to identify the category of $\mathbf{GSO}_n^Z$-torsors on the étale site

$X_{\text{ét}}$ with the category of regular oriented (see §1.10) line bundle-valued quadratic forms of rank n and discriminant δ on X.

We prove that the invariant respects Grothendieck–Witt equivalence, and so defines an étale cohomological invariant on the second part of the "fundamental" filtration of the total quadratic Grothendieck–Witt group (see §2.7). Let $GW(X,\mathscr{L})$ denote the Grothendieck–Witt group of $\mathscr{L}$-valued quadratic forms on X and $GW^{\text{tot}}(X) = \oplus_{\mathscr{L}} GW(X,\mathscr{L})$ denote the total quadratic Grothendieck-Witt group (which is actually a group fibered over the Picard groupoid of X). The signed discriminant defines a (surjective) cohomological invariant

$$d_{\pm} : GW^{\text{tot}}(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2).$$

Define $GI_2^{\text{tot}}(X) = \oplus_{\mathscr{L}} GI_2(X,\mathscr{L})$ as the subgroup of $GW^{\text{tot}}(X)$ of classes of quadratic forms of rank $n \equiv 0 \bmod 4$ and trivial discriminant.

Theorem 2 (Theorem 2.11)**.** *Let X be a scheme with 2 invertible. The similarity Clifford invariant respects Grothendieck–Witt equivalence and so defines an invariant*

$$gc : GI_2^{\text{tot}}(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_4).$$

Furthermore, if $(\mathscr{E}, q, \mathscr{L})$ is a quadratic form of rank $n \equiv 0 \bmod 4$ and trivial discriminant on X, then

$$gc((\mathscr{E}, q, \mathscr{L}) \perp H_{\mathscr{L}}(\mathscr{V})) = c(\mathscr{E}, q, \mathscr{L}) + i^2 c_1(\mathscr{V}, \boldsymbol{\mu}_2)$$

for any $\mathscr{L}$-valued hyperbolic form $H_{\mathscr{L}}(\mathscr{V})$ with lagrangian $\mathscr{V}$ of even rank.

In §3 we relate the similarity Clifford invariant to "Brauer classes" associated to the even Clifford algebra of line bundle-valued form defined in Bichsel [18] and Bichsel–Knus [17]. Parimala–Srinivas [85] define an "involutive" Brauer group $\mathrm{Br}^*(X)$ of Azumaya $\mathscr{O}_X$-algebras with involution of the first kind as well as a "Z/X-unitary involutive" Brauer group $\mathrm{Br}^*(Z/X)$ of Azumaya $\mathscr{O}_Z$-algebras with Z/X-unitary involution for $f : Z \to X$ étale quadratic (see §3.3). If $(\mathscr{E}, q, \mathscr{L})$ is a regular line bundle-valued quadratic form of even rank n, then its even Clifford algebra $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$—together with its canonical Clifford involution τ_0—defines a class $[\widetilde{\mathscr{C}_0}(\mathscr{E}, q, \mathscr{L}), \tau_0]$ in $\mathrm{Br}^*(Z)$ if $n \equiv 0 \bmod 4$ or in $\mathrm{Br}^*(Z/X)$ if $n \equiv 2 \bmod 4$ (see 3.4). Here, the étale quadratic $f : Z \to X$ is the given by the center of the even Clifford algebra.

Theorem 3 (Theorems 3.13, 3.17)**.** *Let X be a scheme with 2 invertible, $(\mathscr{E}, q, \mathscr{L})$ a regular line bundle-valued quadratic form of even rank n and étale quadratic $f : Z \to X$ given by the center of its even Clifford algebra.*

a) If $n \equiv 0 \bmod 4$, then

$$\varphi^2 gc(\mathscr{E}, q, \mathscr{L}) = i^2[\widetilde{\mathscr{C}_0}(\mathscr{E}, q, \mathscr{L}), \tau_0] + c_1(f^*\mathscr{L}, \boldsymbol{\mu}_4)$$

in $H^2_{\text{ét}}(Z, \boldsymbol{\mu}_4)$, where $i^2 : \mathrm{Br}^*(Z) \to H^2_{\text{ét}}(Z, \boldsymbol{\mu}_4)$ and $\varphi^2 : H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \to H^2_{\text{ét}}(Z, \boldsymbol{\mu}_4)$ are certain canonical comparison maps.

b) If $n \equiv 2 \bmod 4$, then

$$\phi^2 gc(\mathscr{E}, q, \mathscr{L}) = [\widetilde{\mathscr{C}_0}(\mathscr{E}, q, \mathscr{L})]$$

in $H^2_{\text{ét}}(Z, \mathbf{G}_m)$, where $\phi^2 : H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \to H^2_{\text{ét}}(Z, \mathbf{G}_m)$ is a certain canonical comparison map.

In §4 we proceed to exactly calculate the similarity Clifford invariant in terms of "involutive" cohomology classes arising from what we call *Clifford data.* A Clifford datum mixes the canonical involution of the even Clifford algebra together with so-called *torsion data* arising from the Clifford bimodule. One should think of a torsion datum as a generalization of an involution. A Clifford datum $(Z/X, \mathscr{A}, \sigma, \mathscr{P}, \varphi)$ combines an Azumaya algebra with involution $(\mathscr{A}, \sigma)$ of the first kind (resp. of unitary type) together with a torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of unitary type (resp. of the first kind). The interaction of both these structures enables the explicit calculation of the similarity Clifford invariant.

A torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ on X gives rise to a cohomology class $[\mathscr{A}, \mathscr{P}, \varphi]$ in $H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)$ or in $H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}})$, depending on whether it is of the first type or of unitary type, respectively.

Theorem 4. *Let X be a scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a regular line bundle-valued quadratic form of even rank n and Clifford datum $(Z/X, \mathscr{A}, \sigma, \mathscr{P}, \varphi)$.*

a) If $n \equiv 2 \bmod 4$ then $gc(\mathscr{E}, q, \mathscr{L}) = [\mathscr{A}, \mathscr{P}, \varphi]$ in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)$

b) If $n \equiv 0 \bmod 4$ then $i^2 gc(\mathscr{E}, q, \mathscr{L}) = [\mathscr{A}, \mathscr{P}, \varphi]$ in $H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m)$.

where $i^2 : H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \to H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m)$ is a certain canonical comparison map.

In the case where m is even, we also have an exact formula (see Theorem 4.12) expressing the similarity Clifford invariant in terms of Clifford data. To achieve this, we establish a concrete presentation for the cohomology group $H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4)$, generalizing Colliot-Thélène–Gille–Parimala [28, Prop. 2.10], as a certain fibered product of involutive Brauer groups.

The results in §3 and §4 indicate that the (refined) Tits algebra captures the even Clifford algebra with it's canonical involution while the similarity Clifford invariant constructed here captures the Clifford bimodule and its associated involutive structures.

Finally, in §5 we use the classification of line bundle-valued quadratic forms of rank 2, 4, and 6 in terms of twisted norm and pfaffian functors to explicitly compute the similarity Clifford invariant and give necessary and sufficient conditions as to its vanishing. The classification of regular quadratic forms of low rank over rings was initiated by Kneser, Knus, Ojanguren, Parimala, Paques, and Sridharan [60], [62], [63], [61], [64], [65]. Bichsel [18] and Bichsel–Knus [17] provide an extension of this theory to line bundle-valued forms of trivial discriminant over rings. In the context of quadratic forms over schemes, *low rank* usually means of rank ≤ 6. In this interval, the exceptional isomorphisms of Dynkin diagrams $\mathsf{A}_1 = \mathsf{B}_1 = \mathsf{C}_1$, $\mathsf{D}_2 = \mathsf{A}_1^2$, $\mathsf{B}_2 = \mathsf{C}_2$, and $\mathsf{A}_3 = \mathsf{D}_3$, have beautiful reverberations in the theory of quadratic forms of rank 3, 4, 5, and 6, respectively. Now, a standard reference on this work is Knus [68, Ch. V]. Over fields, a wonderful reference is [69, IV §15]. Much of the work in §5 can be viewed as generalization, from fields to schemes, of facts from [69]. While much of the existing theory over rings can be globalized, a unified treatment using the category of Clifford data (see §4.3) appears in Auel [4].

Theorem 5 (Corollaries 5.6, 5.10, 5.14)**.** *Let X be a scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a regular line bundle-valued quadratic form of even rank n and étale quadratic $f : Z \to X$ arising from the center of the even Clifford algebra with ι the nontrivial element of the Galois group of Z/X.*

- *If $n = 2$ then $(\mathscr{E}, q, \mathscr{L})$ has trivial discriminant if and only if it's similar to a hyperbolic form $H_{\mathscr{L}}(\mathscr{V})$ for some invertible $\mathscr{O}_X$-module $\mathscr{V}$. Furthermore, $gc(\mathscr{E}, q, \mathscr{L})$ is trivial if and only if both $\mathscr{L}$ and $\mathscr{V}$ are squares in* $\mathrm{Pic}(X)$.
- *If $n = 4$ then $gc(\mathscr{E}, q, \mathscr{L})$ is trivial if and only if there exists a locally free $\mathscr{O}_Z$-module $\mathscr{V}$ of rank 2 satisfying $\det \mathscr{V} \cong \iota^* \det \mathscr{V}$ such that $(\mathscr{E}, q, \mathscr{L})$ is similar to the norm $N_{Z/X}(\mathscr{V}, \wedge, \det \mathscr{V})$ of the canonical skew-symmetric form $\mathscr{V} \otimes \mathscr{V} \xrightarrow{\wedge} \det \mathscr{V}$. Furthermore, if $(\mathscr{E}, q, \mathscr{L})$ has trivial discriminant, then it's similar to the canonical quadratic form $\mathscr{W} \otimes \mathscr{W}' \xrightarrow{\wedge \otimes \wedge} \det \mathscr{W} \otimes \det \mathscr{W}'$ for locally free $\mathscr{O}_X$-modules $\mathscr{W}$ and $\mathscr{W}'$ of rank 2 with $\det \mathscr{W} \cong \det \mathscr{W}'$.*
- *If $n = 6$ then $gc(\mathscr{E}, q, \mathscr{L})$ is trivial if and only if there exists a regular line bundle-valued Z/X-hermitian form $(\mathscr{H}, h, \mathscr{M})$ of rank 4 with trivial hermitian discriminant such that $f^*(\mathscr{E}, q, \mathscr{L})$ is similar (with the pullback Z/X-hermitian structure) to the canonical quadratic form $\bigwedge^2 \mathscr{H} \xrightarrow{\wedge} \det \mathscr{H}$. Furthermore, if $(\mathscr{E}, q, \mathscr{L})$ has trivial discriminant, then it's similar to the canonical quadratic form $\bigwedge^2 \mathscr{W} \xrightarrow{\wedge} \det \mathscr{W}$ for a locally free $\mathscr{O}_X$-module $\mathscr{W}$ of rank 4 with $\det \mathscr{W}$ a square in* $\mathrm{Pic}(X)$.

While much of the motivation to construct such a theory of cohomological invariants for line bundle-valued quadratic forms (and eventually Azumaya algebras with orthogonal involution) comes from classification problems, there are many other potential applications. As a complement to a series of papers by Parimala, Scharlau, and Sridharan [82], [83], [81], the author [2] uses the similarity Clifford invariant to prove the validity of Merkurjev's theorem for line bundle-valued quadratic forms, where it was known to fail for ($\mathscr{O}_X$-valued) quadratic forms. There are also applications to the "orthogonal Riemann-Roch" problem on computing cohomological invariants of pushforwards in derived Grothendieck–Witt groups. Finally, a version of the four-fold covering of the orthogonal similitude group can be adapted to Galois representations—analogously to Deligne's [31] use of the spin group—to investigate obstructions to embedding problems and local constants associated to essentially self-dual Galois representations of orthogonal type.

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1. Torsorial theory of line bundle-valued forms

Let X be a scheme and $\mathscr{L}$ and invertible $\mathscr{O}_X$-module. An *($\mathscr{L}$-valued) bilinear form* on X is a triple $(\mathscr{E}, b, \mathscr{L})$, where $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module of finite rank and $b : T^2\mathscr{E} = \mathscr{E} \otimes \mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-module morphism, equivalently, b is a global section of $\mathscr{H}om(T^2\mathscr{E}, \mathscr{L})$. An $\mathscr{L}$-valued bilinear form is *symmetric* if b factors through the canonical epimorphism $T^2\mathscr{E} \to S^2\mathscr{E}$, equivalently, b is a section of $\mathscr{H}om(S^2\mathscr{E}, \mathscr{L}) \subset \mathscr{H}om(T^2\mathscr{E}, \mathscr{L})$. An $\mathscr{L}$-valued bilinear form is *skew-symmetric* if b factors through the canonical epimorphism $T^2\mathscr{E} \to \bigwedge^2\mathscr{E}$, equivalently, b is a section of $\mathscr{H}om(\bigwedge^2\mathscr{E}, \mathscr{L}) \subset \mathscr{H}om(T^2\mathscr{E}, \mathscr{L})$. An *($\mathscr{L}$-valued) quadratic form* on X is a triple $(\mathscr{E}, q, \mathscr{L})$, where $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module and $q : S_2\mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-module morphism, equivalently, q is a global section of $\mathscr{H}om(S_2\mathscr{E}, \mathscr{L}) \cong S^2(\mathscr{E}^\vee) \otimes \mathscr{L}$. Here $S^2\mathscr{E}$ and $S_2\mathscr{E}$ denote the second symmetric power and submodule of symmetric second tensors of $\mathscr{E}$, respectively.

Conventions. Denote by $X_{\text{ét}}$ the (large) étale site of X. We consider many sheaves of groups on $X_{\text{ét}}$, such as $\mathbf{GL}(\mathscr{E})$ and $\mathbf{O}(\mathscr{E}, q, \mathscr{L})$, that depend on a regular line bundle-valued bilinear form $(\mathscr{E}, q, \mathscr{L})$. When writing exact sequences and commutative diagrams of such groups, we will often suppress this dependence. We will also suppress the dependence on the base X, when writing common sheaves of groups, such as $\mathbf{G}_{\text{m}}$ and $\boldsymbol{\mu}_n$, that are defined over $\operatorname{Spec}\mathbb{Z}$.

If $\alpha : \mathbf{G} \to \mathbf{G}$ is a homomorphism of sheaves of groups on $X_{\text{ét}}$, then denote by $\alpha^i : H^i_{\text{ét}}(X, \mathbf{G}) \to H^i_{\text{ét}}(X, \mathbf{G}')$ the induced map on étale cohomology, where we assume $\mathbf{G}$ and $\mathbf{G}'$ are abelian for $i \geq 2$.

All algebras and sheaves of algebras will be unital and associative. The sheaf of algebra automorphisms of an $\mathscr{O}_X$-algebra $\mathscr{A}$ will be denoted by $\mathbf{Aut}(\mathscr{A})$. For a commutative $\mathscr{O}_X$-algebra $\mathscr{Z}$, and a $\mathscr{Z}$-module (resp. algebra) $\mathscr{M}$, we will denote by $\widetilde{\mathscr{M}}$ the associated $\mathscr{O}_Z$-module (resp. algebra) on $Z = \mathbf{Spec}\,\mathscr{Z}$. A finite étale morphism $f : Z \to X$ of degree 2 will be called *étale quadratic.*

1.1. Quadratic and symmetric bilinear forms.

Over an arbitrary scheme, the above notion of quadratic form agrees with the "classical" notion. See also Swan [92, Lemma 2.1] and Wood [97, §2.6].

Lemma 1.1. *Let $\mathscr{Q}uad(\mathscr{E}, \mathscr{L})$ be the Zariski sheaf associated with the presheaf assigning to $U \to X$ the abelian group of maps $q : \mathscr{E}|_U \to \mathscr{L}|_U$ that satisfy*

- *$q(a\,v) = a^2\,q(v)$ on local sections a of $\mathscr{O}_U$ and v of $\mathscr{E}|_U$, and*
- *the map $b_q : \mathscr{E}|_U \times \mathscr{E}|_U \to \mathscr{L}|_U$, defined locally on sections by $b(v, w) = q(v+w) - q(v) - q(w)$, is $\mathscr{O}_U$-bilinear.*

Then there is a canonical isomorphism of $\mathscr{O}_X$-modules $\mathscr{Q}uad(\mathscr{E}, \mathscr{L}) \to \mathscr{H}om(S_2\mathscr{E}, \mathscr{L})$.

There's also a canonical $\mathscr{O}_X$-module morphism $\Gamma^2\mathscr{E} \to S_2\mathscr{E}$ from the module of second divided powers, which is an isomorphism is $\mathscr{E}$ is locally free, see Deligne [30, 5.5.2.5]. Composing the isomorphism constructed in Lemma 1.1 with the dual of this isomorphism yields an isomorphism $\mathscr{Q}uad(\mathscr{E}, \mathscr{L}) \to \mathscr{H}om(\Gamma^2\mathscr{E}, \mathscr{L})$, yielding an additional general tensorial interpretation of quadratic forms. Also, there's a canonical isomorphism $S^2(\mathscr{E}^\vee) \otimes \mathscr{L} \to \mathscr{H}om(S_2\mathscr{E}, \mathscr{L})$.

As in the classical case, if 2 is invertible on X, then every quadratic form is the associated quadratic form of a (symmetric) bilinear form on X, i.e. the canonical $\mathscr{O}_X$-module morphism $\mathscr{H}om(S^2\mathscr{E}, \mathscr{L}) \to \mathscr{H}om(S_2\mathscr{E}, \mathscr{L})$ is an isomorphism. Note that this morphism is *not* in general equivariant with respect to the natural $\mathbf{GL}(\mathscr{E}) \times \mathbf{GL}(\mathscr{L})$ actions.

1.2. Adjoint morphism.

An $\mathscr{L}$-valued bilinear form $b : T^2\mathscr{E} \to \mathscr{L}$ has an $\mathscr{O}_X$-module *adjoint morphism* $\psi_b : \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L})$ defined on sections by $v \mapsto \big(w \mapsto b(v \otimes w)\big)$. A bilinear form b is called *regular* if ψ_b is an isomorphism of $\mathscr{O}_X$-modules. An $\mathscr{L}$-valued quadratic form $q : \mathscr{E} \to \mathscr{L}$ is called regular if its associated bilinear form b_q is regular.

On the category of coherent $\mathscr{O}_X$-modules, denote by $(-)^{\vee\mathscr{L}}$ the exact (contravariant) functor $\mathscr{H}om(-, \mathscr{L})$. There is a canonical evaluation morphism of functors

$$\mathrm{ev}^{\mathscr{L}} : \mathrm{id} \to ((-)^{\vee\mathscr{L}})^{\vee\mathscr{L}},$$

which is an isomorphism on the subcategory $\mathsf{VB}(X)$ of locally free $\mathscr{O}_X$-modules of finite rank. The triple $(\mathsf{VB}(X), (-)^{\vee\mathscr{L}}, \mathrm{ev}^{\mathscr{L}})$ forms an *exact category with duality* in the language of Balmer [6].

1.3. Similarities and isometries.

A *similarity (transformation)* or *similitude* between bilinear forms $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}')$ is a pair (φ, λ) of $\mathscr{O}_X$-module isomorphisms $\varphi : \mathscr{E} \to \mathscr{E}'$ and

$\lambda : \mathscr{L} \to \mathscr{L}'$ such that either of the following (equivalent) diagrams,

$$\text{(2)} \qquad \begin{array}{ccc} T^2\mathscr{E} & \xrightarrow{b} & \mathscr{L} \\ {\scriptstyle T^2\varphi}\downarrow & & \downarrow{\scriptstyle \lambda} \\ T^2\mathscr{E}' & \xrightarrow{b'} & \mathscr{L}' \end{array} \qquad \begin{array}{ccc} \mathscr{E} & \xrightarrow{\psi_b} & \mathscr{H}om(\mathscr{E},\mathscr{L}) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \lambda^{-1}\varphi^{\vee\mathscr{L}}} \\ \mathscr{E}' & \xrightarrow{\psi_{b'}} & \mathscr{H}om(\mathscr{E}',\mathscr{L}') \end{array}$$

of $\mathscr{O}_X$-modules commute, where $\lambda^{-1}\varphi^{\vee\mathscr{L}}(\psi) = \lambda^{-1}\circ\psi\circ\varphi$ on sections. Note that the commutativity of the left-hand diagram (2) takes on the familiar formula $b'(\varphi(v), \varphi(w)) = \lambda\circ b(v, w)$ on sections, and can be adapted to define similarity transformations between quadratic forms. A similarity transformation (φ, λ) is an *isometry* if $\mathscr{L} = \mathscr{L}'$ and λ is the identity map.

Denote by $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$ (resp. $\mathbf{Iso}(\mathscr{E}, b, \mathscr{L})$) the presheaf of similitudes (resp. isometries) of a regular $\mathscr{L}$-valued quadratic or bilinear form $(\mathscr{E}, b, \mathscr{L})$. In fact, these presheaves are sheaves on $X_{\text{ét}}$ and are representable by smooth affine reductive group schemes over X (see Demazure–Gabriel [33, III §5.2.3]). When b is quadratic or symmetric, we denote this group by $\mathbf{GO}(\mathscr{E}, b, \mathscr{L})$ (resp. $\mathbf{O}(\mathscr{E}, b, \mathscr{L})$) and call them the *orthogonal similitudes group* (resp. *orthogonal group*) of $(\mathscr{E}, b, \mathscr{L})$. When b is skew-symmetric, we denote these groups by $\mathbf{GSp}(\mathscr{E}, b, \mathscr{L})$ (resp. $\mathbf{Sp}(\mathscr{E}, b, \mathscr{L})$). For $\epsilon \in H^0_{\text{ét}}(X, \boldsymbol{\mu}_2)$ we will say that an $\mathscr{L}$-valued bilinear form is *ϵ-symmetric* if on each connected component X' of X, it is either symmetric and $\epsilon|_{X'} = 1$ or skew-symmetric and $\epsilon|_{X'} = -1$. We will often omit the dependence of the groups of similitudes and isometries on the form $(\mathscr{E}, q, \mathscr{L})$ when no confusion may arise. Even when these sheaves of groups are representable by schemes over X, we will still think of them as sheaves of groups on $X_{\text{ét}}$.

1.4. **Torsor interpretations.** For the abstract notion of a (right) torsor for a sheaf of groups on a site, see Giraud [47].

Proposition 1.2. *Let X be a scheme. Let $(\mathscr{E}, b, \mathscr{L})$ be a fixed $\mathscr{L}$-valued ϵ-symmetric bilinear form of rank n on X.*

- *a) The groupoid of $\mathbf{Iso}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the groupoid whose objects are regular $\mathscr{L}$-valued ϵ-symmetric bilinear forms of rank n and whose morphisms are isometries.*
- *b) The groupoid of $\mathbf{Sim}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the groupoid whose objects are regular line bundle-valued ϵ-symmetric bilinear forms of rank n and whose morphisms are similarity transformations.*

Proof. For *a)*, the statement is a twist of [33, III, §5.2.1]. For further details, and for *b)*, see Auel [3, Appendix A]. □

Remark 1.3. If $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module of rank n, we will denote by $\mathbf{GL}(\mathscr{E})$ its *general linear group* on $X_{\text{ét}}$. The sheaves of groups $\mathbf{GL}(\mathscr{E})$ and $\mathbf{GL}(\mathscr{E}')$ are isomorphic on $X_{\text{ét}}$ if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$ for some invertible $\mathscr{O}_X$-module $\mathscr{L}$. For any invertible $\mathscr{O}_X$-module $\mathscr{L}$, there's a canonical isomorphism $\mathbf{G}_{\text{m}} \xrightarrow{\sim} \mathbf{GL}(\mathscr{L})$, through which we shall identify these groups. A subtlety inherent in this identification arises when considering the pointed set of isomorphism classes of $\mathbf{GL}(\mathscr{L})$-torsors, for which the distinguished point is canonically associated to the isomorphism class of $\mathscr{L}$. A similar remark should be made for the identification $\boldsymbol{\mu}_2 = \mathbf{O}(\mathscr{E}, b, \mathscr{L})$ where $(\mathscr{E}, b, \mathscr{L})$ is a regular form of rank 1. In what follows, we will keep track of these identifications, if not in our notation, then in any statement concerning torsors.

1.5. **The multiplier sequence.** The map assigning $(\varphi, \lambda) \mapsto \lambda$ on sections defines the *multiplier coefficient* homomorphism $\mu : \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{GL}(\mathscr{L}) = \mathbf{G}_{\text{m}}$.

Proposition 1.4. *For any scheme X with 2 invertible, the sequence of sheaves of groups,*

$$1 \to \mathbf{Iso}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\mu} \mathbf{G}_m \to 1,$$

is exact on $X_{\text{ét}}$ and is called the multiplier sequence*.*

Proof. The only part needing an argument is that μ is an epimorphism. This follows from the fact that μ restricted to the central subgroup of homotheties is the squaring map, i.e. there's a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_{\text{m}} & \xrightarrow{2} & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbf{Iso} & \longrightarrow & \mathbf{Sim} & \xrightarrow{\mu} & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \end{array}$$

of sheaves of groups on $X_{\text{ét}}$ with exact rows. Of course, the (top horizontal) Kummer sequence in exact on $X_{\text{ét}}$ if 2 is invertible. □

Remark 1.5. The interpretation of the multiplier sequence on cohomology is as follows. If $(\mathscr{E}, b, \mathscr{L})$ is a fixed $\mathscr{L}$-valued ϵ-symmetric bilinear form of rank n on X, then the map

$$H^1_{\text{ét}}(X, \mathbf{Iso}(\mathscr{E}, b, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbf{Sim}(\mathscr{E}, b, \mathscr{L}))$$

takes the isometry class of a regular $\mathscr{L}$-valued ϵ-symmetric bilinear form of rank n on X to its similarity class. Under the identification $\mathbf{GL}(\mathscr{L}) = \mathbf{G}_{\mathrm{m}}$ the map

$$H^1_{\text{ét}}(X, \mathbf{Sim}(\mathscr{E}, b, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) \cong \mathrm{Pic}(X)$$

takes the similarity class of a regular $\mathscr{L}'$-valued symmetric bilinear form of rank n on X to the class of $\mathscr{L}'$ in $\mathrm{Pic}(X)$.

1.6. **Forms of odd rank.** The theory of regular line bundle-valued bilinear forms of odd rank essentially reduces to the study of regular $\mathscr{O}_X$-valued bilinear forms.

Lemma 1.6. *If $\mathscr{L}$ is not a square in the Picard group* $\mathrm{Pic}(X)$*, then any regular $\mathscr{L}$-valued bilinear form has even rank.*

Proof. Let $(\mathscr{E}, b, \mathscr{L})$ be a regular bilinear form of rank n. Comparing determinants yields an isomorphism $\det(E)^{\otimes 2} \cong \mathscr{L}^{\otimes n}$ of line bundles. Thus we see that either n must be even or $\mathscr{L}$ is a square in $\mathrm{Pic}(X)$ (up to an n-torsion element, which is itself a square). An alternate proof in the symmetric case, using the even Clifford algebra (see §1.8), can be found in [17, Theorem 3.7]. □

Thus every regular line bundle-valued bilinear form of odd rank has values in the square of some line bundle. In fact, for symmetric forms, there is even a way to take a canonical square root of the value bundle, see Auel [3, §1.3]. This can also be seen as a consequence of the direct product structures $\mathbf{G}_{\mathrm{m}} \times \mathbf{SO}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{GO}(\mathscr{E}, b, \mathscr{L})$ of orthogonal similitude groups in odd rank. In view of this fact, we will only consider regular line bundle-valued forms of even rank.

1.7. **Metabolic forms.** The notion of a metabolic form—the correct generalization to schemes of the notion of hyperbolic form—is necessary to the construction of the Grothendieck–Witt and Witt groups of schemes. The formalism of $\mathscr{O}_X$-valued metabolic forms over arbitrary schemes was first introduced by Knebusch [58, I §3] and then extended to the context of triangulated and exact categories with duality by Balmer [6], [7], and [8]. We will summarize the formalism of metabolic forms in the context of line bundle-valued symmetric bilinear forms.

Definition 1.7. Let $(\mathscr{E}, b, \mathscr{L})$ be a line bundle-valued symmetric bilinear form on X and $\mathscr{V} \xrightarrow{j} \mathscr{E}$ a locally free $\mathscr{O}_X$-submodule with locally free quotient. The *($\mathscr{L}$-valued) orthogonal complement* $\mathscr{V}^\perp \xrightarrow{j^\perp} \mathscr{E}$ is the kernel of the composition $\mathscr{E} \xrightarrow{j^{\vee_{\mathscr{L}}} \circ \psi_b} \mathscr{H}om(\mathscr{V}, \mathscr{L})$.

Note that the restriction of the adjoint map yields an isomorphism $\mathscr{V}^\perp \xrightarrow{\psi_b|_{\mathscr{V}^\perp}} \mathscr{H}om(\mathscr{E}/\mathscr{V}, \mathscr{L})$.

Definition 1.8. Let $(\mathscr{E}, b, \mathscr{L})$ be a line bundle-valued symmetric bilinear form on X and $\mathscr{V} \xrightarrow{j} \mathscr{E}$ a locally free $\mathscr{O}_X$-submodule. Then $\mathscr{V}$ is an *($\mathscr{L}$-valued) lagrangian* if $\mathscr{V} = \mathscr{V}^\perp$, i.e. if there's an isomorphism of short exact sequences,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{V} & \xrightarrow{j = j^\perp} & \mathscr{E} & \xrightarrow{j^{\vee_{\mathscr{L}}} \circ \psi_b} & \mathscr{H}om(\mathscr{V}, \mathscr{L}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{can}^{\mathscr{L}}} & & \downarrow{\scriptstyle \psi_b} & & \| & & \\
0 & \longrightarrow & \mathscr{H}om(\mathscr{H}om(\mathscr{V}, \mathscr{L}), \mathscr{L}) & \xrightarrow{(j^{\vee_{\mathscr{L}}} \circ \psi_b)^{\vee_{\mathscr{L}}}} & \mathscr{H}om(\mathscr{E}, \mathscr{L}) & \xrightarrow{j^{\vee_{\mathscr{L}}}} & \mathscr{H}om(\mathscr{V}, \mathscr{L}) & \longrightarrow & 0
\end{array}$$

of locally free $\mathscr{O}_X$-modules. An $\mathscr{L}$-valued symmetric bilinear form on X is called *metabolic* if it has an $\mathscr{L}$-valued lagrangian. An $\mathscr{L}$-valued quadratic form on X is called *metabolic* if it's associated bilinear form is metabolic. Any metabolic form is regular.

The *$\mathscr{L}$-valued hyperbolic* quadratic form $H_{\mathscr{L}}(\mathscr{V})$ with lagrangian $\mathscr{V}$ is the quadratic form $(\mathscr{V} \oplus \mathscr{H}om(\mathscr{V}, \mathscr{L}), h, \mathscr{L})$ where h is the canonical evaluation pairing $h(v, f) = f(v)$ on sections. We will often consider the associated $\mathscr{L}$-valued hyperbolic symmetric bilinear form as well.

We call an $\mathscr{L}$-valued metabolic form $(\mathscr{E}, b, \mathscr{L})$ a *split metabolic form* if some lagrangian $\mathscr{V} \to \mathscr{E}$ is a direct summand. If 2 is invertible on X, then any split metabolic form $(\mathscr{E}, b, \mathscr{L})$ with direct summand lagrangian $\mathscr{V}$ is isometric to the hyperbolic form $H_{\mathscr{L}}(\mathscr{V})$, see [58, §3 Prop. 1].

Splitting principle for metabolic forms. Over an affine scheme, every (line bundle-valued) metabolic form is split. This follows by an adaptation of Knebusch [58, I §3, Corollary 1]. Over a general scheme, this is no longer the case. However, there's a splitting principle for metabolic forms, in analogy with the classical splitting principle for locally free sheaves.

Theorem 1.9. *Let X be a scheme and let $(\mathscr{E}, b, \mathscr{L})$ be a metabolic form on X with lagrangian $\mathscr{V} \xrightarrow{j} \mathscr{E}$.*

a) There exists a morphism of schemes $f : Y \to X$ such that $f^* : H^i_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^i_{\text{ét}}(X, \boldsymbol{\mu}_2)$ is injective and $f^*(\mathscr{E}, b, \mathscr{L})$ is a split metabolic form with lagrangian $f^*\mathscr{V}$.

b) If 2 is invertible on X, then there exists a morphism of schemes $f : Y \to X$ such that $f^* : H^i_{\text{ét}}(X, \boldsymbol{\mu}_2) \to H^i_{\text{ét}}(Y, \boldsymbol{\mu}_2)$ is injective and $f^*(\mathscr{E}, b, \mathscr{L})$ is isometric to an orthogonal sum of hyperbolic planes

$$H_{f^*\mathscr{L}}(\mathscr{V}_1) \perp \cdots \perp H_{f^*\mathscr{L}}(\mathscr{V}_m),$$

for invertible $\mathscr{O}_X$-modules $\mathscr{V}_1, \ldots, \mathscr{V}_m$ on Y.

Proof. As for *a)*, this is a classical construction. Following Fulton [40, §2], let $\mathscr{V} \xrightarrow{j} \mathscr{E}$ be a lagrangian and let $\mathscr{P} \subset \mathscr{H}om(\mathscr{E}, \mathscr{V})$ be the subbundle whose sections over $U \to X$ are $\mathscr{O}_U$-module morphisms $\varphi : \mathscr{E}|_U \to \mathscr{V}|_U$ such that $j|_U \circ \varphi = \mathrm{id}_U$. Let $Y = \mathbb{V}(\mathscr{P})$ be the corresponding affine bundle, and $f : Y \to X$ the natural projection. Then $f^*\mathscr{E}$ has a tautological projection to $f^*\mathscr{V}$, which is a section of f^*j, hence $f^*(\mathscr{E}, b, \mathscr{L})$ is a split metabolic form with lagrangian $f^*\mathscr{V}$. It is a standard fact that f^* is an injection on cohomology for a Zariski locally trivial affine bundle.

Now *b)* is a direct result of *a)* and the classical splitting principle for locally free sheaves. Also see Fulton [40, §2] for a proof using the isotropic flag bundle of a metabolic bundle. Pulling back to this flag variety yields the required splitting. Also see Esnault–Kahn–Viehweg [34, §5] for a proof using iterated projective bundles. □

1.8. **Even Clifford algebra.** In his thesis, Bichsel [18] constructed an even Clifford algebra of a line bundle-valued quadratic form on an affine scheme. Bichsel–Knus [17], Caenepeel–van Oystaeyen [23] and Parimala–Sridharan [84, §4] gave alternate constructions, all of which we shall recall below. Let $(\mathscr{E}, q, \mathscr{L})$ be a line bundle-valued quadratic form on X, not necessarily assumed to be regular.

Splitting construction. Let $L\mathscr{L} = \oplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}$ (called the *Laurent polynomial algebra* or *generalized Rees ring* in the literature) and define $Y = \mathbf{Spec}\, L\mathscr{L}$. Then $p : Y \to X$ is identified with the total space of the line bundle $\mathscr{L}$ on X with the zero section removed. There is a canonical identification $p^*\mathscr{L} = \mathscr{O}_Y$ defined using the multiplication in the Rees ring, with respect to which, $p^*(\mathscr{E}, q, \mathscr{L})$ has a natural structure of ($\mathscr{O}_Y$-valued) quadratic form on Y. The quasi-coherent sheaf of $\mathscr{O}_X$-algebras $\mathscr{C}'(\mathscr{E}, q, \mathscr{L}) = p_*\mathscr{C}(p^*(\mathscr{E}, q, \mathscr{L}))$, where $\mathscr{C}$ is the standard Clifford algebra functor, is called the *generalized Clifford algebra* in [17]. As an $L\mathscr{L}$-algebra, $\mathscr{C}'(\mathscr{E}, q, \mathscr{L})$ inherits a natural $\mathbb{Z}$-graded $\mathscr{O}_X$-algebra structure. The *even Clifford algebra* $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$ is defined to be its 0th degree submodule. This construction is due to Caenepeel–van Oystaeyen [23].

Tensorial construction. Let $T\mathscr{E} = \oplus_{n\geq 0}\mathscr{E}^{\otimes n}$ be the tensor algebra on $\mathscr{E}$. Define an ideal $\mathscr{J} \subset T\mathscr{E} \otimes_{\mathscr{O}_X} L\mathscr{L}$, locally generated by $v \otimes v \otimes 1_{L\mathscr{L}} - 1_{T\mathscr{E}} \otimes q(v)$ for sections v of $\mathscr{E}$. Define a $\mathbb{Z}$-grading on $T\mathscr{E} \otimes_{\mathscr{O}_X} L\mathscr{L}$ by putting $\mathscr{E}$ in degree 1 and $\mathscr{L}$ in degree 2. Then $\mathscr{J}$ is a $2\mathbb{Z}$-graded ideal and $\mathscr{C}'(\mathscr{E}, q, \mathscr{L}) = T\mathscr{E} \otimes_{\mathscr{O}_X} L\mathscr{L}/\mathscr{J}$ is a $\mathbb{Z}$-graded (as well as a $\mathbb{Z}/2\mathbb{Z}$-graded) $\mathscr{O}_X$-algebra, defining the generalized Clifford algebra. The even Clifford algebra is defined to be its 0th degree submodule. This construction is due to Bichsel–Knus [17, §3].

Alternatively, following [69, II Lemma 8.1], we can directly define ideals $\mathscr{J}_1$ and $\mathscr{J}_2$ of $T(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee)$ locally generated by

$$v \otimes v \otimes f - f(q(v)), \quad \text{and} \quad u \otimes v \otimes f \otimes v \otimes w \otimes g - u \otimes w \otimes f(q(v)) \cdot g,$$

respectively, for sections u, v, w of $\mathscr{E}$ and f, g of $\mathscr{L}^\vee$. Then $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) = T(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee)/(\mathscr{J}_1 + \mathscr{J}_2)$.

Gluing construction. Choose a Zariski affine open cover, $\mathscr{U} = \{U_i\}_{i\in I}$, of X trivializing $\mathscr{L}$ via $\varphi_i : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$. Then $a_{ij} = \varphi_{i\underline{j}}{}^{-1}\varphi_{\underline{i}j} \in \mathbf{G}_{\mathrm{m}}(U_{ij})$ for $(i, j) \in I^2$ (where $U_{ij} = U_i \times_X U_j$ and $\varphi_{i\underline{j}} = \varphi_i|_{U_{ij}}$, etc.) is a Čech 1-cocycle representing $\mathscr{L}$. For each $i \in I$, the composition,

$$\varphi_i^{-1} \circ b|_{U_i} : \mathscr{E}|_{U_i} \to \mathscr{L}|_{U_i} \to \mathscr{O}_{U_i},$$

defines an $\mathscr{O}_{U_i}$-valued quadratic form on U_i, and for each $(i, j) \in I^2$, one checks that the identity map on $\mathscr{E}|_{U_{ij}}$ defines a similarity transformation,

$$(\mathrm{id}|_{U_{ij}}, a_{ij}) : (\mathscr{E}|_{U_{ij}}, \varphi_j^{-1} \circ q|_{U_{ij}}, \mathscr{O}_{U_{ij}}) \to (\mathscr{E}|_{U_{ij}}, \varphi_i^{-1} \circ q|_{U_{ij}}, \mathscr{O}_{U_{ij}}).$$

These similarities lift to $\mathscr{O}_{U_{ij}}$-algebra isomorphisms of the standard even Clifford algebras (see Knus [68, IV Prop. 7.1.1]),

$$\mathscr{C}_0(\mathrm{id}|_{U_{ij}}, a_{ij}) : \mathscr{C}_0(\mathscr{E}|_{U_{ij}}, \varphi_j^{-1} \circ q|_{U_{ij}}) \to \mathscr{C}_0(\mathscr{E}|_{U_{ij}}, \varphi_i^{-1} \circ q|_{U_{ij}}),$$

yielding a gluing datum constructing the even Clifford algebra $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$. This construction is due to Parimala–Sridharan [84, §4]. The original construction in Bichsel [18], in the affine case, uses faithfully flat descent.

Remark 1.10. For a regular ($\mathscr{O}_X$-valued) quadratic form $(\mathscr{H}, b)$ of rank n and trivial Clifford algebra, the homomorphism of sheaves of groups, $\mathbf{GO}(\mathscr{H}, b) \to \mathbf{Aut}(\mathscr{C}_0(\mathscr{H}, b))$ defined by lifting similarities to automorphisms of the even Clifford algebra, induces a map

$$H^1_{\text{ét}}(X, \mathbf{GO}(\mathscr{H}, b)) \to H^1_{\text{ét}}(X, \mathbf{Aut}(\mathscr{C}_0(\mathscr{H}, b)))$$

with the following interpretation: the class associated (by Proposition 1.2 and Proposition 3.3) to the similarity class of a line bundle-valued quadratic form of rank n maps to the isomorphism class of its even Clifford algebra.

Functorial properties. Some consequence of the splitting construction and the classical properties of the even Clifford algebra are (assuming that $(\mathscr{E}, q, \mathscr{L})$ is regular and of rank n on X):

a) If n is odd, $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is a central $\mathscr{O}_X$-algebra. If $n = 2m$ is even, the center $\mathscr{Z}(\mathscr{E}, q, \mathscr{L})$ of $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an étale quadratic $\mathscr{O}_X$-algebra (i.e. the morphism $f : \operatorname{Spec} \mathscr{Z} = Z \to X$ is étale quadratic).

b) If n is odd, $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an Azumaya $\mathscr{O}_X$-algebra of rank 2^{n-1}; if n is even, $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ gives rise to an Azumaya $\mathscr{O}_Z$-algebra of rank 2^{n-2}

c) There is a canonical embedding of locally free $\mathscr{O}_X$-modules

$$i : \mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee \to \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$$

and a unique *canonical involution* τ_0 of $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ that restricts to the naïve switch map on (the first two tensor factors of) $\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee$.

d) Any similarity transformation $(\varphi, \lambda) : (\mathscr{E}, q, \mathscr{L}) \to (\mathscr{E}', q', \mathscr{L}')$ induces an $\mathscr{O}_X$-algebra isomorphism

$$\mathscr{C}_0(\varphi, \lambda) : \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \to \mathscr{C}_0(\mathscr{E}', q', \mathscr{L}').$$

e) Any regular bilinear form $(\mathscr{N}, n, \mathscr{N}^{\otimes 2})$ of rank 1, induces an $\mathscr{O}_X$-algebra isomorphism

$$\mathscr{C}_0(n \otimes \mathrm{id}) : \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \to \mathscr{C}_0(\mathscr{N} \otimes \mathscr{E}, n \otimes q, \mathscr{N}^{\otimes 2} \otimes \mathscr{L}).$$

f) For any morphism of schemes $g : X' \to X$, there's a canonical $\mathscr{O}_X$-module isomorphism

$$g^* \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \xrightarrow{\sim} \mathscr{C}_0(g^*(\mathscr{E}, q, \mathscr{L})).$$

1.9. **The discriminant and Arf invariant.** The classical signed discriminant of a quadratic form generalizes to line bundle-valued bilinear forms of even rank, see Parimala–Srinivas [85, §2.2] and Parimala–Sridharan [84, §4]. We will review this construction here.

By a *discriminant module* $(\mathscr{N}, b)$ on X we mean a regular $\mathscr{O}_X$-valued bilinear form $b : T^2\mathscr{N} \to \mathscr{O}_X$ of rank 1. The group (under tensor product and with identity $\langle 1 \rangle : T^2\mathscr{O}_X \to \mathscr{O}_X$ given by multiplication) of isometry classes of discriminant modules on X is canonically isomorphic to $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$, see Milne [76, III §4]. Given an étale quadratic $f : Z \to X$, let $\mathscr{N}$ be the kernel of the trace map $f_*\mathscr{O}_Z \xrightarrow{\mathrm{Tr}} \mathscr{O}_X$. Then the multiplication in the $\mathscr{O}_X$-algebra $f_*\mathscr{O}_Z$ induces a bilinear form $m|_{\mathscr{N}} : T^2\mathscr{N} \to \mathscr{O}_X$, yielding a discriminant module on X, see Knus [68, III §4.2]. The X-isomorphism classes of étale quadratic $f : Z \to X$ is canonically isomorphic to the group $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, and we've just defined a map $\chi^1 : H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$. This map is the homomorphism induced on cohomology from the canonical homomorphism of group schemes $\chi : \mathbb{Z}/2\mathbb{Z} \to \boldsymbol{\mu}_2$ and is an isomorphism if 2 is invertible on X, see [68, III Prop. 4.2.4].

Let $(\mathscr{E}, b, \mathscr{L})$ be a regular $\mathscr{L}$-valued bilinear form of rank n on X. Applying the determinant functor to the adjoint morphism yields an $\mathscr{O}_X$-module isomorphism

$$\det \mathscr{E} \xrightarrow{\det \psi_b} \det \mathscr{H}om(\mathscr{E}, \mathscr{L}) \xrightarrow{\mathrm{can}} \mathscr{H}om(\det \mathscr{E}, \mathscr{L}^{\otimes n}),$$

of $\mathscr{O}_X$-modules, where the canonical isomorphism at right given by

$$f_1 \wedge \cdots \wedge f_n \mapsto \big(v_1 \wedge \cdots \wedge v_n \mapsto \det(f_i(v_j))_{ij}\big) \tag{3}$$

on sections. Write $\det b$ for the $\mathscr{L}^{\otimes n}$-valued bilinear form of rank 1 on X whose adjoint morphism is the composition $\mathrm{can} \circ \det \psi_b$. Note that under the above identification, $\det b$ is given by

$$\begin{array}{ccc} \det \mathscr{E} \otimes \det \mathscr{E} & \xrightarrow{\det b} & \mathscr{L}^{\otimes n} \\ v_1 \wedge \cdots \wedge v_n \otimes w_1 \wedge \cdots \wedge w_n & \longmapsto & \det(b(v_i, w_j))_{ij} \end{array}$$

on sections.

Definition 1.11. Let $(\mathscr{E}, q, \mathscr{L})$ be a regular $\mathscr{L}$-valued bilinear form of even rank $n = 2m$ on X. The *unsigned discriminant form* of $(\mathscr{E}, b, \mathscr{L})$ is the discriminant module $d(\mathscr{E}, b, \mathscr{L}) = (\mathscr{L}^{\vee \otimes m} \otimes \det \mathscr{E}, d(b))$ given by the composition

$$d(b) : T^2(\mathscr{L}^{\vee\otimes m} \otimes \det \mathscr{E}) \to \mathscr{L}^{\vee\otimes n} \otimes T^2 \det \mathscr{E} \xrightarrow{\mathrm{id}\otimes \det b} \mathscr{L}^{\vee\otimes n} \otimes \mathscr{L}^{\otimes n} \xrightarrow{\mathrm{ev}} \mathscr{O}_X,$$

The *signed discriminant form* is the discriminant module $d_\pm(\mathscr{E}, b, \mathscr{L}) = (\mathscr{L}^{\vee\otimes m} \otimes \det \mathscr{E}, (-1)^m d(b))$. The *signed discriminant invariant* is the isometry class of the signed discriminant form in $H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

Definition 1.12. The *Arf covering* of a line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ will refer to the morphism $f : \mathbf{Spec}\, \mathscr{Z}(\mathscr{E}, q, \mathscr{L}) = Z \to X$. The Arf covering is étale quadratic if and only if $(\mathscr{E}, q, \mathscr{L})$ is regular (see [68, IV Prop. 4.8.9]), and then its isomorphism class $[Z/X]$ in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ is called the *Arf invariant.*

Proposition 1.13. *Let X be a scheme and $(\mathscr{E}, q, \mathscr{L})$ be a regular line bundle-valued quadratic form of even rank on X with Arf covering $f : Z \to X$. Then $\chi^1[Z/X] = d_\pm(\mathscr{E}, q, \mathscr{L})$.*

Proof. The statement holds on any Zariski open cover of X trivializing $\mathscr{L}$ by [68, IV Prop. 4.6.3]. Since the transition functions are similarity transformations, the statement glues via Knus [68, IV Prop. 7.1.2]. □

Any similitude $(\varphi, \lambda) : (\mathscr{E}, b, \mathscr{L}) \to (\mathscr{E}', b', \mathscr{L}')$ between forms of even rank on X induces an isometry $d(\varphi, \lambda) : d(\mathscr{E}, b, \mathscr{L}) \to d(\mathscr{E}', b', \mathscr{L}')$ between discriminant forms and an $\mathscr{O}_X$-algebra isomorphism $\mathscr{Z}(\varphi, \lambda) : \mathscr{Z}(\mathscr{E}, q, \mathscr{L}) \to \mathscr{Z}(\mathscr{E}', q', \mathscr{L}')$ between even Clifford algebra centers. These give rise to homomorphisms $\det/\mu^m : \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{O}(d(\mathscr{E}, b, \mathscr{L})) = \boldsymbol{\mu}_2$ and $\Delta : \mathbf{GO}(\mathscr{E}, q, \mathscr{L}) \to \mathbf{Aut}(\mathscr{Z}(\mathscr{E}, q, \mathscr{L})) = \mathbb{Z}/2\mathbb{Z}$, respectively. The induced maps

$$(\det/\mu^m)^1 : H^1_{\text{ét}}(X, \mathbf{GO}(\mathscr{E}, b, \mathscr{L})) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$$
$$\Delta^1 : H^1_{\text{ét}}(X, \mathbf{GO}(\mathscr{E}, q, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$$

then have the following interpretation: $(\mathscr{E}', b', \mathscr{L}') \mapsto d_\pm(\mathscr{E}', b', \mathscr{L}') - d_\pm(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}') \mapsto \mathscr{Z}(\mathscr{E}', b', \mathscr{L}') - \mathscr{Z}(\mathscr{E}, q, \mathscr{L})$, respectively.

Define the group of *proper* similitudes $\mathbf{GSO}(\mathscr{E}, q, \mathscr{L})$ to be the sheaf kernel of the homomorphism Δ. Then $\chi \circ \Delta = \det/\mu^m$, so that when 2 is invertible on X, $\mathbf{GSO}(\mathscr{E}, b, \mathscr{L})$ is also equal to the sheaf kernel of $\det/\mu^m$.

When 2 is invertible on X, there's a short exact sequence of sheaves of groups

(4)
$$1 \to \mathbf{GSO}(\mathscr{E}, b, \mathscr{L}) \to \mathbf{GO}(\mathscr{E}, b, \mathscr{L}) \xrightarrow{\det/\mu^m} \boldsymbol{\mu}_2 \to 1$$

called the *similitude discriminant sequence* fitting into a commutative diagram with exact rows and columns,

(5)
$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mathbf{SO} & \to & \mathbf{GSO} & \xrightarrow{\mu} & \mathbf{G}_{\mathrm{m}} & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \mathbf{O} & \to & \mathbf{GO} & \xrightarrow{\mu} & \mathbf{G}_{\mathrm{m}} & \to & 1 \\
 & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det/\mu^m} & & & & \\
 & & \boldsymbol{\mu}_2 & = & \boldsymbol{\mu}_2 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$.

Lemma 1.14. *Any $\mathscr{L}$-valued metabolic quadratic form $(\mathscr{E}, b, \mathscr{L})$ of rank $n = 2m$ on X has trivial Arf invariant. Moreover, any choice of lagrangian $\mathscr{V} \xrightarrow{j} \mathscr{E}$ induces an $\mathscr{O}_X$-algebra isomorphism $\zeta_{\mathscr{V}} : \mathscr{O}_X \times \mathscr{O}_X \to \mathscr{Z}(\mathscr{E}, q, \mathscr{L})$.*

Proof. In the case that 2 is invertible on X, we can appeal to Proposition 1.13, then use a straightforward adaptation of Knebusch [58, IV Proposition 3.2] or [59, Satz 4.1.2] to line bundle-valued metabolic forms. □

1.10. **Proper torsor interpretation.** Let $\mathscr{Z}$ be an étale quadratic $\mathscr{O}_X$-algebra. An *$\mathscr{Z}$-orientation* on a line bundle-valued quadratic form $(\mathscr{E}', q', \mathscr{L}')$ is an $\mathscr{O}_X$-algebra isomorphism $\zeta' : \mathscr{Z} \to \mathscr{Z}(\mathscr{E}', q', \mathscr{L}')$. If $f : Z \to X$ is étale quadratic, we also say use the term Z/X-orientation to mean an $f_*\mathscr{O}_Z$-orientation. When 2 is invertible on X and $\mathscr{Z} = \mathscr{Z}(\mathscr{E}, q, \mathscr{L})$, then a $\mathscr{Z}$-orientation on $(\mathscr{E}', q', \mathscr{L}')$ is equivalent to an isometry $\zeta' : d_\pm(\mathscr{E}, q, \mathscr{L}) \to d_\pm(\mathscr{E}', q', \mathscr{L}')$ of discriminant modules, by Proposition 1.13.

Proposition 1.15. *Let X be a scheme and $(\mathscr{E}, b, \mathscr{L})$ a regular $\mathscr{L}$-valued quadratic form of even rank n on X.*

a) The groupoid of $\mathbf{SO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are $\mathscr{Z}(\mathscr{E}, q, \mathscr{L})$-oriented regular $\mathscr{L}$-valued quadratic forms $((\mathscr{E}', b', \mathscr{L}), \zeta')$ of rank n, and whose morphisms between objects $((\mathscr{E}', b', \mathscr{L}), \zeta')$ and $((\mathscr{E}'', b'', \mathscr{L}), \zeta'')$ are isometries $\varphi : (\mathscr{E}', b', \mathscr{L}) \to (\mathscr{E}'', b'', \mathscr{L})$ such that $\zeta'' = \mathscr{Z}(\varphi) \circ \zeta'$.

b) The groupoid of $\mathbf{GSO}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category whose objects are $\mathscr{Z}(\mathscr{E}, q, \mathscr{L})$-oriented regular line bundle-valued quadratic forms $((\mathscr{E}', b', \mathscr{L}'), \zeta')$ of rank n, and whose morphisms between objects $((\mathscr{E}', b', \mathscr{L}'), \zeta')$ and $((\mathscr{E}'', b'', \mathscr{L}''), \zeta'')$ are similarities $\varphi : (\mathscr{E}', b', \mathscr{L}') \to (\mathscr{E}'', b'', \mathscr{L}'')$ such that $\zeta'' = \mathscr{Z}(\varphi) \circ \zeta'$.

Proof. This is a straightforward adaptation of Proposition 1.2. □

2. Clifford invariants of line bundle-valued forms

In this section, after reviewing étale cohomological invariants related to 1st Chern classes (in §2.1), we will construct the four-fold covering of the orthogonal similitude group (in §2.3) that gives rise to an étale cohomological invariant of line bundle-valued quadratic forms extending the classical Clifford invariant. We then show that our invariant respects Grothendieck–Witt equivalence of line bundle-valued forms (in §2.7).

2.1. **Chern classes.** Let $l \geq 2$ be such that l is invertible on X. Recall Grothendieck's [48] construction of *Chern classes modulo l* in étale cohomology, $c_i(\mathscr{E}, \boldsymbol{\mu}_l) \in H^{2i}_{\text{ét}}(X, \boldsymbol{\mu}_l^{\otimes i})$, of a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank. If $\mathscr{E}$ is invertible, define $c_1(\mathscr{E}, \boldsymbol{\mu}_l) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_l)$ to be the image of the isomorphism class of $\mathscr{E}$ under the coboundary map $\mathrm{Pic}(X) \cong H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_l)$ arising from the Kummer sequence,

$$1 \to \boldsymbol{\mu}_l \to \mathbf{G}_{\mathrm{m}} \xrightarrow{l} \mathbf{G}_{\mathrm{m}} \to 1,$$

on $X_{\text{ét}}$. For a general locally free $\mathscr{O}_X$-module, the construction is reduced to the case of invertible modules by the splitting principle. In particular, $c_1(\mathscr{E}, \boldsymbol{\mu}_l) = c_1(\det \mathscr{E}, \boldsymbol{\mu}_l) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_l)$. Note that in view of the long exact sequence in étale cohomology associated to the Kummer sequence, $c_1(\mathscr{E}, \boldsymbol{\mu}_l)$ is trivial if and only if $\det \mathscr{E}$ is an lth power in the Picard group $\mathrm{Pic}(X)$.

For the rest of this section we will assume that 2 is invertible on X. The scalar multiplication homomorphism $m : \mathbf{GL}(\mathscr{L}) \times \mathbf{O}(\mathscr{E}, q, \mathscr{L}) \to \mathbf{GO}(\mathscr{E}, q, \mathscr{L})$, given by $(\lambda, \varphi) \mapsto (\lambda \circ \varphi, \lambda^2)$ on sections, yields a commutative diagram of sheaves of groups with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_{\mathrm{m}} \times \mathbf{O} & \xrightarrow{m} & \mathbf{GO} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \mu} & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \xrightarrow{2} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \end{array} \tag{6}$$

on $X_{\text{ét}}$, where $\boldsymbol{\mu}_2 \to \mathbf{G}_{\mathrm{m}} \times \mathbf{O}(\mathscr{E}, q, \mathscr{L})$ is the diagonal inclusion and the center vertical map is the projection onto the first factor. The commutativity of diagram (6) implies that the coboundary map of the first row $H^1_{\text{ét}}(X, \mathbf{GO}(\mathscr{E}, q, \mathscr{L})) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ has the following interpretation (via Proposition 1.2): the class associated to $(\mathscr{E}', b', \mathscr{L}')$ is mapped to $c_1(\mathscr{L}', \boldsymbol{\mu}_2) - c_1(\mathscr{L}, \boldsymbol{\mu}_2)$.

Remark 2.1. What about the 1st Chern class $c_1(\mathscr{E}, \boldsymbol{\mu}_2)$ of the underlying locally free module? For regular line bundle-valued symmetric bilinear forms of even rank $n = 2m$, the discriminant and 1st Chern class (modulo 2) of the value line bundle determine the 1st Chern class of $\mathscr{E}$. Indeed, we have the equations $\det \mathscr{E} = \mathscr{L}^m \otimes d(\mathscr{E}, b, \mathscr{L})$ in $\mathrm{Pic}(X)$ (see the proof of Lemma 1.6), and also

$$c_1(d(\mathscr{E}, b, \mathscr{L}), \boldsymbol{\mu}_2) = d(\mathscr{E}, b, \mathscr{L}) \smile (d(\mathscr{E}, b, \mathscr{L}) + (-1))$$

in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \cong H^2_{\text{ét}}(X, \boldsymbol{\mu}_2^{\otimes 2})$ due to Esnault–Kahn–Viehweg [34, Lemma 5.3].

2.2. **Weil restriction.** Let $f : Z \to X$ be finite étale and $\mathbf{G}$ a sheaf of groups on $Z_{\text{ét}}$ represented by a Z-scheme, which by abuse of notation, we shall also denote by $\mathbf{G}$. Then $f_*\mathbf{G}$ is a sheaf of groups on $X_{\text{ét}}$ represented by an X-scheme $\mathbf{R}_{Z/X}\mathbf{G}$, called the *Weil restriction* or *corestriction* of $\mathbf{G}$ through $f : Z \to X$. By further abuse of notation, we often write $\mathbf{R}_{Z/X}\mathbf{G}$ for the sheaf of groups $f_*\mathbf{G}$. A general reference on Weil restriction is [33, I §1.6.6].

The counit of adjunction provides a canonical morphism $\varepsilon_{\mathbf{G}} : f^{-1}\mathbf{R}_{Z/X}\mathbf{G} \to \mathbf{G}$ of sheaves of groups on $Z_{\text{ét}}$ (or of Z-schemes), and the Weil restriction is universal for this property: given any sheaf of groups on $X_{\text{ét}}$ (or X-scheme) T and a morphism $\phi : f^{-1}T \to \mathbf{G}$, there exists a unique morphism $\varphi : T \to \mathbf{R}_{Z/X}\mathbf{G}$ of sheaves of groups on $X_{\text{ét}}$ (or X-schemes) such that $\varepsilon_{\mathbf{G}} \circ f^{-1}\varphi = \phi$.

If $\mathbf{G}$ is a sheaf of groups on $X_{\text{ét}}$ representable by an X-scheme, then $f^{-1}\mathbf{G}$ is a sheaf of groups on $Z_{\text{ét}}$ representable by the Z-scheme $Z \times_X \mathbf{G}$, and the unit of adjunction provides a canonical morphism of sheaves of groups on $X_{\text{ét}}$ (or X-schemes) $\eta_{\mathbf{G}} : \mathbf{G} \to \mathbf{R}_{Z/X}f^{-1}\mathbf{G}$.

When $\mathbf{G}$ is abelian, there's a *norm* homomorphism $N = N_{Z/X} : \mathbf{R}_{Z/X}f^{-1}\mathbf{G} \to \mathbf{G}$ of sheaves of groups on $X_{\text{ét}}$. The sheaf kernel of the norm homomorphism is denoted by $\mathbf{R}^1_{Z/X}f^{-1}\mathbf{G}$, and there's an exact sequence

$$1 \to \mathbf{R}^1_{Z/X}f^{-1}\mathbf{G} \to \mathbf{R}_{Z/X}f^{-1}\mathbf{G} \xrightarrow{N} \mathbf{G} \to 1 \tag{7}$$

of sheaves of abelian groups on $X_{\text{ét}}$.

Now let $f : Z \to X$ be étale quadratic with nontrivial automorphism ι. Then the cokernel of the counit of adjunction can be identified with $\mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}$, and we have an exact sequence

$$1 \to \mathbf{G} \xrightarrow{\eta_{\mathbf{G}}} \mathbf{R}_{Z/X}f^{-1}\mathbf{G} \xrightarrow{\mathrm{id}/\iota} \mathbf{R}^1_{Z/X}f^{-1}\mathbf{G} \to 1 \tag{8}$$

of sheaves of abelian groups on $X_{\text{ét}}$, where we can identify the cokernel map with $x \mapsto x/\iota(x)$ for sections x of $\mathbf{R}_{Z/X}f^{-1}\mathbf{G}$.

2.3. **Finite coverings of orthogonal similitude groups.** We now construct the four-fold covering of the orthogonal similitude group by the Clifford group that will be used in defining the similarity Clifford invariant. Fixing a regular $\mathscr{O}_X$-valued quadratic form $(\mathscr{H}, b)$ of even rank $n = 2m$, let $\mathscr{Z} = \mathscr{Z}(\mathscr{H}, b)$ with associated étale quadratic Arf covering $f : Z = \mathbf{Spec}\,\mathscr{Z} \to X$.

Let $\mathbf{\Gamma} = \mathbf{\Gamma}(\mathscr{H}, b)$ be the Clifford group of $(\mathscr{H}, b)$ (see [34, §1.9]) and recall the vector representation $r : \mathbf{\Gamma} \to \mathbf{O}$ and the Clifford norm $N : \mathbf{\Gamma} \to \mathbf{G}_{\mathrm{m}}$. The cartesian product of the Clifford norm and the vector representation yields an exact sequence,

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{\Gamma} \xrightarrow{N \times r} \mathbf{G}_{\mathrm{m}} \times \mathbf{O} \to 1, \tag{9}$$

of sheaves of groups on $X_{\text{ét}}$, where $\boldsymbol{\mu}_2 \to \mathbf{\Gamma}$ denotes the canonical inclusion of constants. The exactness at right may be checked locally in the étale topology. Finally, the composite homomorphism $s = m \circ (N \times r) : \mathbf{\Gamma} \to \mathbf{GO}$ is an epimorphism of sheaves of groups on $X_{\text{ét}}$.

Definition 2.2. Define $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\mathscr{H}, b)$ to be the sheaf kernel of the homomorphism $s : \mathbf{\Gamma} \to \mathbf{GO}$. Equivalently, thinking of $\boldsymbol{\mu}_2 \hookrightarrow \mathbf{O}$ as the subgroup of homotheties, $\boldsymbol{\kappa}$ is the sheaf defined by

$$\boldsymbol{\kappa}(U) = \{x \in \mathbf{\Gamma}(h)(U) : N(x) \in \boldsymbol{\mu}_2(U),\ r(x) \in \boldsymbol{\mu}_2(U),\ N(x) = r(x)\}$$

on $X_{\text{ét}}$. The resulting exact sequence,

$$1 \to \boldsymbol{\kappa} \to \mathbf{\Gamma} \xrightarrow{s} \mathbf{GO} \to 1, \tag{10}$$

on $X_{\text{ét}}$ is called the *similitude Clifford sequence.*

By construction, the similitude Clifford sequence is an extension of the Kummer sequence by the pinor sequence, forming a *fundamental Clifford diagram*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{Pin} & \xrightarrow{r} & \mathbf{O} & \longrightarrow & 1 \\
 & & {\scriptstyle i}\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\kappa} & \longrightarrow & \mathbf{\Gamma} & \xrightarrow{s} & \mathbf{GO} & \longrightarrow & 1 \\
 & & {\scriptstyle p}\downarrow & & {\scriptstyle N}\downarrow & & {\scriptstyle \lambda}\downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \xrightarrow{2} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{11}$$

of sheaves of groups with exact rows and columns on $X_{\text{ét}}$. In particular, $\boldsymbol{\kappa}$ is a étale group scheme of order 4, which we shall now identify.

Proposition 2.3. *Let $(\mathscr{H}, b)$ be a regular $\mathscr{O}_X$-valued quadratic form of even rank n and Arf covering $f : Z \to X$. Then there's a group scheme isomorphism*

$$\boldsymbol{\kappa} \cong \begin{cases} \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \textit{if } n \equiv 0 \bmod 4 \\ \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \textit{if } n \equiv 2 \bmod 4 \end{cases}.$$

Proof. Every symmetric bilinear form over a scheme with 2 invertible is locally diagonalizable in the Zariski topology. We can assume, without loss of generality, that X is connected. Let $\mathscr{U} = \{U_i \to X\}_{i\in I}$ be a Zariski open cover of X diagonalizing the symmetric bilinear form $(\mathscr{H}, b)$ via isometries

$$\Phi_i : (\mathscr{O}^n_{U_i}, \langle b^1_i, \ldots, b^n_i\rangle) \to (\mathscr{H}|_{U_i}, h|_{U_i})$$

for $b^1_i, \ldots, b^n_i \in \mathbf{G}_{\mathrm{m}}(U_i)$ corresponding to orthogonal bases $e^1_i, \ldots, e^n_i \in \mathscr{O}^n_X(U_i)$. Let $d_i = b^1_i \cdots b^n_i \in \mathbf{G}_{\mathrm{m}}(U_i)$, so that there are isometries

$$\det \Phi_i : (\mathscr{O}_{U_i}, \langle d_i\rangle) \to \det(\mathscr{H}, b)|_{U_i}.$$

For each $i, j \in I$, there are isometries

$$\varphi_{ij} = \Phi^{-1}_{i\underline{j}} \circ \Phi_{\underline{i}j} : (\mathscr{O}^n_{U_{ij}}, \langle b^1_j, \ldots, b^n_j\rangle) \to (\mathscr{O}^n_{U_{ij}}, \langle b^1_i, \ldots, b^n_i\rangle)$$

and

$$\det \varphi_{ij} = \det \Phi^{-1}_{i\underline{j}} \circ \det \Phi_{\underline{i}j} : (\mathscr{O}_{U_{ij}}, \langle d_j\rangle) \to (\mathscr{O}_{U_{ij}}, \langle d_i\rangle)$$

yielding the equality

$$(\det \varphi_{ij})^2 d_i = d_j \tag{12}$$

in $\mathbf{G}_{\mathrm{m}}(U_{ij})$.

Now let $p : \widetilde{X} \to X$ be the étale quadratic covering defined by the class of

$$\delta' = -(-1)^{n(n-1)/2}\delta \in H^1_{\text{ét}}(X, \boldsymbol{\mu}_2).$$

For each $i \in I$, fix an isomorphism of $p^{-1}(U_i)$ with $\widetilde{U}_i = U_i \times_{\operatorname{Spec} \mathscr{O}_X(U_i)} \operatorname{Spec} \mathscr{O}_X(U_i)[\sqrt{-d_i}]$. Then $\widetilde{\mathscr{U}} = \{\widetilde{U}_i\}_{i\in I}$ is a Zariski open cover of $\widetilde{X}$ and also can be considered as an étale cover of X. We'll identify $\widetilde{U}_{ij} = \widetilde{U}_i \times_X \widetilde{U}_j$ with $U_{ij} \times_{\operatorname{Spec} \mathscr{O}_X(U_{ij})} \operatorname{Spec} \mathscr{O}_X(U_{ij})[\sqrt{-d_i}, \sqrt{-d_j}]$, where $U_{ij} = U_i \cap U_j$. Then

$$\delta'_{ij} = -\det \varphi_{ij}\, d^{-1}_j \otimes \sqrt{d_i d_j} \in \boldsymbol{\mu}_2(\widetilde{U}_{ij})$$

defines a Čech 1-cocycle representing δ'. Clearly, any étale cover of X splitting $\mathscr{H}$ and δ' is a refinement of such a cover. Finally, for each $i \in I$, define

$$\epsilon_i = \frac{1}{d_i} e^1_i \cdots e^n_i \otimes \sqrt{-d_i} \in \boldsymbol{\Gamma}_i(\widetilde{U}_i),$$

where $\boldsymbol{\Gamma}_i$ is the Clifford group of the diagonal form $(\mathscr{O}_{U_i}, \langle b^1_i, \ldots, b^n_i\rangle)$. We'll argue that

$$N(\epsilon_i) = -1 \in \mathbf{G}_{\mathrm{m}}(\widetilde{U}_i), \quad r(\epsilon_i) = -\mathrm{id} \in \mathbf{O}_i(\widetilde{U}_i), \tag{13}$$

so that $\epsilon_i \in \boldsymbol{\kappa}_i(\widetilde{U}_i)$, where $\mathbf{O}_i$ and $\boldsymbol{\kappa}_i$ are associated to the diagonal form. Recalling that in the Clifford algebra,

$$e^k_i\, e^l_i = \begin{cases} d_i & \text{if } k = l \\ -e^l_i\, e^k_i & \text{if } k \neq l \end{cases},$$

we now compute

$$N(\epsilon_i) = \epsilon_i\, \sigma(\epsilon_i) = \frac{1}{d^2_i}\, e^1_i \cdots e^n_i\, e^n_i \cdots e^1_i \otimes -d_i = -1 \otimes 1.$$

Also, for each $k = 1, \ldots, n$, we have

$$\epsilon_i\, e^k_i = \frac{1}{d_i} e^1_i \cdots e^n_i\, e^k_i \otimes \sqrt{-d_i} = (-1)^{n-1} e^k_i\, \epsilon_i,$$

so that

$$r(\epsilon_i)(e^k_i) = I(\epsilon_i)\, e^k_i\, \epsilon^{-1}_i = (-1)^n\, \epsilon_i\, e^k_i\, \epsilon^{-1} = (-1)^{n-1}(-1)^n\, e^k_i\, \epsilon\epsilon^{-1} = -e_i,$$

hence $r(\epsilon_i) = -\mathrm{id}$.

Thus -1 and ϵ_i generate $\boldsymbol{\kappa}_i(\widetilde{U}_i)$. Using the identity $\sigma(\epsilon) = (-1)^{n(n-1)/2}$, and equation (13), we find that

$$\epsilon^2 = -(-1)^{n(n-1)/2},$$

so that

$$\boldsymbol{\kappa}_i(\widetilde{U}_i) \cong \begin{cases} \mathbb{Z}/4\mathbb{Z} & \text{if } n \equiv 0, 1 \bmod 4 \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } n \equiv 2, 3 \bmod 4 \end{cases},$$

via either $\epsilon_i \mapsto 1$ or $\epsilon_i \mapsto (0,1)$ and $-1 \mapsto (1,0)$, depending on the rank modulo 4.

The isometry Φ_i induces an algebra isomorphism $\mathscr{C}(\Phi_i)$ of Clifford algebras, hence commutes with N and r and induces group scheme isomorphisms $\boldsymbol{\kappa}_i \xrightarrow{\sim} \boldsymbol{\kappa}|_{\widetilde{U}_i}$ and $\mathbf{O}_i \xrightarrow{\sim} \mathbf{O}|_{\widetilde{U}_i}$. We have deduced that $\boldsymbol{\kappa}$ is a form of the corresponding constant group of order 4, which is split by the étale cover $\widetilde{\mathscr{U}}$. We'll now argue that $\boldsymbol{\kappa}$ is split by $p : \widetilde{X} \to X$ by showing that the sections $\mathscr{C}(\Phi_i)(\epsilon_i) \in \boldsymbol{\kappa}(\widetilde{U})$ glue to a section $\epsilon \in \boldsymbol{\kappa}(\widetilde{X})$.

Lemma 2.4. *Let $\mathscr{H}$ be a free $\mathscr{O}_X$-module of rank n on X, $(\mathscr{H}, b_1)$ and $(\mathscr{H}, b_2)$ be regular quadratic forms with orthogonal bases $e_1^1, \ldots, e_1^n$ and $e_2^1, \ldots, e_2^n$, and $\varphi : (\mathscr{H}, b_1) \to (\mathscr{H}, b_2)$ an isometry. Then*

$$\mathscr{C}(\varphi)(e_1^1 \cdots e_1^n) = \varphi(e_1^1) \cdots \varphi(e_1^n) = \det \varphi \, e_2^1 \cdots e_2^n.$$

in the Clifford algebra of $(\mathscr{H}, b_2)$.

Now using equation (12) and Lemma 2.4, compute

$$\begin{aligned}
\mathscr{C}(\varphi_{ij})(\epsilon_{ij}) &= \frac{1}{d_{\underline{i}j}} \varphi_{ij}(e_{\underline{i}j}^1) \cdots \varphi_{ij}(e_{\underline{i}j}^n) \otimes \sqrt{-d_{\underline{i}j}} \\
&= \frac{1}{(\det \varphi_{ij})^2 \, d_{i\underline{j}}} \det \varphi_{ij} \, e_{i\underline{j}}^1 \cdots e_{i\underline{j}}^n \otimes \sqrt{-(\det \varphi_{ij})^2 \, d_{i\underline{j}}} \\
&= \frac{1}{d_{i\underline{j}}} e_{i\underline{j}}^1 \cdots e_{i\underline{j}}^n \otimes \sqrt{-d_{i\underline{j}}} = \epsilon_{i\underline{j}}
\end{aligned}$$

so that we have the equality

$$\mathscr{C}(\Phi_{\underline{i}j})(\epsilon_{\underline{i}j}) = \mathscr{C}(\Phi_{i\underline{j}})(\mathscr{C}(\varphi_{ij})(\epsilon_{\underline{i}j})) = \mathscr{C}(\Phi_{i\underline{j}})(\epsilon_{i\underline{j}})$$

of sections in $\boldsymbol{\Gamma}(\widetilde{U}_{ij})$. Thus the sections $\mathscr{C}(\Phi_i)(\epsilon_i) \in \boldsymbol{\kappa}(\widetilde{U}_i)$ glue to a section $\epsilon \in \boldsymbol{\kappa}(\widetilde{X})$ which splits the group scheme $\boldsymbol{\kappa}$ over the étale quadratic covering $p : \widetilde{X} \to X$. Clearly, the Galois action on the section $\epsilon \in \boldsymbol{\kappa}(\widetilde{X})$ is trivial if and only if the class δ' is trivial. Since any étale quadratic extension splits a unique nontrivial form of $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, the proposition follows. On the étale cover $\widetilde{\mathscr{U}}$, the isomorphism of group schemes constructed above descends.

Alternatively, we can also view the above equality as

$$\mathscr{C}(\Phi_{\underline{i}j})(\epsilon_{\underline{i}j}) = \left(\det \varphi_{ij} \, d_{i\underline{j}}^{-1} \otimes \sqrt{d_{i\underline{j}} d_{\underline{i}j}}\right) \mathscr{C}(\Phi_{i\underline{j}})(\epsilon_{i\underline{j}}) = \delta'_{ij} \mathscr{C}(\Phi_{i\underline{j}})(\epsilon_{i\underline{j}})$$

so that $\boldsymbol{\kappa}$ is identified with the twist of $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ by the the Čech 1-cocyle δ'_{ij} on the étale cover $\widetilde{\mathscr{U}}$ on X. □

Remark 2.5. For regular $\mathscr{O}_X$-valued quadratic forms of odd rank, we have

$$\boldsymbol{\kappa} \cong \begin{cases} \mathbf{R}^1_{Z/X} \boldsymbol{\mu}_4 & \text{if } n \equiv 1 \bmod 4 \\ \mathbf{R}_{Z/X} \boldsymbol{\mu}_2 & \text{if } n \equiv 3 \bmod 4 \end{cases}$$

see Auel [3, Thm. 2.11]. This formula in the odd rank case depends on our convention for the Clifford norm N. We take the convention of Fröhlich [39, Appendix I]. The other convention, taken by Knus [68, IV §6.1], defines N via the *standard Clifford involution* induced by negation on $\mathscr{H}$.

Example 2.6. We continue to assume, without loss of generality, that X is connected. For certain standard quadratic forms, we now make the isomorphism in Proposition 2.3. Let $(\mathscr{H}, b) = H_{\mathscr{O}_X}(\mathscr{O}_X^m)$ be the hyperbolic quadratic form on X with trivial lagrangian, and $e_1, \ldots, e_m, f_1, \ldots, f_m$ a choice of global sections forming a hyperbolic basis, i.e. $h(f_l, e_k) = \delta_{lk}$. An explicit representation of $\boldsymbol{\kappa} = \boldsymbol{\kappa}(H_{\mathscr{O}_X}(\mathscr{O}_X^m))$ is as follows.

For m odd, a straightforward calculation shows that the global section $\epsilon = \prod_{l=1}^m (1 - 2 e_l f_l) \in \boldsymbol{\Gamma}(X)$ of the Clifford group generates, together with $-1 \in \boldsymbol{\Gamma}(X)$, the group scheme $\boldsymbol{\kappa}$. This yields a global isomorphism $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 = \langle -1 \rangle \times \langle \epsilon \rangle \xrightarrow{\sim} \boldsymbol{\kappa}$ of group schemes on $X_{\text{ét}}$, depending on our particular choice of ϵ.

For m even, the section $\tilde{\epsilon} = (-1)^{m/2} \prod_{l=1}^m (1 - 2 e_l f_l) \otimes \sqrt{-1} \in \boldsymbol{\Gamma}(\widetilde{X})$ of the Clifford group over the étale double covering $\widetilde{X} = X \times_{\operatorname{Spec} \mathscr{O}_X(X)} \operatorname{Spec} \mathscr{O}_X(X)[\sqrt{-1}] \to X$ generates $\boldsymbol{\kappa}|_{\widetilde{X}}$. By descent there's an isomorphism $\boldsymbol{\mu}_4 \xrightarrow{\sim} \boldsymbol{\kappa}$ of group schemes on $X_{\text{ét}}$. This isomorphism also depends on our particular choice of ϵ.

More generally, for each even $n = 2m \geq 2$ and each étale quadratic $f : Z \to X$, define $(\mathscr{H}, b) = h_n^Z = (f_* \mathscr{O}_Z, h^Z) \perp H_{\mathscr{O}_X}(\mathscr{O}_X^{m-1})$, where h^Z is the *norm form* associated to $f : Z \to X$ (see §5.1). Then h_n^Z has rank n, Arf invariant canonically isomorphic to $f : Z \to X$, and trivial classical Clifford invariant (see [68, V §2.3]). There's a canonical isometry $f^* h_n^Z \to H_{\mathscr{O}_Z}(\mathscr{O}_Z^m)$ (indeed, for $m = 1$

the canonical $\mathscr{O}_Z$-algebra isomorphism $f^*f_*\mathscr{O}_Z \to \mathscr{O}_Z \times \mathscr{O}_Z$ is an isometry of the quadratic forms $f^*(f_*\mathscr{O}_Z, h^Z) \to H_{\mathscr{O}_Z}(\mathscr{O}_Z)$, which extends, for general m, to the remaining hyperbolic factors by the identity). Hence we have isomorphisms $f^{-1}\boldsymbol{\kappa}(h_n^Z) \cong \boldsymbol{\kappa}(f^*h_n^Z) \cong \boldsymbol{\kappa}(H_{\mathscr{O}_Z}(\mathscr{O}_Z^m))$.

For m odd, we compose with the projection onto the subgroup $\boldsymbol{\kappa}(H_{\mathscr{O}_Z}(\mathscr{O}_Z^m)) \cong \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \to \langle -1 \rangle$ (this depends on our choice of ϵ, as above), yielding a morphism $f^{-1}\boldsymbol{\kappa}(h_n^Z) \to \boldsymbol{\mu}_2$, and hence by adjunction, a morphism $\boldsymbol{\kappa}(h_n^Z) \to \mathbf{R}_{Z/X}\boldsymbol{\mu}_2$, which we can check, locally using Proposition 2.3, is an isomorphism.

For m even, we have an isomorphism $f^{-1}\boldsymbol{\kappa}(h_n^Z) \to \boldsymbol{\mu}_4$, which by adjunction yields a morphism $\boldsymbol{\kappa}(h_n^Z) \to \mathbf{R}_{Z/X}\boldsymbol{\mu}_4$, which is seen to be in the kernel of the norm map $\mathbf{R}_{Z/X}\boldsymbol{\mu}_4 \to \boldsymbol{\mu}_4$. Thus we have a map $\boldsymbol{\kappa}(h_n^Z) \to \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4$, which again we can check is an isomorphism locally. Thus for the form h_n^Z, we have an explicit choice of isomorphism in Proposition 2.3.

2.4. **The similarity Clifford invariant.** By the calculations in the proof of Proposition 2.3, we see that any section of $\boldsymbol{\kappa}$ not in the canonical subgroup $\boldsymbol{\mu}_2 \hookrightarrow \boldsymbol{\kappa}$ has degree (in the Clifford group) the parity n. The structure theory of the center of the Clifford algebra then implies that the similitude Clifford sequence (10) is central for n odd. For n even, its restriction to the even Clifford group,

$$1 \to \boldsymbol{\kappa} \to \mathbf{S\Gamma} \xrightarrow{s} \mathbf{GSO} \to 1, \tag{14}$$

is central and we consider the restriction of the fundamental diagram (11) to the even Clifford group:

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \boldsymbol{\mu}_2 & \to & \mathbf{Spin} & \xrightarrow{r} & \mathbf{SO} & \to & 1 \\
 & & \downarrow{\scriptstyle i} & & \downarrow & & \downarrow & & \\
1 & \to & \boldsymbol{\kappa} & \to & \mathbf{S\Gamma} & \xrightarrow{s} & \mathbf{GSO} & \to & 1 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle N} & & \downarrow{\scriptstyle \lambda} & & \\
1 & \to & \boldsymbol{\mu}_2 & \to & \mathbf{G}_{\mathrm{m}} & \xrightarrow{2} & \mathbf{G}_{\mathrm{m}} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{15}$$

Definition 2.7. For each $n = 2m$, étale quadratic $f : Z \to X$, and each sheaf of groups $\mathbf{G}$ depending on a quadratic form, write $\mathbf{G}_n^Z = \mathbf{G}(h_n^Z)$.

Now, we are ready to define the cohomological invariant we'll primarily be concerned with.

Definition 2.8. Let $f : Z \to X$ be étale quadratic. For a regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank $n = 2m$ and Arf invariant $[Z/X]$, define the *similarity Clifford invariant* $gc(q) = gc(\mathscr{E}, q, \mathscr{L}) \in H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z)$ to be the image of $(\mathscr{E}, q, \mathscr{L}, \zeta)$ under the coboundary map $H^1_{\text{ét}}(X, \mathbf{GSO}_n^Z) \to H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z)$ arising from the even Clifford sequence (14), for any choice of Z/X-orientation ζ. By Lemma 5.15, $gc(q)$ does not depend on the choice of orientation ζ.

Remark 2.9. An analogous similarity Clifford invariant can also be defined for line bundle-valued quadratic forms of odd rank. By Lemma 1.6, the value line bundle is a square, and in fact, has a canonical square root. This can be used to compute the invariants odd rank forms in terms of classical invariants, see Auel [3, §2.2.1].

2.5. **Interpolation property.** The similarity Clifford invariant "interpolates" between the classical étale cohomological Clifford invariant (see Parimala–Srinivas [85, Lemma 6] or Knus–Parimala–Sridharan [66, §3]) and the 1st Chern class (modulo 2) (see §2.1) of the value line bundle.

Theorem 2.10. *Let X be a scheme with 2 invertible. Let $(\mathscr{E}, q, \mathscr{L})$ be a regular $\mathscr{L}$-valued quadratic form of rank $n = 2m$ and Arf invariant $[Z/X]$. We have an exact sequence*

$$\cdots \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \xrightarrow{i^2} H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \xrightarrow{p^2} H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \to \cdots$$

of étale cohomology groups.

a) Then $p^2 gc(\mathscr{E}, q, \mathscr{L}) = c_1(\mathscr{L}, \boldsymbol{\mu}_2)$, i.e. the similarity Clifford invariant maps to the 1st Chern class (modulo 2) of $\mathscr{L}$.

b) If $\mathscr{L} = \mathscr{O}_X$ then $gc(\mathscr{E}, q, \mathscr{O}_X) = i^2 c(\mathscr{E}, q)$, i.e. the classical Clifford invariant of a quadratic form maps to the similarity Clifford invariant of its similarity class.

Proof. In light of the cohomology interpretation of Remark 1.5, one needs only to consider the implications on étale cohomology of the fundamental diagram (15). For *b)*, it remains to be verified that the coboundary map $H^1_{\text{ét}}(X, \mathbf{SO}_n^Z) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ agrees with the Clifford invariant of a quadratic form $(\mathscr{E}, q)$ (together with an irrelevant choice of orientation) with Arf invariant $[Z/X]$. This is verified using a generalization to $X_{\text{ét}}$ of formulas from Lam [73, V Prop. 3.20] and Serre [90, III App. 2, §2.2]. □

2.6. **Functoriality.** Let $f : Y \to X$ be a morphism of schemes. If $(\mathscr{E}, q, \mathscr{L})$ is an $\mathscr{L}$-valued quadratic form of rank n on X, then there's a canonical $f^*\mathscr{L}$-valued quadratic form $f^*(\mathscr{E}, q, \mathscr{L})$ of rank n on Y. By the functoriality of the Clifford group, there's also a canonical isomorphism of group schemes $\boldsymbol{\kappa}_n^{Y\times_X Z} \xrightarrow{\sim} f^{-1}\boldsymbol{\kappa}_n^Z$, where f^{-1} is the pullback in the category of sheaves of groups on $X_{\text{ét}}$ (see Milne [76, II §2]). Finally, we have

$$gc(f^*q) = f^* gc(q)$$

in $H^2_{\text{ét}}(Y, \boldsymbol{\kappa}^m_{Y\times_X Z/Y}) \xrightarrow{\sim} H^2_{\text{ét}}(Y, f^{-1}\boldsymbol{\kappa}^Z_n)$.

2.7. **Grothendieck–Witt equivalence.** Now let $f : Z \to X$ be split, so that $h^Z_n = H_{\mathscr{O}_X}(\mathscr{O}^m_X)$. Write $\mathbf{G}_{m,m}$ instead of $\mathbf{G}^Z_n$ for any sheaf of groups depending on a quadratic form. If n is divisible by four then $gc(q) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_4)$ via the identification $\boldsymbol{\mu}_4 \xrightarrow{\sim} \boldsymbol{\kappa}$ of Remark 2.6.

Theorem 2.11. *Let X be a scheme with 2 invertible. Let $(\mathscr{E}, q, \mathscr{L})$ be a regular quadratic form of rank $n = 2m$ divisible by four and trivial Arf invariant on X.*

a) If $\mathscr{V}$ is a locally free $\mathscr{O}_X$-module of even rank, then

$$gc(q \perp H_{\mathscr{L}}(\mathscr{V})) = gc(q) + i^2 c_1(\mathscr{V}, \boldsymbol{\mu}_2).$$

b) If $(\mathscr{E}, q, \mathscr{L})$ is metabolic with lagrangian $\mathscr{V} \to \mathscr{E}$, then

$$gc(q) = c(H_{\mathscr{L}}(\mathscr{V})) = c_1(\mathscr{L}, \boldsymbol{\mu}_4) + i^2 c_1(\mathscr{V}, \boldsymbol{\mu}_2).$$

Proof. Consider the homomorphism $H : \mathbf{G}_{\mathrm{m}} \times \mathbf{GL}_m \to \mathbf{GSO}_{m,m}$ defined by

$$H(l, \varphi) = \begin{pmatrix} \varphi & 0 \\ 0 & l\,(\varphi^{-1})^\vee \end{pmatrix}.$$

Lemma 2.12. *The induced mapping $H^1 : H^1_{\text{ét}}(X, \mathbf{G}_m \times \mathbf{GL}_m) \to H^1_{\text{ét}}(X, \mathbf{GSO}_{m,m})$ is interpreted as $(\mathscr{L}, \mathscr{V}) \mapsto (H_{\mathscr{L}}(\mathscr{V}), \zeta_{\mathscr{V}})$, where $\zeta_{\mathscr{V}}$ is the orientation from Lemma 1.14.*

Proof. Let $\mathscr{U} = \{U_i\}_{i\in I}$ be a Zariski open cover of X splitting $\mathscr{L}$ and $\mathscr{V}$ via $\mathscr{O}_{U_i}$-module isomorphisms

$$\lambda_i : \mathscr{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}, \quad \phi_i : \mathscr{O}^m_{U_i} \xrightarrow{\sim} \mathscr{V}|_{U_i},$$

for each $i \in I$. The $\mathscr{O}_{U_i}$-module morphism

$$\begin{aligned} \psi_i : H_{\mathscr{O}_{U_i}}(\mathscr{O}_{U_i}) &\to H_{\mathscr{L}}(\mathscr{V})|_{U_i} \\ (v, f) &\mapsto (\phi_i(v), \lambda_i \circ f \circ \phi_i^{-1}) \end{aligned}$$

is a proper similarity transformation with similarity factor λ_i. For each $(i, j) \in I^2$, the collections

$$l_{ij} = \lambda^{-1}_{\underline{i}j} \circ \lambda_{i\underline{j}} \in \mathbf{G}_{\mathrm{m}}(U_{ij}), \qquad \varphi_{ij} = \phi^{-1}_{\underline{i}j} \circ \phi_{i\underline{j}} \in \mathbf{GL}_m(U_{ij}),$$

and

$$b_{ij} = \psi^{-1}_{\underline{i}j} \circ \psi_{i\underline{j}} \in \mathbf{GO}_{m,m}(U_{ij}),$$

form Čech étale 1-cocycles representing the classes associated to $\mathscr{L}$, $\mathscr{V}$, and $H_{\mathscr{L}}(\mathscr{V})$ in $H^1_{\text{ét}}(\mathscr{U}, \mathbf{G}_{\mathrm{m}})$, $H^1_{\text{ét}}(\mathscr{U}, \mathbf{GL}_m)$, and $H^1_{\text{ét}}(\mathscr{U}, \mathbf{GO}_{m,m})$, respectively. With respect to the decomposition $H_{\mathscr{O}_X}(\mathscr{O}^m_X) = \mathscr{O}^m_X \oplus \mathscr{O}^{m\vee}_X$, the cocycle b_{ij} is represented by the matrices

$$H(l_{ij}, \varphi_{ij}) = \begin{pmatrix} \varphi_{ij} & 0 \\ 0 & l_{ij}\,(\varphi^{-1}_{ij})^\vee \end{pmatrix}$$

proving the claim. Thus we've shown that $H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}} \times \mathbf{GL}_m) \to H^1_{\text{ét}}(X, \mathbf{GO}_{m,m})$ is interpreted by $(\mathscr{L}, \mathscr{V}) \mapsto H_{\mathscr{L}}(\mathscr{V})$. All that remains is the verification of the orientation statement. If 2 is invertible on X, this is equivalent to the equality

$$(\det/\mu^m)(H(l_{ij}, \varphi_{ij})) \circ \zeta_{\mathscr{V}|_{U_{ij}}} = \zeta_{\mathscr{V}|_{U_{ij}}} : \mathscr{O}_{U_{ij}} = d_{\pm}(H_{\mathscr{O}_{U_{ij}}}(\mathscr{O}^m_{U_{ij}})) \to d_{\pm}(H_{\mathscr{O}_{U_{ij}}}(\mathscr{V}|_{U_{ij}})).$$

This is a straightforward process of unwinding the definition of $\zeta_{\mathscr{V}}$ from Lemma 1.14. □

Furthermore, consider the restriction of H to the maximal torus $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}^{\times m}_{\mathrm{m}} \to \mathbf{GSO}_{m,m}$, which similarly to Lemma 2.12, is interpreted as $(\mathscr{L}, \mathscr{V}_1, \ldots, \mathscr{V}_m) \mapsto H_{\mathscr{L}}(\mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_m)$, where the orientation from Lemma 2.12 is understood. We now construct a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \to & \boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2^{\times m} & \to & \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}^{\times m} & \to & \mathbf{G}_{\mathrm{m}} \times \mathbf{G}_{\mathrm{m}}^{\times m} & \to & 1 \\ & & \downarrow{\scriptstyle \Pi} & & \downarrow{\scriptstyle \tilde{H}} & & \downarrow{\scriptstyle H} & & \\ 1 & \to & \boldsymbol{\mu}_4 & \to & \mathbf{S\Gamma}_{m,m} & \to & \mathbf{GSO}_{m,m} & \to & 1 \end{array} \tag{16}$$

lifting H to the even Clifford group: the top row is the product of appropriate Kummer sequences, the bottom row is the even Clifford sequence (14) for the split similitude group, $\prod$ is the homomorphism given by multiplying a tuple of the elements together, and

$$\tilde{H}(l, a_1, \dots, a_m) = l^{m+1} \prod_{j=1}^{m} a_j^{-1} \Big(1 - \big(1 - (l^{-1}a_j)^2\big) e_j f_j\Big)$$

on sections, where $e_1, \dots, e_m, f_1, \dots, f_m$ are global sections of $H_{\mathscr{O}_X}(\mathscr{O}_X^m)$ forming a hyperbolic basis, i.e. $h(e_i, e_j) = 0$, $h(f_i, f_j) = 0$, and $h(e_i, f_j) = \delta_{ij}$, where δ_{ij} is the Kronecker δ-function.

To verify the commutativity of diagram (16), we will decompose $\tilde{H}$ into a product of homomorphisms. Calculations in the Clifford algebra show that for each $1 \le j \le m$, the map $\chi_j : \mathbf{G}_{\mathrm{m}} \to \mathbf{S\Gamma}_{m,m}$ defined on sections by

$$\chi_j(x) = 1 - (1 - x) e_j f_j$$

forms a one parameter subgroup of $\mathbf{S\Gamma}_{m,m}$ satisfying

$$N(\chi_j(x)) = x, \qquad r(\chi_j(x)) : \begin{array}{ccl} e_k & \mapsto & \big(1 - \delta_{jk}(1 - x)\big) e_k \\ f_k & \mapsto & \big(1 - \delta_{jk}(1 - x^{-1})\big) f_k. \end{array}$$

In particular,

$$N(\tilde{H}(l, a_1, \dots, a_m)) = l^2, \qquad r(\tilde{H}(l, a_1, \dots, a_m)) = l^{-2} H(l^4, a_1^2, \dots, a_m^2)$$

so that $\tilde{H}$ is a homomorphism and the rightmost square of diagram (16) is commutative. The fact that the leftmost square is commutative results from the identity

$$\tilde{H}(\sqrt{-1}, 1, \dots, 1) = (-1)^{m/2} \sqrt{-1} \prod_{j=1}^{m} (1 - e_j f_j)$$

and the identification $\boldsymbol{\mu}_4 \xrightarrow{\sim} \boldsymbol{\kappa}(h)$ of Remark 2.6.

Finally, by the splitting principle for metabolic forms (see Theorem 1.9) and the fact that the formulas of Theorem 2.11 all commute with pullbacks, to prove *b)* we can reduce to the case where $(\mathscr{E}, q, \mathscr{L})$ is a hyperbolic form $H_{\mathscr{L}}(\mathscr{V})$ with lagrangian $\mathscr{V} = \mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_m$ a sum of invertible modules. The coboundary map of the top sequence of diagram (16), composed with the map $\prod^2$ induced from the leftmost vertical map, yields

$$(\mathscr{L}, \mathscr{V}_1, \dots, \mathscr{V}_m) \mapsto (c_1(\mathscr{L}, \boldsymbol{\mu}_4), c_1(\mathscr{V}_1, \boldsymbol{\mu}_2), \dots, c_1(\mathscr{V}_m, \boldsymbol{\mu}_2)) \mapsto c_1(\mathscr{L}, \boldsymbol{\mu}_4) + i^2 c_1(\mathscr{V}, \boldsymbol{\mu}_2)$$

and thus $H_{\mathscr{L}}(\mathscr{V}) \mapsto c_1(\mathscr{L}, \boldsymbol{\mu}_4) + i^2 c_1(\mathscr{V}, \boldsymbol{\mu}_2)$ under the coboundary map of the Clifford sequence for the similitude group, verifying the formula in *b)*.

As for *a)*, let m' be even and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1] & \longrightarrow & \boldsymbol{\mu}_4 \times \boldsymbol{\mu}_2^{\times m'} & \longrightarrow & \mathbf{S\Gamma}_{m,m} \times \mathbf{G}_{\mathrm{m}}^{\times m'} & \longrightarrow & \mathbf{GSO}_{m,m} \times \mathbf{G}_{\mathrm{m}}^{\times m'} & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle \prod} & & \Big\downarrow{\scriptstyle \tilde{H}'} & & \Big\downarrow{\scriptstyle H'} & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_4 & \longrightarrow & \mathbf{S\Gamma}_{m+m',m+m'} & \longrightarrow & \mathbf{GSO}_{m+m',m+m''} & \longrightarrow & 1 \end{array} \tag{17}$$

where $H'((\varphi, l), a_1, \dots, a_{m'}) = (\varphi, l) \perp H(l, a_1, \dots, a_{m'})$ and

$$\tilde{H}'(x, a_1, \dots, a_{m'}) = N(x)^{m'/2}\, x \prod_{j=1}^{m'} a_j^{-1} \chi_j(N(x)^{-1} a_j^2).$$

The commutativity of the diagram is checked as above. Finally, H'^1 is interpreted as

$$\big((\mathscr{E}, q, \mathscr{L}), \mathscr{V}_1, \dots, \mathscr{V}_{m'}\big) \mapsto (\mathscr{E}, q, \mathscr{L}) \perp H_{\mathscr{L}}(\mathscr{V}_1 \oplus \cdots \oplus \mathscr{V}_{m'}),$$

while the coboundary map of the top sequence is interpreted as

$$\big((\mathscr{E}, q, \mathscr{L}), \mathscr{V}_1, \dots, \mathscr{V}_{m'}\big) \mapsto \big(c(\mathscr{E}, q, \mathscr{L}), c_1(\mathscr{V}_1, \boldsymbol{\mu}_2), \dots, c_1(\mathscr{V}_{m'}, \boldsymbol{\mu}_2)\big)$$

so that in total $gc(q \perp H_{\mathscr{L}}(\mathscr{V})) = c(q) + i^2 c_1(\mathscr{V}, \boldsymbol{\mu}_2)$, verifying the formula in *a)*. □

Corollary 2.13. *Let $(\mathscr{E}_0, q_0, \mathscr{L})$, $(\mathscr{E}_1, q_1, \mathscr{L})$, and $(\mathscr{E}_2, q_2, \mathscr{L})$ be $\mathscr{L}$-valued quadratic forms of rank divisible by 4 and trivial Arf invariant. If there's an isometry $q_0 \perp q_1 \cong q_0 \perp q_2$ then $gc(q_1) = gc(q_2)$.*

Proof. If $q_0 \perp q_1 \cong q_0 \perp q_2$ then $(-q_0 \perp q_0) \perp q_1 \cong (-q_0 \perp q_0) \perp q_2$. But $(\mathscr{E}_0, -q_0, \mathscr{L}) \perp (\mathscr{E}_0, q_0, \mathscr{L})$ is $\mathscr{L}$-valued metabolic with lagrangian $\mathscr{E}_0$, so that by Theorem 2.11,

$$gc(q_1) + i^2 c_1(\mathscr{E}_0, \boldsymbol{\mu}_2) = gc(q_2) + i^2 c_1(\mathscr{E}_0, \boldsymbol{\mu}_2)$$

in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_4)$, proving the corollary. □

Let $GW(X, \mathscr{L})$ be the Grothendieck–Witt group of $\mathscr{L}$-valued quadratic forms on X and $GW^{\mathrm{tot}}(X) = \oplus_{\mathscr{L}} GW(X, \mathscr{L})$ the total quadratic Grothendieck-Witt group. Letting $GI_1^{\mathrm{tot}}(X) = \oplus_{\mathscr{L}} GI_1(X, \mathscr{L})$ be the subgroup of quadratic forms of even rank, the signed discriminant defines a (surjective) cohomological invariant (in a sense generalizing Garibaldi–Merkurjev–Serre [43])

$$d_{\pm} : GI_1^{\mathrm{tot}}(X) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2).$$

Letting $GI_2^{\mathrm{tot}}(X) = \oplus_{\mathscr{L}} GI_2(X, \mathscr{L})$ be the subgroup of quadratic forms of rank $n \equiv 0 \bmod 4$ and trivial discriminant, we may interpret Theorem 2.11 as the statement that the similarity Clifford invariant descends to a cohomological invariant

$$gc : GI^{\mathrm{tot}}(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_4).$$

2.8. **Oriented invariants.** We now restrict our attention to line bundle-valued quadratic forms with trivial Arf invariant. As noted in Definition 2.8, the invariant $gc(\mathscr{E}, q, \mathscr{L}) \in H^2_{\text{ét}}(X, \boldsymbol{\kappa}_{m,m})$ does not depend on an orientation of $(\mathscr{E}, q, \mathscr{L})$. In this section, we construct related invariants in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$, but which are orientation dependent.

Definition 2.14. Let $(\mathscr{H}, b)$ be a regular quadratic form of even rank $n = 2m$ on X. Denote by $\mathbf{z} = \mathbf{z}(\mathscr{H}, b)$ be the subgroup scheme of $\mathbf{S\Gamma}(\mathscr{H}, b)$ defined by

$$\mathbf{z}(U) = \{x \in \mathbf{S\Gamma}(\mathscr{H}, b)(U) : N(x) = 1,\ r(x) \in \boldsymbol{\mu}_2(U)\}.$$

Equivalently, $\mathbf{z}$ is the kernel in the short exact sequence of sheaves of groups

$$1 \to \mathbf{z} \to \mathbf{Spin} \to \mathbf{PSO} \to 1 \tag{18}$$

on $X_{\text{ét}}$ (here $\mathbf{PSO} = \mathbf{SO}/\boldsymbol{\mu}_2$, see §3.1). In fact, $\mathbf{z}$ is the center of the simply connected affine algebraic group $\mathbf{Spin}$. It's a classical fact, see [69, VI.26.A] for example, that there are isomorphisms of group schemes

$$\mathbf{z} \cong \begin{cases} \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \text{if } n \equiv 0 \bmod 4 \\ \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \text{if } n \equiv 2 \bmod 4 \end{cases}$$

where $Z \to X$ is the Arf cover of $(\mathscr{H}, b)$.

We retain the notation of §2.7 and assume, for simplicity of exposition, that X is connected. For each $m \geq 1$, the choice of global section $\varepsilon = \prod_{i=1}^m (1 - 2e_i f_i)$ of $\mathbf{S\Gamma}_{m,m}$ fixes a group scheme isomorphism $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \xrightarrow{\sim} \mathbf{z}_{m,m}$ for m even and $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \xrightarrow{\sim} \boldsymbol{\kappa}_{m,m}$ for m odd. Indeed, if m is even, then ε is a global section of $\mathbf{Spin}_{m,m}$, which together with -1, generate $\mathbf{z}_{m,m}$. If m is odd, see Example 2.6.

For m even, define $\mathbf{\Gamma}'_{m,m} = \mathbf{G}_{\mathrm{m}} \times \mathbf{Spin}_{m,m}$ and $\boldsymbol{\gamma}^+_{m,m}$ to be the subgroupscheme generated by the global section $(-1, \varepsilon)$. For m odd, define $\mathbf{\Gamma}'_{m,m} = \mathbf{S\Gamma}_{m,m}$ and $\boldsymbol{\gamma}^+_{m,m}$ to be the subgroupscheme generated by the global section ε.

Definition 2.15. For $m \geq 1$, define $\mathbf{GSpin}^+_{m,m} = \mathbf{\Gamma}'_{m,m}/\boldsymbol{\gamma}^+_{m,m}$. Replacing ε by $-\varepsilon$ yields an analogous subgroup $\boldsymbol{\gamma}^-_{m,m}$ and quotient $\mathbf{GSpin}^-_{m,m}$.

Remark 2.16. For m odd, $\mathbf{GSpin}^{\pm}_{m,m}$ is precisely the image of $\mathbf{S\Gamma}_{m,m}$ under the "half-spin" representations $\rho^{\pm} : \mathbf{S\Gamma}_{m,m} \to \mathbf{GL}_{2^{2(m-1)}}$. In the literature, $\mathbf{GSpin}(q)$ is often another name for the even Clifford group $\mathbf{S\Gamma}(q)$ of a quadratic form. Our notation is somewhat justified since $\mathbf{GSpin}^+_{m,m}$ becomes isomorphic to $\mathbf{S\Gamma}_{m,m}$ upon replacing ε by $(-1)^{m-1}$ in the definition.

The compositions $\mathbf{G}_{\mathrm{m}} \times \mathbf{Spin}_{m,m} \xrightarrow{pr_1} \mathbf{G}_{\mathrm{m}} \xrightarrow{2} \mathbf{G}_{\mathrm{m}}$ and $\mathbf{S\Gamma}_{m,m} \xrightarrow{N} \mathbf{G}_{\mathrm{m}} \xrightarrow{2} \mathbf{G}_{\mathrm{m}}$ descend to homomorphisms $\mu^{\pm} : \mathbf{GSpin}^{\pm}_{m,m} \to \mathbf{G}_{\mathrm{m}}$. The kernel of $\mu^{\pm}$ is isomorphic to $\mathbf{Spin}_{m,m}$, as seen by the Nine Lemma 5.16 applied to the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbf{Spin}_{m,m} & \longrightarrow & \ker \mu^{\pm} & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\gamma}^{\pm}_{m,m} & \longrightarrow & \mathbf{\Gamma}'_{m,m} & \longrightarrow & \mathbf{GSpin}^{\pm}_{m,m} & \longrightarrow & 1 \\
 & & \downarrow{\wr} & & \downarrow & & \downarrow{\scriptstyle \mu^{\pm}} & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$, with exact columns and bottom rows.

Also, the homomorphisms $\mathbf{G}_{\mathrm{m}} \times \mathbf{Spin}_{m,m} \to \mathbf{GSO}_{m,m}$ given by $(a,x) \mapsto (a\,r(x), a^2)$ and $s : \mathbf{S\Gamma}_{m,m} \to \mathbf{GSO}_{m,m}$ from §2.4) factor through homomorphisms $s^{\pm} : \mathbf{GSpin}^{\pm}_{m,m} \to \mathbf{GSO}_{m,m}$, yielding short exact sequences

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{GSpin}^{\pm}_{m,m} \xrightarrow{s^{\pm}} \mathbf{GSO}_{m,m} \to 1 \tag{19}$$

fitting into commutative diagrams with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{Spin}_{m,m} & \xrightarrow{r} & \mathbf{SO}_{m,m} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{GSpin}^{\pm}_{m,m} & \xrightarrow{s^{\pm}} & \mathbf{GSO}_{m,m} & \longrightarrow & 1 \\
 & & & & {\scriptstyle \mu^{\pm}}\downarrow & & {\scriptstyle \mu}\downarrow & & \\
 & & & & \mathbf{G}_{\mathrm{m}} & = & \mathbf{G}_{\mathrm{m}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array} \tag{20}
$$

of sheaves of groups on $X_{\text{ét}}$.

Definition 2.17. For an oriented regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L}, \zeta)$ of rank $n = 2m$ and trivial Arf invariant, define a pair of invariants $gc^{\pm}(q,\zeta) = gc^{\pm}(\mathscr{E}, q, \mathscr{L}, \zeta) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ to be the images of $(\mathscr{E}, q, \mathscr{L}, \zeta)$ under the coboundary maps $H^1_{\text{ét}}(X, \mathbf{GSpin}^{\pm}_{m,m}) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ arising from sequence (19).

Proposition 2.18. *Let $(\mathscr{E}, q, \mathscr{L})$ be a regular line bundle-valued quadratic form of rank $n = 2m$ and trivial Arf invariant. Let ζ be any orientation of $(\mathscr{E}, q, \mathscr{L})$. Then we have:*

- *a) $gc^{\pm}(q,\zeta) = c(q)$ if $\mathscr{L} = \mathscr{O}_X$,*
- *b) $gc^{+}(q,\zeta) + gc^{-}(q,\zeta) = c_1(\mathscr{L}, \boldsymbol{\mu}_2)$,*
- *c) $gc^{\pm}(q,-\zeta) = gc^{\mp}(q,\zeta)$,*
- *d) $gc^{+}(q,\zeta_{\mathscr{V}}) = c_1(\mathscr{V}, \boldsymbol{\mu}_2)$, if $\mathscr{V} \to \mathscr{E}$ is a lagrangian of $(\mathscr{E}, q, \mathscr{L})$ and $\zeta_{\mathscr{V}}$ is the induced orientation from Lemma 1.14.*

Proof. For *a)*, we simply consider the interpretation on cohomology of the top rows of diagram (20).

For *b)*, denote by $\boldsymbol{\gamma}_{m,m}$ the subgroupscheme of $\mathbf{\Gamma}'_{m,m}$ generated by $\boldsymbol{\gamma}^{+}_{m,m}$ and $\boldsymbol{\gamma}^{-}_{m,m}$. Note that when m is odd, $\boldsymbol{\gamma}_{m,m} = \boldsymbol{\kappa}_{m,m}$. We have a fixed isomorphism $\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 \xrightarrow{\sim} \boldsymbol{\gamma}_{m,m}$ and a labeling $\boldsymbol{\gamma}^{\pm}_{m,m}$ and $\boldsymbol{\gamma}^{0}_{m,m}$ of the three subgroupschemes of $\boldsymbol{\gamma}_{m,m}$. For $\bullet \in \{-,0,+\}$, denote by $p^{\bullet}$ the quotient homomorphism in the exact sequence

$$1 \to \boldsymbol{\gamma}^{\bullet}_{m,m} \to \boldsymbol{\gamma}_{m,m} \xrightarrow{p^{\bullet}} \boldsymbol{\mu}_2 \to 1.$$

For each $m \geq 1$ and each $\bullet \in \{-,0,+\}$, it's a straightforward verification that the diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \boldsymbol{\gamma}^{\bullet}_{m,m} & = & \boldsymbol{\gamma}^{\bullet}_{m,m} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \boldsymbol{\gamma}_{m,m} & \longrightarrow & \mathbf{\Gamma}'_{m,m} & \longrightarrow & \mathbf{GSO}_{m,m} & \longrightarrow & 1 \\
 & & {\scriptstyle p^{\bullet}}\downarrow & & {\scriptstyle q^{\bullet}}\downarrow & & \| & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{GSpin}^{\bullet}_{m,m} & \xrightarrow{s^{\bullet}} & \mathbf{GSO}_{m,m} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{21}
$$

of sheaves of groups on $X_{\text{ét}}$ is commutative with exact rows and columns. Here we've used the following notation: for $\bullet \in \{\pm\}$, the map $q^{\bullet} : \mathbf{\Gamma}'_{m,m} \to \mathbf{GSpin}^{\bullet}_{m,m}$ is the quotient homomorphism; for $\bullet = 0$, we set $s^0 = s$ as defined in Definition 2.2, $\mathbf{GSpin}^{0}_{m,m} = \mathbf{G}_{\mathrm{m}} \times \mathbf{SO}_{m,m}$, and $q^0 = N \times r : \mathbf{\Gamma}'_{m,m} = \mathbf{S\Gamma}_{m,m} \to \mathbf{G}_{\mathrm{m}} \times \mathbf{SO}_{m,m}$ when m is odd while $q^0 = \mathrm{id} \times r : \mathbf{\Gamma}'_{m,m} = \mathbf{G}_{\mathrm{m}} \times \mathbf{Spin}_{m,m} \to \mathbf{G}_{\mathrm{m}} \times \mathbf{SO}_{m,m}$ when m is even.

For each $m \geq 1$, denote by $\delta_m : H^1_{\text{ét}}(X, \mathbf{GSO}_{m,m}) \to H^2_{\text{ét}}(X, \boldsymbol{\gamma}_{m,m})$ the coboundary map associated to the central row of diagram (21). Note that for m odd, $\delta_m = gc$, by Definition 2.8.

Consider the split exact sequence of group schemes,

$$1 \to \boldsymbol{\gamma}_{m,m} \xrightarrow{p^- \times p^0 \times p^+} \boldsymbol{\mu}_2^{\times 3} \xrightarrow{\prod} \boldsymbol{\mu}_2 \to 1,$$

uniquely defined up to ordering, where $\prod$ is the total multiplication homomorphism. On étale cohomology, there are thus (split) exact sequences,

$$0 \to H^i_{\text{ét}}(X, \boldsymbol{\gamma}_{m,m}) \xrightarrow{p^{-,i} \oplus p^{0,i} \oplus p^{+,i}} H^i_{\text{ét}}(X, \boldsymbol{\mu}_2)^{\oplus 3} \to H^i_{\text{ét}}(X, \boldsymbol{\mu}_2) \to 0. \tag{22}$$

of abelian groups for each $i \geq 0$.

By the commutativity of the bottom rows of diagram (21), we have

$$p^{\bullet,2}\delta_m(\mathscr{E}, q, \mathscr{L}, \zeta) = \begin{cases} gc^{\bullet}(\mathscr{E}, q, \mathscr{L}, \zeta) & \text{for } \bullet \in \{\pm\} \\ c_1(\mathscr{L}, \boldsymbol{\mu}_2) & \text{for } \bullet = 0 \end{cases}$$

by the definition of $gc^{\pm}$ for $\bullet \in \{\pm\}$ and by the commutativity of diagram (6) for $\bullet = 0$. Note that for m odd and $\bullet = 0$, this is a restatement of Theorem 2.10*a)*. In particular, by exact sequence (22) of cohomology groups (for $i = 2$),

$$gc^+(\mathscr{E}, q, \mathscr{L}) + gc^-(\mathscr{E}, q, \mathscr{L}) = p^{-,2}\delta_m(\mathscr{E}, b, \mathscr{L}, \zeta) + p^{+,2}\delta_m(\mathscr{E}, b, \mathscr{L}, \zeta) = c_1(\mathscr{L}, \boldsymbol{\mu}_2)$$

in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.

Part *c)* follows immediately from the definition of $\mathbf{GSpin}^{\pm}_{m,m}$. For *d)*, we can adapt the proof of Theorem 2.11 ☐

Remark 2.19. By Proposition 2.18*c)*, if $\mathscr{L}$ is not a square in $\mathrm{Pic}(X)$, then $gw^{\pm}(q, \zeta)$ depends on the orientation ζ. In particular, if $\mathrm{Pic}(X)$ is not 2-divisible, then the invariants $gc^{\pm}$ are in general orientation dependent and hence do not descend to $GI_2^{\mathrm{tot}}(X)$.

Example 2.20. If $m = 3$, then $\mathbf{GSpin}^{\pm}_{3,3} \cong \mathbf{GL}_4$ and exact sequence (19) is isomorphic to the familiar sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{GL}_4 \xrightarrow{\wedge^2} \mathbf{GSO}_{3,3} \to 1,$$

for an appropriate choice of $\pm$ and ϵ, see §5.4 for more details.

As an application of Example 2.20, we can give a characterization of locally free $\mathscr{O}_X$-modules of rank 6 that are *exterior squares*, i.e. isomorphic to $\bigwedge^2 \mathscr{V}$ for a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of rank 4.

Proposition 2.21. *Let X be a scheme with 2 invertible. Then a locally free $\mathscr{O}_X$-module $\mathscr{E}$ of rank 6 is an exterior square if and only if $\mathscr{E}$ supports a regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ with trivial Arf invariant and trivial $gc^+(\mathscr{E}, q, \mathscr{L}, \zeta)$ for some orientation ζ.*

Proof. If $\mathscr{E} \cong \bigwedge^2 \mathscr{V}$, then $\mathscr{E}$ supports a regular line bundle-valued quadratic form via the canonical "wedging" form $\bigwedge^2 \mathscr{V} \xrightarrow{\wedge} \det \mathscr{V}$). That such a form has trivial Arf invariant follows from its interpretation as a reduced pfaffian and the explicit calculation of the even Clifford algebra of a reduced pfaffian, see §5.4 and [17, Prop. 4.8]. Since 2 is invertible on X we can argue directly, by showing that the associated symmetric bilinear form has trivial discriminant. Indeed, when $\mathscr{V}$ is free with basis $e_1, \ldots, e_4$, an $\mathscr{O}_X$-module morphism $\zeta : \det(\bigwedge^2 \mathscr{V}) \to (\det \mathscr{V})^{\otimes 3}$ can be given by

$$\bigwedge_{1 \leq i < j \leq 4} (e_i \wedge e_j) \quad \mapsto \quad (e_1 \wedge \cdots \wedge e_4)^{\otimes 3}.$$

A standard computation (compute the similarity factor on the standard middle exterior power of the fundamental representation of $\mathbf{GL}_r$) shows that this morphism does not depend on the choice of basis. Hence for a general locally free $\mathscr{V}$, this morphism patches over a Zariski open cover of X splitting $\mathscr{V}$. Finally, scaling ζ by $\left(\det \mathscr{V}^{\vee}\right)^{\otimes 3}$ yields an $\mathscr{O}_X$-module morphism $\zeta_{\mathscr{V}} : d_{\pm}(\bigwedge^2 \mathscr{V}, \wedge, \det \mathscr{V}) \to <-1>$, which can be checked to be an isometry.

Now the map $\mathbf{GL}_4 \xrightarrow{\wedge^2} \mathbf{GSO}_{3,3}$ can be reinterpreted as the homomorphism of automorphism sheaves (when applied to $\mathscr{V} = \mathscr{O}_X^4$) induced from the functor $\mathscr{V} \mapsto (\bigwedge^2 \mathscr{V}, \wedge, \det \mathscr{V})$. Hence, the exact sequence of cohomology sets

$$H^1_{\text{ét}}(X, \mathbf{GL}_4) \xrightarrow{(\wedge^2)^1} H^1_{\text{ét}}(X, \mathbf{GSO}_{3,3}) \xrightarrow{gc^+} H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

has the following interpretation: $(\wedge^2)^1$ takes the isomorphism class of a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of rank 4 to $(\bigwedge^2 \mathscr{V}, \wedge, \det \mathscr{V})$ with the above orientation $\zeta_{\mathscr{V}}$. In particular, $gc^+(\bigwedge^2 \mathscr{V}, \wedge, \det \mathscr{V}, \zeta_{\mathscr{V}})$ is trivial.

Now suppose that $\mathscr{E}$ supports a regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ with trivial Arf invariant. Then the similarity class of $(\mathscr{E}, q, \mathscr{L})$ defines a class in $H^1_{\text{ét}}(X, \mathbf{GSO}_{3,3})$ and by the exactness of the above sequence, $gc^+(\mathscr{E},q,\mathscr{L},\zeta)$ is trivial for some ζ if and only if $(\mathscr{E},q,\mathscr{L},\zeta)$ is in the image of $(\wedge^2)^1$. In particular, $\mathscr{E} \cong \bigwedge^2 \mathscr{V}$ for some locally free $\mathscr{O}_X$-module $\mathscr{V}$ of rank 4. □

3. Relationship to the even Clifford algebra

In this section we relate the similarity Clifford invariant to classes in étale cohomology arising from the (generalized) even Clifford algebra construction of Bichsel–Knus [17].

3.1. **Involutions on endomorphism algebras and projective similarity.** An *antiautomorphism* of an $\mathscr{O}_X$-algebra $\mathscr{A}$ is defined to be a $\mathscr{O}_X$-algebra isomorphism $\sigma : \mathscr{A} \to \mathscr{A}^{\mathrm{op}}$, where $\mathscr{A}^{\mathrm{op}}$ is the opposite algebra. An antiautomorphism σ is called an *involution* (of the first kind) if $\sigma^{\mathrm{op}} \circ \sigma = \mathrm{id}_{\mathscr{A}}$, identifying $\mathscr{A} = (\mathscr{A}^{\mathrm{op}})^{\mathrm{op}}$. A morphism $\varphi : (\mathscr{A},\sigma) \to (\mathscr{A}',\sigma')$ between $\mathscr{O}_X$-algebras with involution consists of an $\mathscr{O}_X$-algebra homomorphism $\varphi : \mathscr{A} \to \mathscr{A}'$ such that $\varphi \circ \sigma' = \sigma \circ \varphi^{\mathrm{op}}$. If $(\mathscr{E},b,\mathscr{L})$ is a regular $\mathscr{L}$-valued bilinear form, the $\mathscr{O}_X$-algebra anti-automorphism defined by

$$\begin{aligned} \sigma_b : \mathscr{E}nd(\mathscr{E}) &\to \mathscr{E}nd(\mathscr{E})^{\mathrm{op}} \\ \varphi &\mapsto \psi_b^{-1} \circ \varphi^{\vee\mathscr{L}} \circ \psi_b \end{aligned}$$

is an involution (called the *adjoint involution*) if and only if b is ϵ-symmetric (see 1.3), for some $\epsilon \in H^0_{\text{ét}}(X,\boldsymbol{\mu}_2)$. Conversely, any involution (of the first kind) on $\mathscr{E}nd(\mathscr{E})$ arises this way.

Proposition 3.1 (Saltman [88, Thm. 4.2a], Knus–Parimala–Srinivas [67])**.** *Let X be a scheme and $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of finite rank. If σ is an involution (of the first kind) of $\mathscr{E}nd(\mathscr{E})$ then there exists a regular ϵ-symmetric bilinear form $(\mathscr{E},b,\mathscr{L})$ such that $\sigma = \sigma_b$. Moreover, the similarity class of $(\mathscr{E},b,\mathscr{L})$ is uniquely determined by $\mathscr{E}$ and σ.*

Remark 3.2. Note however, that the $\mathscr{O}_X$-algebras $\mathscr{E}nd(\mathscr{N}\otimes\mathscr{E})$ and $\mathscr{E}nd(\mathscr{E})$ are isomorphic for any invertible $\mathscr{O}_X$-module $\mathscr{N}$ (see Remark 1.3). Thus an involution σ on $\mathscr{E}nd(\mathscr{E})$ that corresponds to a bilinear form $(\mathscr{E},b,\mathscr{L})$ will give rise be an involution on $\mathscr{E}nd(\mathscr{N}\otimes\mathscr{E})$ that corresponds to the bilinear form $(\mathscr{N}\otimes\mathscr{E},b',\mathscr{N}^{\otimes 2}\otimes\mathscr{L})$. Thus the isomorphism class (as $\mathscr{O}_X$-algebras with involution) of $(\mathscr{E}nd(\mathscr{E}),\sigma_b)$ only determines the *projective similarity* class of the bilinear form $(\mathscr{E},b,\mathscr{L})$, i.e. the set of similarity classes of bilinear forms $(\mathscr{N}\otimes\mathscr{E},n\otimes b,\mathscr{N}^{\otimes 2}\otimes\mathscr{L})$, where $(\mathscr{N},n,\mathscr{N}^{\otimes 2})$ is a bilinear form of rank 1.

If $(\mathscr{E},b,\mathscr{L})$ is a regular ϵ-symmetric bilinear form, the *projective similitude* group $\mathbf{PSim}(\mathscr{E},b,\mathscr{L})$ is defined as the sheaf of automorphism (as algebras with involution) groups $\mathbf{Aut}(\mathscr{E}nd(\mathscr{E}),\sigma_b)$. There's a natural homomorphism $\mathbf{GO}(\mathscr{E},q,\mathscr{L}) \to \mathbf{PSim}(\mathscr{E},q,\mathscr{L})$ of sheaves of groups on $X_{\text{ét}}$, with kernel the central embedding $\mathbf{G}_{\mathrm{m}} \to \mathbf{Sim}(\mathscr{E},b,\mathscr{L})$ given by homotheties There is an exact sequence

$$1 \to \mathbf{G}_{\mathrm{m}} \to \mathbf{Sim}(\mathscr{E},b,\mathscr{L}) \to \mathbf{PSim}(\mathscr{E},b,\mathscr{L}) \to 1. \tag{23}$$

of sheaves of groups on $X_{\text{ét}}$. The embedding of homotheties restricts to a central embedding $\boldsymbol{\mu}_2 \to \mathbf{Isom}(\mathscr{E},b,\mathscr{L})$, the sheaf cokernel of which defines the *projective orthogonal* group $\mathbf{PIsom}(\mathscr{E},b,\mathscr{L})$. In total, we have a commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \xrightarrow{2} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbf{Isom} & \longrightarrow & \mathbf{Sim} & \xrightarrow{\mu} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & \mathbf{PIsom} & \xrightarrow{\sim} & \mathbf{PSim} & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \tag{24}$$

of sheaves of groups on $X_{\text{ét}}$ by the Nine Lemma 5.16.

A locally free $\mathscr{O}_X$-algebra $\mathscr{A}$ of finite rank is an *Azumaya $\mathscr{O}_X$-algebra* if the canonical $\mathscr{O}_X$-algebra homomorphism

$$\begin{array}{ccc} \mathscr{A}\otimes\mathscr{A}^{\mathrm{op}} & \to & \mathscr{E}nd(\mathscr{A}) \\ a\otimes b & \mapsto & c \mapsto acb \end{array}$$

is an isomorphism, where $\mathscr{E}nd(\mathscr{A})$ is the algebra of $\mathscr{O}_X$-module endomorphisms of $\mathscr{A}$. Every Azumaya algebra is locally isomorphic on $X_{\text{ét}}$ to an endomorphism algebra of a locally free sheaf, see Milne [76, IV Proposition 2.3]. In particular, locally on $X_{\text{ét}}$, an Azumaya algebra has rank n^2 for some n, giving rise to a global section $\deg(\mathscr{A}) \in H^0_{\text{ét}}(X,\mathbb{Z})$ called the *degree* of $\mathscr{A}$.

An involution of the first kind σ on an Azumaya $\mathscr{O}_X$-algebra $\mathscr{A}$ is locally isomorphic on $X_{\text{ét}}$, by Proposition 3.1, to an adjoint involution associated to an ϵ-symmetric bilinear form for some $\epsilon \in H^0_{\text{ét}}(X, \boldsymbol{\mu}_2)$, called the *type* of the involution. On each connected component, the type is called *orthogonal* or *symplectic* depending on the sign of ϵ.

Proposition 3.3. *Let X be a scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ be a regular ϵ-symmetric bilinear form of rank n on X. Then the groupoid of $\mathbf{PSim}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the category of Azumaya $\mathscr{O}_X$-algebras of degree n with involution of type ϵ together with isomorphisms of algebras with involution.*

Remark 3.4. The map on cohomology

$$H^1_{\text{ét}}(X, \mathbf{Sim}(\mathscr{E}, b, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbf{PSim}(\mathscr{E}, b, \mathscr{L}))$$

associated to sequence (23) has the following interpretation: the similarity class of a line bundle-valued ϵ-symmetric bilinear form is mapped to the adjoint involution on its endomorphism algebra. The coboundary map

$$H^1_{\text{ét}}(X, \mathbf{PSim}(\mathscr{E}, b, \mathscr{L})) \to H^2_{\text{ét}}(X, \mathbf{G}_{\text{m}})$$

takes an Azumaya algebra with with involution of type ϵ to the Brauer equivalence class of the algebra. The Brauer class of any Azumaya $\mathscr{O}_X$-algebra $\mathscr{A}$ supporting an involution of the first kind is 2-torsion. A refinement of this coboundary map is given via the *involutive Brauer group*, see 3.3.

3.2. **Unitary involutions and hermitian forms.** Let $f : Z \to X$ be étale quadratic and ι the nontrivial X-automorphism of Z. Since ι is an automorphism of order 2, $\iota_*^{-1} = \iota_*$ and ι^* are isomorphic functors on the category of $\mathscr{O}_Z$-modules. Indeed, for any $\mathscr{O}_Z$-module $\mathscr{H}$, a canonical isomorphism $\alpha_{\mathscr{H}} : \iota^*\mathscr{H} \to \iota_*\mathscr{H}$ is given by adjunction applied to the identity map $\mathscr{H} \to \iota_*\iota_*\mathscr{H}$.

For any $\mathscr{O}_X$-module $\mathscr{L}$, the unit of adjunction defines an $\mathscr{O}_Z$-module isomorphism $f^*\mathscr{L} \xrightarrow{\eta} \iota_*\iota^* f^*\mathscr{L} = \iota_* f^*\mathscr{L}$, which we shall (by abuse of notation) also denote by $\iota^\sharp : f^*\mathscr{L} \to \iota_* f^*\mathscr{L}$. Note that this also coincides with the composition

$$f^*\mathscr{L} \cong f^*\mathscr{L} \otimes \mathscr{O}_Z \xrightarrow{\mathrm{id}_{f^*\mathscr{L}} \otimes \iota^\sharp} f^*\mathscr{L} \otimes \iota_*\mathscr{O}_Z \xrightarrow{\sim} \iota_*(\iota^* f^*\mathscr{L} \otimes \mathscr{O}_Z) \cong \iota_* f^*\mathscr{L},$$

where the third isomorphism is from the projection formula.

Hermitian forms. A *line bundle-valued Z/X-hermitian form* is a triple $(\mathscr{H}, h, \mathscr{L})$, where $\mathscr{H}$ is a locally free $\mathscr{O}_Z$-module of finite rank, $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module, and $h : \mathscr{H} \otimes \iota_*\mathscr{H} \to f^*\mathscr{L}$ is an $\mathscr{O}_Z$-module morphism such that the following diagram of $\mathscr{O}_Z$-modules commute

$$\begin{array}{ccc} \mathscr{H} \otimes \iota_*\mathscr{H} & \xrightarrow{h} & f^*\mathscr{L} \\ \downarrow & & \downarrow{\scriptstyle \iota^\sharp} \\ \iota_*(\mathscr{H} \otimes \iota_*\mathscr{H}) & \xrightarrow{\iota_* h} & \iota_* f^*\mathscr{L} \end{array} \tag{25}$$

where the left vertical map is the composition

$$\mathscr{H} \otimes \iota_*\mathscr{H} \to \iota_*\mathscr{H} \otimes \mathscr{H} \to \iota_*(\mathscr{H} \otimes \iota^*\mathscr{H}) \xrightarrow{\iota_*(\mathrm{id}_{\mathscr{H}} \otimes \alpha_{\mathscr{H}})} \iota_*(\mathscr{H} \otimes \iota_*\mathscr{H})$$

of canonical $\mathscr{O}_Z$-module morphisms: naïve tensor switch, projection formula, and induced from α. The commutativity of diagram (25) represents the usual formula "$\overline{h(x,y)} = h(y,x)$" defining a hermitian form.

As in the bilinear form case, a Z/X-hermitian form $(\mathscr{H}, h, \mathscr{L})$ has an $\mathscr{O}_Z$-module *adjoint* morphism $\psi_h : \mathscr{H} \to \mathscr{H}om(\iota_*\mathscr{H}, f^*\mathscr{L})$. It is called *regular* if ψ_h is an $\mathscr{O}_Z$-module isomorphism. The commutativity of diagram (25) is equivalent to $\psi_h = \psi_h^\dagger$, where $\psi_h^\dagger$ is defined as the composition

$$\psi_h^\dagger : \mathscr{H} \xrightarrow{\mathrm{ev}^{*\mathscr{L}}} \mathscr{H}om(\iota_* \mathscr{H}om(\iota_*\mathscr{H}, f^*\mathscr{L}), f^*\mathscr{L}) \xrightarrow{\psi_h^{\star\mathscr{L}}} \mathscr{H}om(\iota_*\mathscr{H}, f^*\mathscr{L}).$$

Here, $(-)^{\star\mathscr{L}}$ is the contravariant functor $\mathscr{H}om(\iota_*(-), f^*\mathscr{L})$ and

$$\mathrm{ev}^{\star\mathscr{L}} : \mathrm{id} \to ((-)^{\star\mathscr{L}})^{\star\mathscr{L}}$$

is the canonical morphism of functors on the category of coherent $\mathscr{O}_Z$-modules, which is an isomorphism on the subcategory $\mathsf{VB}(Z)$ of locally free $\mathscr{O}_Z$-modules. The triple $(\mathsf{VB}(Z), (-)^{\star\mathscr{L}}, \mathrm{ev}^{\star\mathscr{L}})$ forms an exact category with duality, the Witt group of which is studied in Gille [46]. This definition of hermitian form is equivalent to the one found in [67, §3] and (upon taking ι_*) [85, §1.2].

Unitary groups. A *similarity (transformation)* or *similitude* between line bundle-valued Z/X-hermitian forms $(\mathscr{H}, h, \mathscr{L})$ and $(\mathscr{H}', h', \mathscr{L}')$ is a pair (φ, λ), where $\varphi : \mathscr{H} \to \mathscr{H}'$ is an $\mathscr{O}_Z$-module isomorphism and $\lambda : \mathscr{L} \to \mathscr{L}'$ is an $\mathscr{O}_X$-module isomorphism such that either of the following (equivalent) diagrams,

$$
(26) \qquad
\begin{array}{ccc}
\mathscr{H} \otimes \iota_* \mathscr{H} & \xrightarrow{\;h\;} & f^* \mathscr{L} \\
{\scriptstyle \varphi \otimes \iota_* \varphi} \downarrow & & \downarrow {\scriptstyle f^* \lambda} \\
\mathscr{H}' \otimes \iota_* \mathscr{H}' & \xrightarrow{\;h'\;} & f^* \mathscr{L}'
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathscr{H} & \xrightarrow{\;\psi_h\;} & \mathscr{H}om(\iota_* \mathscr{H}, f^* \mathscr{L}) \\
{\scriptstyle \varphi} \downarrow & & \downarrow {\scriptstyle f^* \lambda^{-1} \varphi^{* \mathscr{L}}} \\
\mathscr{H}' & \xrightarrow{\;\psi_{h'}\;} & \mathscr{H}om(\iota_* \mathscr{E}', f^* \mathscr{L}')
\end{array}
$$

of $\mathscr{O}_Z$-modules commute, where $f^* \lambda^{-1} \varphi^{* \mathscr{L}}(\psi) = f^* \lambda^{-1} \circ \psi \circ \varphi$ on sections. A similarity transformation (φ, λ) is an *isometry* if $\mathscr{L} = \mathscr{L}'$ and λ is the identity map.

For $U \to X$, denote by $(\mathscr{H}, h, \mathscr{L})|_U$ the $Z \times_X U/U$-hermitian form $(\mathscr{H}|_{f^{-1}(U)}, h|_{f^{-1}(U)}, \mathscr{L}|_U)$. Denote by $\mathbf{GU}(\mathscr{H}, h, \mathscr{L})$ (resp. $\mathbf{U}(\mathscr{H}, h, \mathscr{L})$) the presheaf on $X_{\text{ét}}$ of similarities (resp. isometries)

$$U \mapsto \{\text{similarities (resp. isometries) } (\varphi, \lambda) : (\mathscr{H}, h, \mathscr{L})|_U \to (\mathscr{H}, h, \mathscr{L})|_U\}$$

of the regular $\mathscr{L}$-valued Z/X-hermitian form $(\mathscr{H}, h, \mathscr{L})$. In fact, these presheaves are sheaves on $X_{\text{ét}}$ and are representable by smooth affine reductive group schemes over X, called the *general unitary* (resp. *unitary*) group of the line bundle-valued Z/X-hermitian form $(\mathscr{H}, h, \mathscr{L})$. We will often omit the dependence of these groups on the form $(\mathscr{H}, h, \mathscr{L})$ when no confusion may arise. Even though these sheaves of groups are representable by schemes over X, we will still think of them as sheaves of groups on $X_{\text{ét}}$.

The map assigning $(\varphi, \lambda) \mapsto \lambda$ on sections defines the *multiplier coefficient* homomorphism $\mu : \mathbf{GU}(\mathscr{H}, h, \mathscr{L}) \to \mathbf{GL}(\mathscr{L}) = \mathbf{G}_{\text{m}}$. As in Proposition 1.4, for any scheme X with 2 invertible, the sequence of sheaves of groups,

$$1 \to \mathbf{U}(\mathscr{H}, h, \mathscr{L}) \to \mathbf{GU}(\mathscr{H}, h, \mathscr{L}) \xrightarrow{\mu} \mathbf{G}_{\text{m}} \to 1,$$

is exact on $X_{\text{ét}}$ and is called the *multiplier sequence.*

Proposition 3.5. *Let X be a scheme, $f : Z \to X$ étale quadratic, and $(\mathscr{H}, h, \mathscr{L})$ a fixed regular $\mathscr{L}$-valued Z/X-hermitian form of rank n.*

- *a) The groupoid of $\mathbf{U}(\mathscr{H}, h, \mathscr{L})$-torsors is equivalent to the groupoid whose objects are regular $\mathscr{L}$-valued Z/X-hermitian forms of rank n and whose morphisms are isometries.*
- *b) The groupoid of $\mathbf{GU}(\mathscr{E}, b, \mathscr{L})$-torsors is equivalent to the groupoid whose objects are regular line bundle-valued Z/X-hermitian forms of rank n and whose morphisms are similarity transformations.*

Remark 3.6. The multiplier sequence has a cohomological interpretation in analogy with Remark 1.5. If $(\mathscr{H}, h, \mathscr{L})$ is a fixed $\mathscr{L}$-valued Z/X-hermitian form of rank n, then the map

$$H^1_{\text{ét}}(X, \mathbf{U}(\mathscr{H}, h, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbf{GU}(\mathscr{H}, h, \mathscr{L}))$$

takes the isometry class of a regular $\mathscr{L}$-valued Z/X-hermitian form of rank n to its similarity class. Under the identification $\mathbf{GL}(\mathscr{L}) = \mathbf{G}_{\text{m}}$ the map

$$H^1_{\text{ét}}(X, \mathbf{GU}(\mathscr{H}, h, \mathscr{L})) \to H^1_{\text{ét}}(X, \mathbf{G}_{\text{m}}) \cong \text{Pic}(X)$$

takes the similarity class of a regular $\mathscr{L}'$-valued of rank n to the class of $\mathscr{L}'$ in $\text{Pic}(X)$.

Unitary involutions. If $\mathscr{B}$ is an $\mathscr{O}_Z$-algebra, then an *ι-semilinear antiautomorphism* of $\mathscr{B}$ is defined to be an $\mathscr{O}_Z$-algebra isomorphism $\sigma : \mathscr{B} \to \iota_* \mathscr{B}^{\text{op}}$ such that the restriction $\sigma|_{\mathscr{O}_Z}$ equals the canonical sheaf isomorphism $\iota^\sharp : \mathscr{O}_Z \to \iota_* \mathscr{O}_Z$. An ι-semilinear antiautomorphism σ is called a *Z/X-unitary involution* if $\iota_* \sigma^{\text{op}} \circ \sigma = \text{id}_{\mathscr{B}}$. An $\mathscr{O}_Z$-algebra isomorphism $\sigma : \mathscr{B} \to \iota^* \mathscr{B}^{\text{op}}$ specifies an ι-semilinear antiautomorphism $\alpha_{\mathscr{B}^{\text{op}}} \circ \sigma$ of $\mathscr{B}$.

A morphism $\varphi : (\mathscr{B}, \sigma) \to (\mathscr{B}', \sigma')$ of $\mathscr{O}_Z$-algebras with Z/X-unitary involution consists of an $\mathscr{O}_Z$-algebra homomorphism $\varphi : \mathscr{B} \to \mathscr{B}'$ such that $\sigma' \circ \varphi = \iota_* \varphi^{\text{op}} \circ \sigma$.

Given a regular line bundle-valued Z/X-hermitian form, the ι-semilinear antiautomorphism defined by

$$
\begin{aligned}
\sigma_h : \mathscr{E}nd(\mathscr{H}) &\to \mathscr{E}nd(\iota_* \mathscr{H})^{\text{op}} \cong \iota_* \mathscr{E}nd(\mathscr{H})^{\text{op}} \\
\varphi &\mapsto \psi_h^{-1} \circ \varphi^{\star \mathscr{L}} \circ \psi_h
\end{aligned}
$$

is a Z/X-unitary involution called the *adjoint involution* of $(\mathscr{H}, h, \mathscr{L})$. Conversely, every Z/X-unitary involution on $\mathscr{E}nd(\mathscr{H})$ arises this way.

Proposition 3.7 (Saltman [88, Thm. 4.2b] or Knus–Parimala–Srinivas [67]). *Let X be a scheme, $f : Z \to X$ be étale quadratic, and $\mathscr{H}$ a locally free $\mathscr{O}_Z$-module of finite rank. If σ is a Z/X-unitary involution on $\mathscr{E}nd(\mathscr{H})$ then there exists a regular Z/X-hermitian form $(\mathscr{H}, h, \mathscr{L})$ such that $\sigma = \sigma_h$. Moreover, the similarity class of $(\mathscr{H}, h, \mathscr{L})$ is uniquely determined by $\mathscr{H}$ and σ.*

Remark 3.8. Similarly to Remark 3.2, the isomorphism class of the Z/X-unitary involution $(\mathscr{E}nd(\mathscr{H}), \sigma_h)$ determines the projective similarity class of $(\mathscr{H}, h, \mathscr{L})$.

Let $(\mathscr{H}, h, \mathscr{L})$ be a regular Z/X-hermitian form. For every étale $U \to X$, denote by $(\mathscr{E}nd(\mathscr{H}), \sigma_h)|_U$ the $\mathscr{O}_U$-algebra with $Z \times_X U/U$-unitary involution $(\mathscr{E}nd(\mathscr{H}|_{f^{-1}(U)}), \sigma_h|_{f^{-1}(U)})$. Denote by $\mathbf{PGU}(\mathscr{H}, h, \mathscr{L})$ the presheaf of automorphism groups

$$U \mapsto \{Z \times_X U/U\text{-unitary automorphisms } \varphi : (\mathscr{E}nd(\mathscr{H}), \sigma_h)|_U \to (\mathscr{E}nd(\mathscr{H}), \sigma_h)|_U\}$$

on $X_{\text{ét}}$. In fact, this presheaves is a sheaves on $X_{\text{ét}}$, representable as an affine reductive algebraic group, called the *projective unitary similitude* group $\mathbf{PGU}(\mathscr{H}, h, \mathscr{L}) = \mathbf{Aut}_{Z/X}(\mathscr{E}nd(\mathscr{H}), \sigma_h)$. There's a natural homomorphism $\mathbf{GU}(\mathscr{H}, h, \mathscr{L}) \to \mathbf{PGU}(\mathscr{H}, q, \mathscr{L})$ of sheaves of groups on $X_{\text{ét}}$, with kernel the central embedding $\mathbf{R}_{Z/X}\mathbf{G}_{\text{m}} \to \mathbf{GU}(\mathscr{E}, b, \mathscr{L})$ given by homotheties. There is an exact sequence

$$1 \to \mathbf{R}_{Z/X}\mathbf{G}_{\text{m}} \to \mathbf{GU}(\mathscr{H}, h, \mathscr{L}) \to \mathbf{PGU}(\mathscr{H}, h, \mathscr{L}) \to 1.$$

of sheaves of groups on $X_{\text{ét}}$. The multiplier of a homothety is its norm, thus the inclusion of homotheties restricts to a central inclusion $\mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} \to \mathbf{U}(\mathscr{H}, h, \mathscr{L})$, the sheaf cokernel of which defines the *projective unitary* group $\mathbf{PU}(\mathscr{H}, h, \mathscr{L})$. In total, we have a commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\text{m}} & \xrightarrow{N} & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{U} & \longrightarrow & \mathbf{GU} & \xrightarrow{\mu} & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathbf{PU} & \xrightarrow{\sim} & \mathbf{PGU} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{27}$$

of sheaves of groups on $X_{\text{ét}}$.

Finally, if the $\mathscr{O}_Z$-rank of $\mathscr{H}$ is m, then $N(\det(\varphi)) = \lambda^{2m}$. Thus we have an induced homomorphism $\det/\mu^m : \mathbf{GU}(\mathscr{H}, h, \mathscr{L}) \to \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}}$, whose sheaf kernel is the *proper general unitary* group $\mathbf{SGU}(\mathscr{H}, h, \mathscr{L})$. In analogy with (4), we have an exact sequence

$$1 \to \mathbf{SGU}(\mathscr{H}, h, \mathscr{L}) \to \mathbf{GU}(\mathscr{H}, h, \mathscr{L}) \xrightarrow{\det/\mu^m} \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} \to 1. \tag{28}$$

Restricting $\det/\mu^m$ to the unitary group yields a homomorphism $\det : \mathbf{U}(\mathscr{H}, h, \mathscr{L}) \to \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}}$, whose sheaf kernel is the *special unitary* group $\mathbf{SU}(\mathscr{H}, h, \mathscr{L})$. In total, we have a commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbf{SU} & \longrightarrow & \mathbf{SGU} & \xrightarrow{\mu} & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \mathbf{U} & \longrightarrow & \mathbf{GU} & \xrightarrow{\mu} & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\
 & & {\scriptstyle\det}\downarrow & & {\scriptstyle\det/\mu^m}\downarrow & & & & \\
 & & \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} & = & \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$, in analogy with (5).

3.3. **Involutive Brauer group.** Parimala–Srinivas [85] construct an "involutive" Brauer group $\mathrm{Br}^*(X)$ of isomorphism classes $[\mathscr{A},\sigma]$ of Azumaya $\mathscr{O}_X$-algebras with involution of the first kind modulo classes of adjoint involutions associated to $\mathscr{O}_X$-valued symmetric bilinear forms. For an étale quadratic $f : Z \to X$, they also construct a "Z/X-unitary" Brauer group $\mathrm{Br}^*(Z/X)$ of isomorphism classes of Azumaya $\mathscr{O}_Z$-algebras with Z/X-unitary involution modulo classes of adjoint involutions associated to $\mathscr{O}_X$-valued Z/X-hermitian forms. The following result shows that $\mathrm{Br}^*(X)$ is a refinement of the 2-torsion subgroup ${}_2\mathrm{Br}(X)$ and provides a new interpretation for $H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)$.

We have canonical isomorphisms $H^i_{\text{ét}}(X,\mathbf{R}_{Z/X}\mathbf{G}_\mathrm{m}) \xrightarrow{\sim} H^i_{\text{ét}}(Z,\mathbf{G}_\mathrm{m})$ from the degeneration of the Leray spectral sequence associated to the étale morphism $f : Z \to X$. Combined with the long exact sequence of étale cohomology groups associated to (7) and the canonical identification $H^1_{\text{ét}}(X,\mathbf{G}_\mathrm{m}) = \mathrm{Pic}(X)$, we have the exact sequence

$$\cdots \to H^1_{\text{ét}}(X,\mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) \to \mathrm{Pic}(Z) \xrightarrow{N} \mathrm{Pic}(X) \to H^1_{\text{ét}}(X,\mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) \to H^2_{\text{ét}}(Z,\mathbf{G}_\mathrm{m}) \to \cdots$$

which defines, for every invertible $\mathscr{O}_X$-module $\mathscr{L}$ a class $c_1(\mathscr{L},\mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) \in H^2_{\text{ét}}(X,\mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m})$.

Theorem 3.9 (Parimala–Srinivas [85, Th. 1, Th. 2])**.** *Let X be a scheme with 2 invertible.*

a) There is a canonical homomorphism $\Psi : \mathrm{Br}^*(X) \to H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}^*(X) & \xrightarrow{\Psi} & H^2_{\text{ét}}(X,\boldsymbol{\mu}_2) \\ \downarrow & & \downarrow \\ {}_2\mathrm{Br}(X) & \longrightarrow & {}_2H^2_{\text{ét}}(X,\mathbf{G}_m) \end{array}$$

where the vertical maps are the natural forgetful maps. If $(\mathscr{E},b,\mathscr{L})$ is a regular line bundle-valued (skew-)symmetric bilinear form on X, then $\Psi[\mathscr{E}nd(\mathscr{E}),\sigma_b] = c_1(\mathscr{L},\boldsymbol{\mu}_2)$.

b) If $f : Z \to X$ is étale quadratic, then there is a canonical homomorphism $\Psi_{Z/X} : \mathrm{Br}^*(Z/X) \to H^2_{\text{ét}}(X,\mathbf{R}^1_{Z/X}\mathbf{G}_m)$ fitting into a commutative diagram

$$\begin{array}{ccc} \mathrm{Br}^*(Z/X) & \xrightarrow{\Psi_{Z/X}} & H^2_{\text{ét}}(X,\mathbf{R}^1_{Z/X}\mathbf{G}_m) \\ \downarrow & & \downarrow \\ \mathrm{Br}(Z) & \longrightarrow & H^2_{\text{ét}}(Z,\mathbf{G}_m) \end{array}$$

where the left vertical map is the natural forgetful map and the right vertical map is induced from cohomology applied to (7). If $(\mathscr{H},h,\mathscr{L})$ is a regular line bundle-valued Z/X-hermitian form, then $\Psi_{Z/X}[\mathscr{E}nd(\mathscr{H}),\sigma_h] = -c_1(\mathscr{L},\mathbf{R}^1_{Z/X}\mathbf{G}_m)$.

Remark 3.10. The forgetful maps $\mathrm{Br}^*(X) \to {}_2\mathrm{Br}(X)$ and $\mathrm{Br}^*(Z/X) \to \ker\big(H^2_{\text{ét}}(Y,\mathbf{G}_\mathrm{m}) \xrightarrow{N} H^2_{\text{ét}}(X,\mathbf{G}_\mathrm{m})\big)$ are surjective, since every 2-torsion (resp. trivial norm) Brauer class is represented by an Azumaya algebra with involution of the first kind (resp. unitary involution). This follows from the construction in Knus–Parimala–Srinivas [67], which generalize results of Saltman [88] and Albert [1, X.9 Thm. 19], see also [69, I.3].

Now suppose that the map ${}_2\mathrm{Br}(X) \to {}_2H^2_{\text{ét}}(X,\mathbf{G}_\mathrm{m})$ is surjective, i.e. that every 2-torsion class in the cohomological Brauer group is represented by an Azumaya algebra. This is the case when X is a quasi-compact quasi-separated scheme admitting an ample invertible sheaf by de Jong's extension [29] (see also [74, Th. 2.2.2.1]) of a result of Gabber [41]. Then by the above remark and Theorem 3.9, $\mathrm{Br}^*(X) \to H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)$ is surjective. Furthermore, restricting Theorem 3.9 to the subgroup $\mathrm{Br}^+(X) \subset \mathrm{Br}^*(X)$ consisting of classes of Azumaya algebras with orthogonal involution yields an isomorphism $\mathrm{Br}^+(X) \xrightarrow{\sim} H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)$.

One can reinterpret this in the language of twisted sheaves: while a torsion $\mathbf{G}_\mathrm{m}$-twisted sheaf on a sufficiently nice scheme is represented by an Azumaya algebra, a $\boldsymbol{\mu}_2$-twisted sheaf is represented by an Azumaya algebra with (orthogonal) involution.

Proof of Theorem 3.9. We will review details of the proof inasmuch as they are necessary for our purposes. We will also provide new torsorial proofs of the statements concerning classes arising from adjoint involutions. The sign discrepancy between the final formula of *b)* and the final formula of Parimala–Srinivas [85, Thm. 2] (the proof of which was left to the reader) is most likely due to different conventions for associating $\mathbf{G}_\mathrm{m}$-torsors to invertible sheaves.

As for *a)*, for each $n \geq 1$ and $\epsilon \in H^0_{\text{ét}}(X,\boldsymbol{\mu}_2)$, denote by $\mathbf{Isom}_{\epsilon,n}$ the isometry group of the form $\langle 1,\ldots,1\rangle$ on connected components where $\epsilon = 1$ and the standard skew-symmetric hyperbolic form of rank n (if n is even) on connected components where $\epsilon = -1$. Similarly to Remark 3.4, the map $H^1_{\text{ét}}(X,\mathbf{Isom}_{n,\epsilon}) \to H^1_{\text{ét}}(X,\mathbf{PIsom}_{n,\epsilon})$ associated to (24) is interpreted as: the isometry class of a regular ϵ-symmetric form $(\mathscr{E}, b)$ maps to the isomorphism class of its adjoint involution $(\mathscr{E}nd(\mathscr{E}), \sigma_b)$. In particular, the coboundary map $\Psi_n : H^1_{\text{ét}}(X, \mathbf{PIsom}_{n,\epsilon}) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ vanishes on adjoint algebras of $\mathscr{O}_X$-valued ϵ-symmetric bilinear forms and so factors through $\mathrm{Br}^*(X)$. The map $\Psi : \mathrm{Br}^*(X) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ is defined to be the limit of these coboundary maps over classes of Azumaya algebras with involution of fixed degree. The fact that the map is a group homomorphism is verified in [85]. The commutativity of the diagram in *a)* follows from the interpretation on cohomology of the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{Isom}_{n,\epsilon} & \longrightarrow & \mathbf{PIsom}_{n,\epsilon} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{GL}_n & \longrightarrow & \mathbf{PGL}_n & \longrightarrow & 1 \end{array}$$

of sheaves of groups on $X_{\text{ét}}$.

As for the final statement of *a)*, first note that since $c_1(\mathscr{L}) = c_1(\mathscr{N}^{\otimes 2} \otimes \mathscr{L})$, the 1st Chern class of the value line bundle is an invariant of the projective similarity class (see §3.1) of a line bundle-valued bilinear form, and is therefore an invariant of the $\mathscr{O}_X$-algebra isomorphism class of its adjoint involution. Now we apply the Roman IX Lemma 5.17 to the Roman IX rearrangement (Remark 5.18) of diagram (24), which yields the (anti)commutative pentagon

$$\begin{array}{ccccc} & \nearrow & H^1_{\text{ét}}(X, \mathbf{PSim}_{n,\epsilon}) & \xrightarrow{\Psi_n} & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \\ H^1_{\text{ét}}(X, \mathbf{Sim}_{n,\epsilon}) & & & & \| \\ & \searrow & H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) & \xrightarrow{c_1} & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) \end{array}$$

of cohomology sets. Following the similarity class of a line bundle-valued ϵ-symmetric bilinear form $(\mathscr{E}, b, \mathscr{L})$ around the (anti)commuting pentagon yields

$$\Psi_n[\mathscr{E}nd(\mathscr{E}), \sigma_b] = -c_1(\mathscr{L}, \boldsymbol{\mu}_2) = c_1(\mathscr{L}, \boldsymbol{\mu}_2) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2).$$

A similar technique has been used in Căldăraru [24, Prop. 5.2.4].

As for *b)*, the same argument carries over upon replacing $\mathbf{Isom}_{n,\epsilon}$ by $\mathbf{U}^Z_n$, the unitary group of the standard Z/X-hermitian form of $\mathscr{O}_Z$-rank n. In this case, short exact sequence (23) is replaced by

$$1 \to \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} \to \mathbf{U}^Z_n \to \mathbf{PU}^Z_n \to 1$$

whose coboundary map $\Psi^Z_n : H^1_{\text{ét}}(X, \mathbf{PU}^Z_n) \to H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}})$ factor through $\mathrm{Br}^*(Z/X)$. We define $\Psi_{Z/X} : \mathrm{Br}^*(Z/X) \to H^1_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}})$ as a limit of the maps Ψ^Z_n. Diagram (24) is replaced by the corresponding diagram (27), and the the Roman IX Lemma 5.17 and rearrangement (Remark 5.18) yields the anticommutative pentagon

$$\begin{array}{ccccc} & \nearrow & H^1_{\text{ét}}(X, \mathbf{PGU}^Z_n) & \xrightarrow{\Psi^Z_n} & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}) \\ H^1_{\text{ét}}(X, \mathbf{GU}^Z_n) & & & & \| \\ & \searrow & H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}}) & \xrightarrow{c_1} & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}) \end{array}$$

of cohomology sets. Then following the similarity class of a regular line bundle-valued Z/X-hermitian form $(\mathscr{H}, h, \mathscr{L})$ around the anticommuting pentagon yields

$$\Psi_{Z/X}[\mathscr{E}nd(\mathscr{H}), \sigma_h] = -c_1(\mathscr{L}, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}) \in H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}).$$

Note that constructing a proof of 3.9*b)* was left to the reader in Parimala–Srinivas [85]. □

3.4. **Involutive Brauer class of the even Clifford algebra.** We present a generalization of results of Parimala–Srinivas [85, §2] to even Clifford algebras of line bundle-valued quadratic forms. Given a line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$, the type of the *canonical involution* (see 1.8) τ_0 on $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is given as in the classical case over fields.

Proposition 3.11. *Let $(\mathscr{E}, q, \mathscr{L})$ be a line bundle-valued quadratic form on X.*

a) If n is odd, then $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an Azumaya $\mathscr{O}_X$-algebra and
  - (a) *if $n \equiv 1, 7 \bmod 8$ then τ_0 is of orthogonal type, while*
  - (b) *if $n \equiv 3, 5 \bmod 8$ then τ_0 is of symplectic type.*

b) If n is even, then $\widetilde{\mathscr{C}}_0(\mathscr{E}, q, \mathscr{L})$ is an Azumaya $\mathscr{O}_Z$-algebra, where $Z \to X$ is the Arf covering of $(\mathscr{E}, q, \mathscr{L})$.

(a) *if* $n \equiv 0 \bmod 8$ *then* τ_0 *is of orthogonal type on* Z*,*
(b) *if* $n \equiv 4 \bmod 8$ *then* τ_0 *is of symplectic type on* Z*, while*
(c) *if* $n \equiv 2 \bmod 4$ *then* τ_0 *is of* Z/X*-unitary type.*

Proof. Since the type of an involution is local in the étale topology, we may reduce to the case of symmetric bilinear forms on affine schemes, where we can appeal to [85, Lemma 2]. Of course, the same proposition holds for even Clifford algebras of Azumaya algebras with orthogonal involution. □

Recall the notation of Example 2.6. For each even $n = 2m \geq 2$ and each étale quadratic $f : Z \to X$, let $(\mathscr{H}, b) = h_n^Z = (f_*\mathscr{O}_Z, h^Z) \perp H_{\mathscr{O}_X}(\mathscr{O}_X^{m-1})$, where $(f_*\mathscr{O}_Z, h^Z)$ is the *norm form* associated to $f : Z \to X$ (see §5.1).

3.4.1. *Canonical involutions of the first type.* If $n = 2m \equiv 0 \bmod 4$, the even Clifford algebra gives rise, via the involutive Brauer group, to a class

$$[\widetilde{\mathscr{C}}_0(\mathscr{E}, q, \mathscr{L}), \tau_0] \in H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)$$

where $Z \to X$ is the Arf invariant of $(\mathscr{E}, q, \mathscr{L})$. If $Z \to X$ is split, then any choice of global splitting idempotent induces a decomposition of $\mathscr{O}_X$-algebras with involution $(\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}), \tau_0) \cong (\mathscr{C}_0^+(\mathscr{E}, q, \mathscr{L}), \tau_0^+) \times (\mathscr{C}_0^-(\mathscr{E}, q, \mathscr{L}), \tau_0^-)$, where $\tau_0^\pm$ are the restrictions of τ_0. Note that $\mathscr{C}_0^\pm(\mathscr{E}, q, \mathscr{L})$ are Azumaya $\mathscr{O}_X$-algebras of degree 2^{m-1}, where $n = 2m$.

Recall that there's an isomorphism of group schemes $\mathbf{z}_n^Z \cong \mathbf{R}_{Z/X}\boldsymbol{\mu}_2$ for m even, see [69, VI.26.A] for example. For Z/X split, the choice of an isomorphism $\mathbf{z}_n^Z = \mathbf{z}_{m,m} \cong \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2$ (as in §2.8) yields a fixed isomorphism $\mathbf{z}_n^Z \xrightarrow{\sim} \mathbf{R}_{Z/X}\boldsymbol{\mu}_2$. We now provide a generalization of [69, VII.31.11] to our setting, which can be viewed as an involutive refinement of the Tits algebra in the ${}^1\mathsf{D}_m$ and ${}^2\mathsf{D}_m$ cases, for m even.

Proposition 3.12. *Let* X *be a scheme with* 2 *invertible,* $f : Z \to X$ *étale quadratic, and* $n = 2m \equiv 0 \bmod 4$*. Then the coboundary map*

$$H^1_{\text{ét}}(X, \mathbf{PSO}_n^Z) \to H^2_{\text{ét}}(X, \mathbf{z}_n^Z) \cong H^2_{\text{ét}}(X, \mathbf{R}_{Z/X}\boldsymbol{\mu}_2) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)$$

has the following interpretation: a class associated to the adjoint involution of $(\mathscr{E}, q, \mathscr{L})$ *(for some choice of orientation) maps to the involutive Brauer class* $[\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}), \tau_0]$*.*

Proof. As in Remark 1.10, the natural homomorphism $\mathbf{PSO}_n^Z \to \mathbf{R}_{Z/X}\mathbf{PIso}(\widetilde{\mathscr{C}}_0(h_n^Z), \tau_0)$ induces a map

$$H^1_{\text{ét}}(X, \mathbf{PSO}_n^Z) \to H^1_{\text{ét}}(Z, \mathbf{PIso}(\mathscr{C}_0(h_n^Z), \tau_0))$$

with the following interpretation: a class associated to the adjoint involution of $(\mathscr{E}, q, \mathscr{L})$ maps to the $\mathscr{O}_Z$-isomorphism class of its even Clifford algebra $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$. This homomorphism fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{z}_n^Z & \longrightarrow & \mathbf{Spin}_n^Z & \longrightarrow & \mathbf{PSO}_n^Z & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{Iso}(\widetilde{\mathscr{C}}_0(h_n^Z), \tau_0) & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{PIso}(\widetilde{\mathscr{C}}_0(h_n^Z), \tau_0) & \longrightarrow & 1
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$ with exact rows. The proposition then follows from chasing the class of $(\mathscr{E}nd(\mathscr{E}), \sigma_q)$ around the diagram

$$\begin{array}{ccc}
H^1_{\text{ét}}(X, \mathbf{PSO}_n^Z) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{z}_n^Z) \\
\downarrow & & \downarrow \\
H^1_{\text{ét}}(Z, \mathbf{PIso}(\mathscr{C}_0(h_n^Z), \tau_0)) & \longrightarrow & H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)
\end{array}$$

of coboundary maps. The statement (and proof) of the proposition hold, more generally, for even Clifford algebras of Azumaya algebras of degree $\equiv 0 \bmod 4$ with orthogonal involution. □

Finally, we come the main result of this section. We will denote the following composition

$$H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \cong H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4) \to H^2_{\text{ét}}(X, \mathbf{R}_{Z/X}\boldsymbol{\mu}_4) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_4)$$

by φ^2, where the first isomorphism is induced from Example 2.6. Denote by $i : \boldsymbol{\mu}_2 \to \boldsymbol{\mu}_4$ the canonical homomorphism.

Theorem 3.13. *Let X be a scheme with 2 invertible, $f : Z \to X$ étale quadratic, and $n = 2m \equiv 0 \bmod 4$. Let $(\mathscr{E}, q, \mathscr{L})$ be a regular quadratic form of rank n and Arf invariant $[Z/X]$. Then we have*

$$\varphi^2 gc(\mathscr{E}, q, \mathscr{L}) = i^2[\widetilde{\mathscr{C}}_0(\mathscr{E}, q, \mathscr{L}), \tau_0] + c_1(f^*\mathscr{L}, \boldsymbol{\mu}_4)$$

in $H^2_{\text{ét}}(Z, \boldsymbol{\mu}_4)$.

Proof. This proof may be seen as a refinement of [69, VII Exer. 15a]. Let $\tilde{\boldsymbol{\kappa}}_n^Z$ be the subgroup scheme of $\mathbf{S\Gamma}_n^Z$ defined by

$$\tilde{\boldsymbol{\kappa}}_n^Z(U) = \{x \in \mathbf{S\Gamma}_n^Z(U) : N(x), r(x) \in \boldsymbol{\mu}_2(U)\}.$$

Note that $\tilde{\boldsymbol{\kappa}}_n^Z$ is the kernel in the short exact sequence of group schemes

$$1 \to \tilde{\boldsymbol{\kappa}}_n^Z \to \mathbf{S\Gamma}_n^Z \to \mathbf{GSO}_n^Z/\boldsymbol{\mu}_2 \to 1 \tag{29}$$

on $X_{\text{ét}}$. Restricting the homomorphisms N and $s = N \cdot r$ to $\tilde{\boldsymbol{\kappa}}_n^Z$ yields exact sequences

$$\begin{array}{c} 1 \to \mathbf{z}_n^Z \xrightarrow{\tilde{\imath}_{\mathbf{z}}} \tilde{\boldsymbol{\kappa}}_n^Z \xrightarrow{N} \boldsymbol{\mu}_2 \to 1 \\ 1 \to \boldsymbol{\kappa}_n^Z \xrightarrow{\tilde{\imath}_{\boldsymbol{\kappa}}} \tilde{\boldsymbol{\kappa}}_n^Z \xrightarrow{s} \boldsymbol{\mu}_2 \to 1 \end{array}$$

fitting into commutative diagrams with exact rows and columns

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbf{z} & \to & \tilde{\boldsymbol{\kappa}} & \to & \boldsymbol{\mu}_2 & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbf{Spin} & \to & \mathbf{S\Gamma} & \to & \mathbf{G}_{\mathrm{m}} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \mathbf{PSO} & \to & \mathbf{GSO}/\boldsymbol{\mu}_2 & \to & \mathbf{G}_{\mathrm{m}} & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array} \qquad \begin{array}{ccccccccc} & & & & & & 1 & & \\ & & & & & & \downarrow & & \\ & & 1 & & & & \boldsymbol{\mu}_2 & & \\ & & \downarrow & & & & \downarrow & & \\ 1 & \to & \boldsymbol{\kappa} & \to & \mathbf{S\Gamma} & \to & \mathbf{GSO} & \to & 1 \\ & & \downarrow & & \| & & \downarrow & & \\ 1 & \to & \tilde{\boldsymbol{\kappa}} & \to & \mathbf{S\Gamma} & \to & \mathbf{GSO}/\boldsymbol{\mu}_2 & \to & 1 \\ & & \downarrow & & & & \downarrow & & \\ & & \boldsymbol{\mu}_2 & & & & 1 & & \\ & & \downarrow & & & & & & \\ & & 1 & & & & & & \end{array} \tag{30}$$

of sheaves of groups on $X_{\text{ét}}$. The existence of these diagrams is guaranteed by the Nine Lemma 5.16.

Proposition 3.14. *Let X be a scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a regular $\mathscr{L}$-valued quadratic form of rank n on X. The groupoid of $\mathbf{GO}(\mathscr{E}, q, \mathscr{L})/\boldsymbol{\mu}_2$-torsors is equivalent to the category of* 2-torsion data with involution, *whose objects are tuples $(\mathscr{A}, \sigma, \mathscr{V}, b, \mathscr{M}, \varphi)$, where $(\mathscr{A}, \sigma)$ is an Azumaya $\mathscr{O}_X$-algebra of degree n with orthogonal involution, $(\mathscr{V}, b, \mathscr{M})$ is a regular $\mathscr{M}$-valued quadratic form of rank n^2, and $\varphi : (\mathscr{A}, \sigma) \otimes (\mathscr{A}, \sigma) \to (\mathscr{E}nd(\mathscr{V}), \sigma_b)$ is an isomorphism of $\mathscr{O}_X$-algebras with involution and whose morphisms between objects $(\mathscr{A}, \sigma, \mathscr{V}, b, \mathscr{M}, \varphi)$ and $(\mathscr{A}', \sigma', \mathscr{V}', b', \mathscr{M}', \varphi')$ are tuples (ψ, g, λ) consisting of an isomorphism $\psi : (\mathscr{A}, \sigma) \to (\mathscr{A}', \sigma')$ of $\mathscr{O}_X$-algebras with involution and a similarity $(g, \lambda) : (\mathscr{V}, b, \mathscr{M}) \to (\mathscr{V}', b', \mathscr{M}')$ satisfying $\varphi' \circ (\psi \otimes \psi) = \mathrm{Ad}_g \circ \varphi$.*

Proof. This is similar (but slightly more involved) to the description of $\mathbf{GL}_n/\boldsymbol{\mu}_2$-torsors in terms of 2-torsion data, see 4.2 and Auel [4]. □

The map of pointed sets $\alpha : H^1_{\text{ét}}(X, \mathbf{GO}) \to H^1_{\text{ét}}(X, \mathbf{GO}/\boldsymbol{\mu}_2)$ induced from the quotient homomorphism has the following interpretation: the class of $(\mathscr{E}, q, \mathscr{L})$ maps to the tuple $(\mathscr{E}nd(\mathscr{E}), \sigma_q, \mathscr{E} \otimes \mathscr{E}, q \otimes q, \mathscr{L}^{\otimes 2}, \varphi_{\mathscr{E}})$, where $\varphi_{\mathscr{E}} : \mathscr{E}nd(\mathscr{E}) \otimes \mathscr{E}nd(\mathscr{E}) \to \mathscr{E}nd(\mathscr{E} \otimes \mathscr{E})$ is the canonical $\mathscr{O}_X$-algebra isomorphism.

The map of pointed sets $\beta : H^1_{\text{ét}}(X, \mathbf{PGO}) \cong H^1_{\text{ét}}(X, \mathbf{PO}) \to H^1_{\text{ét}}(X, \mathbf{GO}/\boldsymbol{\mu}_2)$ has the following interpretation: the class of $(\mathscr{A}, \sigma)$ maps to the tuple $(\mathscr{A}, \sigma, \mathscr{A}, \mathrm{Trd}_{(\mathscr{A},\sigma)}, \mathscr{O}_X, \varphi_{\mathscr{A}})$, where $\mathrm{Trd}_{(\mathscr{A},\sigma)}$ is the trace form

$$\begin{aligned} \mathrm{Trd}_{(\mathscr{A},\sigma)} : \mathscr{A} \otimes \mathscr{A} &\to \mathscr{O}_X \\ a \otimes b &\mapsto \mathrm{Trd}_{\mathscr{A}}(a\,\sigma(b)). \end{aligned}$$

Consider the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & \boldsymbol{\mu}_2 & = & \boldsymbol{\mu}_2 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \to & \mathbf{G}_{\mathrm{m}} & \to & \mathbf{GO} & \to & \mathbf{PGO} & \to & 1 \\ & & \downarrow{\scriptstyle 2} & & \downarrow & & \| & & \\ 1 & \to & \mathbf{G}_{\mathrm{m}} & \to & \mathbf{GO}/\boldsymbol{\mu}_2 & \to & \mathbf{PGO} & \to & 1 \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array}$$

of sheaves of groups on $X_{\text{ét}}$. There's an induced action of $\operatorname{Pic} X = H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ on $H^1_{\text{ét}}(X, \mathbf{GO}/\boldsymbol{\mu}_2)$, which is described by

$$\mathscr{L} \cdot (\mathscr{A}, \sigma, \mathscr{V}, b, \mathscr{M}, \varphi) = (\mathscr{A}, \sigma, \mathscr{L} \otimes \mathscr{V}, \mathrm{id} \otimes b, \mathscr{L}^{\otimes 2} \otimes M, \varphi)$$

where we consider $\varphi : (\mathscr{A}, \sigma) \otimes (\mathscr{A}, \sigma) \to (\mathscr{E}nd(\mathscr{V}), \sigma_b) = (\mathscr{E}nd(\mathscr{L} \otimes \mathscr{V}), \sigma_{\mathrm{id} \otimes b})$ via the canonical identification $\mathscr{E}nd(\mathscr{L} \otimes \mathscr{V}) = \mathscr{E}nd(\mathscr{V})$.

By Remark 3.4, the map of pointed sets $H^1_{\text{ét}}(X, \mathbf{GO}) \to H^1_{\text{ét}}(X, \mathbf{PGO}) \cong H^1_{\text{ét}}(X, \mathbf{PO})$ induced from the quotient homomorphism has the following interpretation: the class $(\mathscr{E}, q, \mathscr{L})$ maps to its adjoint involution $(\mathscr{E}nd(\mathscr{E}), \sigma_q)$.

Claim 3.15. *Let $(\mathscr{E}, q, \mathscr{L})$ be a regular symmetric bilinear form. We have the following formula*

$$\beta(\mathscr{E}nd(\mathscr{E}), \sigma_q) = \mathscr{L}^\vee \cdot \alpha(\mathscr{E}, q, \mathscr{L}).$$

Proof. By a generalization of [69, II Prop. 11.4], the composition of $\mathscr{O}_X$-module isomorphisms

$$\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee \xrightarrow{\psi_q^{-1}} \mathscr{E} \otimes \mathscr{E}^\vee \xrightarrow{\text{can}} \mathscr{E}nd(\mathscr{E})$$

is an isometry of the symmetric bilinear forms

$$(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee, q \otimes q \otimes \mathrm{id}_{\mathscr{L}^\vee}, \mathscr{O}_X) \cong (\mathscr{E}nd(\mathscr{E}), \mathrm{Trd}_{(\mathscr{E}nd(\mathscr{E}), \sigma_q)}, \mathscr{O}_X),$$

where

$$q \otimes q \otimes \mathrm{id}_{\mathscr{L}^\vee} : \big(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee\big) \otimes \big(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee\big) \to \mathscr{L}^{\otimes 2} \otimes (\mathscr{L}^\vee)^{\otimes 2} \xrightarrow{\text{ev}} \mathscr{O}_X.$$

Through this isometry, the map β applied to the adjoint involution of $(\mathscr{E}, q, \mathscr{L})$ is identified with $(\mathscr{E}nd(\mathscr{E}), \sigma_q, \mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee, q \otimes q \otimes \mathrm{id}_{\mathscr{L}^\vee}, \mathscr{O}_X, \varphi_{\mathscr{E}})$, where

$$\varphi_{\mathscr{E}} : \mathscr{E}nd(\mathscr{E}) \otimes \mathscr{E}nd(\mathscr{E}) \to \mathscr{E}nd(\mathscr{E} \otimes \mathscr{E}) \cong \mathscr{E}nd(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee).$$

But this is precisely α applied to $(\mathscr{E}, q, \mathscr{L})$ scaled by the class of $\mathscr{L}^\vee$. □

We leave it as an exercise to the reader to make the adjustments to Proposition 3.14, Claim 3.15, and the above cohomological interpretations necessary to describe torsors corresponding to $\mathbf{PGSO}/\boldsymbol{\mu}_2$ (all objects and morphisms are oriented, preserving the discriminant form or center of the Clifford algebra).

Let $\widetilde{gc}(\mathscr{A}, \sigma, \mathscr{V}, b, \mathscr{M}, \varphi) \in H^2_{\text{ét}}(X, \tilde{\boldsymbol{\kappa}}_n^Z)$ be the coboundary map arising from sequence (29) applied to a choice of Z/X-orientation on the tuple $(\mathscr{A}, \sigma, \mathscr{V}, b, \mathscr{M}, \varphi)$. Similarly as in Definition 2.8, by Lemma 5.15, the invariant $\widetilde{gc}$ is independent of the choice of orientation. Considering the central rows of the right-hand diagram (30), we have $\widetilde{gc} \circ \alpha = \tilde{\imath}^2_{\boldsymbol{\kappa}} \circ gc$. Hence, by Proposition 3.12 and the commutativity two left columns of the left-hand diagram (30), we have

$$\widetilde{gc}(\mathscr{L}^\vee \cdot \alpha(\mathscr{E}, q, \mathscr{L})) = \widetilde{gc}(\beta(\mathscr{E}nd(\mathscr{E}), \sigma_q)) = \tilde{\imath}^2_{\mathbf{z}}[\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}), \tau_0] \tag{31}$$

in $H^2_{\text{ét}}(X, \tilde{\boldsymbol{\kappa}}_n^Z)$.

Now we identify the effect of scaling by $H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$ on the coboundary map of sequence (29). Restricting the homomorphism r to $\tilde{\boldsymbol{\kappa}}_n^Z$ yields an exact sequence

$$1 \to \boldsymbol{\mu}_4 \xrightarrow{\tilde{\imath}_{\boldsymbol{\mu}_4}} \tilde{\boldsymbol{\kappa}}_n^Z \xrightarrow{r} \boldsymbol{\mu}_2 \to 1$$

fitting into the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_4 & \longrightarrow & \tilde{\boldsymbol{\kappa}} & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{S\Gamma} & \longrightarrow & \mathbf{SO} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{GSO}/\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{PSO} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

of sheaves of groups on $X_{\text{ét}}$. For $\xi \in H^1_{\text{ét}}(X, \mathbf{GSO}/\boldsymbol{\mu}_2)$ and $\mathscr{L} \in H^1_{\text{ét}}(X, \mathbf{G}_{\mathrm{m}})$, we then have

$$\widetilde{gc}(\mathscr{L}^\vee \cdot \xi) = c_1(\mathscr{L}^\vee, \boldsymbol{\mu}_4) + \widetilde{gc}(\xi)$$

in $H^2_{\text{ét}}(X, \tilde{\boldsymbol{\kappa}}^Z_n)$, by an application of Lemma 5.19 to the above diagram. Applying this to (31) yields

$$\tilde{\imath}^2_{\boldsymbol{\mu}_4} c_1(\mathscr{L}^\vee, \boldsymbol{\mu}_4) + \widetilde{gc}(\alpha(\mathscr{E}, q, \mathscr{L})) = \tilde{\imath}^2_{\mathbf{z}} \, [\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}), \tau_0] \tag{32}$$

in $H^2_{\text{ét}}(X, \tilde{\boldsymbol{\kappa}}^Z_n)$. This is, in fact, a refinement of the statement of the theorem. To finally finish the proof, we construct an embedding $j : \tilde{\boldsymbol{\kappa}}^Z_n \hookrightarrow \mathbf{R}_{Z/X}\boldsymbol{\mu}_4$, so that applying $j^2 : H^2_{\text{ét}}(X, \tilde{\boldsymbol{\kappa}}^Z_n) \to H^2_{\text{ét}}(X, \mathbf{R}_{Z/X}\boldsymbol{\mu}_4) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_4)$ to (32) yields the statement of the theorem.

To this end, first note that the diagram

$$\begin{array}{ccc} \tilde{\boldsymbol{\kappa}}^Z_n & \xleftarrow{\tilde{\imath}_{\boldsymbol{\kappa}}} & \boldsymbol{\kappa}^Z_n \\ \uparrow{\scriptstyle \tilde{\imath}_{\mathbf{z}}} & & \uparrow \\ \mathbf{z}^Z_n & \longleftarrow & \boldsymbol{\mu}_2 \end{array}$$

is a pushout in the category of sheaves of abelian groups on $X_{\text{ét}}$. Indeed, the internal product of the maps $\tilde{\imath}_{\boldsymbol{\kappa}}$ and $\tilde{\imath}_{\mathbf{z}}$ fits into an exact sequence

$$1 \to \boldsymbol{\mu}_2 \xrightarrow{\Delta} \boldsymbol{\kappa}^Z_n \times \mathbf{z}^Z_n \xrightarrow{\tilde{\imath}_{\boldsymbol{\kappa}} \cdot \tilde{\imath}_{\mathbf{z}}} \tilde{\boldsymbol{\kappa}}^Z_n \to 1$$

presenting $\tilde{\boldsymbol{\kappa}}^Z_n$ as the explicit pushout of the canonical inclusions of $\boldsymbol{\mu}_2 \hookrightarrow \mathbf{S\Gamma}^Z_n$ into $\boldsymbol{\kappa}^Z_n$ and $\mathbf{z}^Z_n$. Considering the homomorphisms $\boldsymbol{\kappa}^Z_n \cong \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 \to \mathbf{R}_{Z/X}\boldsymbol{\mu}_4$ and $\mathbf{z}^Z_n \cong \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 \to \mathbf{R}_{Z/X}\boldsymbol{\mu}_4$, the universal property of the pushout supplies a unique homomorphism $j : \tilde{\boldsymbol{\kappa}}^Z_n \to \mathbf{R}_{Z/X}\boldsymbol{\mu}_4$.

Now denoting by N^2 the composition of homomorphisms $\mathbf{R}_{Z/X}\boldsymbol{\mu}_4 \xrightarrow{N} \boldsymbol{\mu}_4 \xrightarrow{2} \boldsymbol{\mu}_2$, we can complete the diagrams provided by the universal property of the pushout to the following commutative diagrams with exact rows and columns

$$\begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\kappa}^Z_n & \xrightarrow{\tilde{\imath}_{\boldsymbol{\kappa}}} & \tilde{\boldsymbol{\kappa}} & \xrightarrow{s} & \boldsymbol{\mu}_2 & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_4 & \xrightarrow{N} & \boldsymbol{\mu}_4 & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle N^2} & & \downarrow{\scriptstyle 2} & & \\ & & & & \boldsymbol{\mu}_2 & = & \boldsymbol{\mu}_2 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{z}^Z_n & \xrightarrow{\tilde{\imath}_{\mathbf{z}}} & \tilde{\boldsymbol{\kappa}} & \xrightarrow{N} & \boldsymbol{\mu}_2 & \longrightarrow & 1 \\ & & \downarrow\wr & & \downarrow{\scriptstyle j} & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_4 & \xrightarrow{2} & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & 1 \\ & & & & \downarrow{\scriptstyle N^2} & & \downarrow{\scriptstyle N} & & \\ & & & & \boldsymbol{\mu}_2 & = & \boldsymbol{\mu}_2 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array}$$

of sheaves of abelian groups on $X_{\text{ét}}$. Similarly, we have the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_4 & \xrightarrow{\tilde{\imath}_{\boldsymbol{\mu}_4}} & \tilde{\boldsymbol{\kappa}} & \xrightarrow{r} & \boldsymbol{\mu}_2 & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle j} & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_4 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_4 & \xrightarrow{\mathrm{id}/\iota} & \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle N^2} & & \downarrow{\scriptstyle 2} & & \\
 & & & & \boldsymbol{\mu}_2 & = & \boldsymbol{\mu}_2 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

of sheaves of abelian groups on $X_{\text{ét}}$, the bottom square of which commutes since $x^2\iota(x)^2 = (x^2/\iota(x)^2)\iota(x)^4 = x^2/\iota(x)^2$ for sections x of $\mathbf{R}_{Z/X}\boldsymbol{\mu}_4$, where ι is the nontrivial automorphism of $Z \to X$ (see §2.2). In particular, the compositions $j \circ \tilde{\imath}_{\boldsymbol{\kappa}}$, $j \circ \tilde{\imath}_{\mathbf{z}}$, and $j \circ \tilde{\imath}_{\boldsymbol{\mu}_4}$ are identified with the canonical inclusions of the respective subgroups into $\mathbf{R}_{Z/X}\boldsymbol{\mu}_4$. As a consequence, we have identifications

$$j^2 \circ \tilde{\imath}^2_{\boldsymbol{\kappa}} = \varphi^2, \quad j^2 \circ \tilde{\imath}^2_{\mathbf{z}} = i^2, \quad j^2 \circ \tilde{\imath}^2_{\boldsymbol{\mu}_4} = f^*$$

of the respective maps after taking $H^2_{\text{ét}}$. Finally, applying j^2 to (32) yields the statement of the theorem. □

3.4.2. *Canonical involutions of the second type.* If $n = 2m \equiv 2 \bmod 4$, the even Clifford algebra gives rise, via the Z/X-unitary involutive Brauer group, to a class

$$[\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}), \tau_0] \in H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}})$$

where $Z \to X$ is the Arf covering of $(\mathscr{E}, q, \mathscr{L})$. If $Z \to X$ is split (equivalently, $d_{\pm}(\mathscr{E}, q, \mathscr{L})$ is trivial), then any choice of global splitting idempotent induces a decomposition of $\mathscr{O}_X$-algebras $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{C}_0^+(\mathscr{E}, q, \mathscr{L}) \times \mathscr{C}_0^-(\mathscr{E}, q, \mathscr{L})$. By a globalization of Bichsel [18, §5.3] (also see [68, IV Thm. 9.2.2] or [69, I Prop. 2.14]), the canonical involution τ_0 induces an $\mathscr{O}_X$-algebra isomorphism $\mathscr{C}_0^+(\mathscr{E}, q, \mathscr{L})^{\mathrm{op}} \xrightarrow{\sim} \mathscr{C}_0^-(\mathscr{E}, q, \mathscr{L})$, and so $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is a *hyperbolic ring*.

There's an isomorphism of group schemes $\mathbf{z}_n^Z \cong \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4$ for m odd, see [69, VI.26.A]. In fact, for Z/X split, the choice of an isomorphism $\mathbf{z}_n^Z = \mathbf{z}_{m,m} \cong \boldsymbol{\mu}_4$ (as in §2.8) yields a fixed isomorphism $\mathbf{z}_n^Z \xrightarrow{\sim} \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4$ (using the method of Remark 2.6). We now provide a generalization of [69, VII.31.9] to our setting, which can be viewed as a unitary involutive refinement of the Tits algebra in the ${}^1\mathsf{D}_m$ and ${}^2\mathsf{D}_m$ cases, for m odd.

Proposition 3.16. *Let X be a scheme with 2 invertible, $f : Z \to X$ étale quadratic, and $n = 2m \equiv 2 \bmod 4$. Then the composition of maps*

$$H^1_{\text{ét}}(X, \mathbf{PSO}_n^Z) \to H^2_{\text{ét}}(X, \mathbf{z}_n^Z) \cong H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4) \to H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m)$$

has the following interpretation: a class associated to the adjoint involution of $(\mathscr{E}, q, \mathscr{L})$ maps to the Z/X-unitary involutive Brauer class $[\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}), \tau_0]$.

Proof. The argument is similar to the proof of Proposition 3.12. As in Remark 1.10, the natural homomorphism $\mathbf{PSO}_n^Z \to \mathbf{PU}(\widetilde{\mathscr{C}_0}(h_n^Z), \tau_0)$ induces a map

$$H^1_{\text{ét}}(X, \mathbf{PSO}_n^Z) \to H^1_{\text{ét}}(X, \mathbf{PU}(\mathscr{C}_0(h), \tau_0))$$

with the following interpretation: a class associated to the adjoint involution of $(\mathscr{E}, q, \mathscr{L})$ maps to the $\mathscr{O}_Z$-algebra $(\widetilde{\mathscr{C}_0}(\mathscr{E}, q, \mathscr{L}), \tau_0)$ with Z/X-unitary involution.

This homomorphism fits into a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{z}_n^Z & \longrightarrow & \mathbf{Spin}_n^Z & \longrightarrow & \mathbf{PSO}_n^Z & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{U}(\widetilde{\mathscr{C}_0}(h_n^Z), \tau_0) & \longrightarrow & \mathbf{PU}(\widetilde{\mathscr{C}_0}(h_n^Z), \tau_0) & \longrightarrow & 1
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$ with exact rows. The proposition then follows by chasing the class of $(\mathscr{E}nd(\mathscr{E}), \sigma_q)$ around the diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(X, \mathbf{PSO}_n^Z) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{z}_n^Z) \\ \downarrow & & \downarrow \\ H^1_{\text{ét}}(Z, \mathbf{PU}(\mathscr{C}_0(h_n^Z), \tau_0)) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}}) \end{array}$$

of coboundary maps. The statement (and proof) of the proposition hold, more generally, for even Clifford algebras of Azumaya algebras of degree $\equiv 2 \bmod 4$ with orthogonal involution. □

Finally, we come the main result of this section, which is analogous to, but weaker than, Theorem 3.13. We will denote the following composition

$$H^2_{\text{ét}}(X, \boldsymbol{\kappa}_n^Z) \cong H^2_{\text{ét}}(X, \mathbf{R}_{Z/X}\boldsymbol{\mu}_2) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2) \to H^2_{\text{ét}}(Z, \mathbf{G}_{\text{m}})$$

by ϕ^2, where the first isomorphism is induced from Example 2.6. Denote by $i : \boldsymbol{\mu}_2 \to \mathbf{G}_{\text{m}}$ the canonical homomorphism.

Theorem 3.17. *Let X be a scheme with 2 invertible, $f : Z \to X$ étale quadratic, and $n = 2m \equiv 2 \bmod 4$. Let $(\mathscr{E}, q, \mathscr{L})$ be a regular quadratic form of rank n and Arf invariant $[Z/X]$. Then we have*

$$\phi^2 gc(\mathscr{E}, q, \mathscr{L}) = [\widetilde{\mathscr{C}}_0(\mathscr{E}, q, \mathscr{L})]$$

in $H^2_{\text{ét}}(Z, \mathbf{G}_m)$.

Proof. The proof is similar to that of Theorem 3.13. Let $\widetilde{\mathbf{G}}_n^Z$ be the subgroup scheme of $\mathbf{S\Gamma}_n^Z$ defined by

$$\widetilde{\mathbf{G}}_n^Z(U) = \{x \in \mathbf{S\Gamma}_n^Z(U) : r(x) \in \boldsymbol{\mu}_2(U)\}.$$

Note that $\widetilde{\mathbf{G}}_n^Z$ is the kernel in the short exact sequence of group schemes

$$1 \to \widetilde{\mathbf{G}}_n^Z \to \mathbf{S\Gamma}_n^Z \to \mathbf{PSO}_n^Z \to 1 \tag{33}$$

on $X_{\text{ét}}$. Restricting the homomorphisms N and $s = N \cdot r$ to $\widetilde{\mathbf{G}}_n^Z$ yields exact sequences

$$\begin{array}{c} 1 \to \mathbf{z}_n^Z \xrightarrow{\tilde{i}_{\mathbf{z}}} \widetilde{\mathbf{G}}_n^Z \xrightarrow{N} \mathbf{G}_{\text{m}} \to 1 \\ 1 \to \boldsymbol{\kappa}_n^Z \xrightarrow{\tilde{i}_{\boldsymbol{\kappa}}} \widetilde{\mathbf{G}}_n^Z \xrightarrow{s} \mathbf{G}_{\text{m}} \to 1 \end{array}$$

fitting into commutative diagrams with exact rows and columns

$$\begin{array}{ccccccccc} & & 1 & & 1 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mathbf{z} & \longrightarrow & \widetilde{\mathbf{G}}_n^Z & \longrightarrow & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbf{Spin} & \longrightarrow & \mathbf{S\Gamma} & \longrightarrow & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & & & \\ 1 & \longrightarrow & \mathbf{PSO} & = & \mathbf{PSO} & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 1 & & 1 & & & & \end{array} \qquad \begin{array}{ccccccccc} & & & & 1 & & 1 & & \\ & & & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\kappa} & \longrightarrow & \widetilde{\mathbf{G}}_n^Z & \longrightarrow & \mathbf{G}_{\text{m}} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\kappa} & \longrightarrow & \mathbf{S\Gamma} & \longrightarrow & \mathbf{GSO} & \longrightarrow & 1 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathbf{PSO} & \xrightarrow{\sim} & \mathbf{PGSO} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 1 & & 1 & & \end{array} \tag{34}$$

of sheaves of groups on $X_{\text{ét}}$. The inclusion of the scalar torus $\mathbf{G}_{\text{m}} \to \widetilde{\mathbf{G}}_n^Z$ can be completed to an exact sequence

$$1 \to \mathbf{G}_{\text{m}} \xrightarrow{\tilde{i}_{\mathbf{G}_{\text{m}}}} \widetilde{\mathbf{G}}_n^Z \to \boldsymbol{\mu}_2 \to 1$$

which fits into the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \widetilde{\mathbf{G}}_n^Z & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{S\Gamma} & \longrightarrow & \mathbf{SO} & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathbf{PSO} & = & \mathbf{PSO} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$. The existence of these diagrams is guaranteed by the Nine Lemma 5.16.

Let $t : H^1_{\text{ét}}(X, \mathbf{PSO}_n^Z) \cong H^1_{\text{ét}}(X, \mathbf{PGO}_n^Z) \to H^2_{\text{ét}}(X, \mathbf{z}_n^Z)$ be the coboundary map of (18). We will write $t(\mathscr{E}, q, \mathscr{L}) = t(\mathscr{E}nd(\mathscr{E}), \sigma_q)$. Now chasing the class of $(\mathscr{E}nd(\mathscr{E}), \sigma_q)$ through the upper rows of the right-hand diagram and the left-most columns of the left-hand diagram (34), we have

$$\tilde{\imath}^2_{\boldsymbol{\kappa}}\, gc(\mathscr{E}, q, \mathscr{L}) = \tilde{\imath}^2_{\mathbf{z}}\, t(\mathscr{E}, q, \mathscr{L}) \tag{35}$$

in $H^2_{\text{ét}}(X, \widetilde{\mathbf{G}}_n^Z)$. This is, in fact, a refinement of the statement of the theorem. To finally finish the proof, we construct an embedding $j : \widetilde{\mathbf{G}}_n^Z \to \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}}$, so that applying $j^2 : H^2_{\text{ét}}(X, \widetilde{\mathbf{G}}_n^Z) \to H^2_{\text{ét}}(X, \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}}) \cong H^2_{\text{ét}}(Z, \mathbf{G}_{\mathrm{m}})$ to (35) yields the statement of the theorem.

To this end, first note that the diagram

$$\begin{array}{ccc}
\widetilde{\mathbf{G}}_n^Z & \xleftarrow{\tilde{\imath}_{\boldsymbol{\kappa}}} & \boldsymbol{\kappa}_n^Z \\
{\scriptstyle \tilde{\imath}_{\mathbf{G}_{\mathrm{m}}}}\uparrow & & \uparrow \\
\mathbf{G}_{\mathrm{m}} & \longleftarrow & \boldsymbol{\mu}_2
\end{array}$$

is a pushout in the category of sheaves of abelian groups on $X_{\text{ét}}$. Indeed, the internal product of the maps $\tilde{\imath}_{\boldsymbol{\kappa}}$ and $\tilde{\imath}_{\mathbf{G}_{\mathrm{m}}}$ fits into an exact sequence

$$1 \to \boldsymbol{\mu}_2 \xrightarrow{\Delta} \boldsymbol{\kappa}_n^Z \times \mathbf{G}_{\mathrm{m}} \xrightarrow{\tilde{\imath}_{\boldsymbol{\kappa}} \cdot \tilde{\imath}_{\mathbf{G}_{\mathrm{m}}}} \widetilde{\mathbf{G}}_n^Z \to 1$$

presenting $\widetilde{\mathbf{G}}_n^Z$ as the explicit pushout of the canonical inclusions of $\boldsymbol{\mu}_2 \hookrightarrow \mathbf{S\Gamma}_n^Z$ into $\boldsymbol{\kappa}_n^Z$ and $\mathbf{G}_{\mathrm{m}}$. Considering the homomorphisms $\boldsymbol{\kappa}_n^Z \cong \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 \to \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}}$ and $\mathbf{G}_{\mathrm{m}} \to \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}}$, the universal property of the pushout supplies a unique homomorphism $j : \widetilde{\mathbf{G}}_n^Z \to \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}}$.

Now denoting by $(\mathrm{id}/\iota)^2$ the composition of homomorphisms $\mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} \xrightarrow{\mathrm{id}/\iota} \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} \xrightarrow{2} \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}$, we can complete the diagrams provided by the universal property of the pushout to the following commutative diagrams with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\kappa}_n^Z & \xrightarrow{\tilde{\imath}_{\boldsymbol{\kappa}}} & \widetilde{\mathbf{G}}_n^Z & \xrightarrow{s} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \wr} & & \downarrow{\scriptstyle j} & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \xrightarrow{2} & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle (\mathrm{id}/\iota)^2} & & \downarrow{\scriptstyle \mathrm{id}/\iota} & & \\
 & & & & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & = & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \xrightarrow{\tilde{\imath}_{\mathbf{G}_{\mathrm{m}}}} & \widetilde{\mathbf{G}}_n^Z & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & 1 \\
 & & \| & & \downarrow{\scriptstyle j} & & \downarrow & & \\
1 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \xrightarrow{\mathrm{id}/\iota} & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & & & \downarrow{\scriptstyle (\mathrm{id}/\iota)^2} & & \downarrow{\scriptstyle 2} & & \\
 & & & & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & = & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

of sheaves of abelian groups on $X_{\text{ét}}$. Similarly, we have the commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbf{z}_n^Z & \xrightarrow{\tilde{\imath}_{\mathbf{z}}} & \widetilde{\mathbf{G}}_n^Z & \xrightarrow{N} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \downarrow\wr & & & & \Vert & & \\
 & & \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & & \downarrow j & & \Vert & & \\
 & & \downarrow & & & & \Vert & & \\
1 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \xrightarrow{N} & \mathbf{G}_{\mathrm{m}} & & \\
 & & \downarrow & & \downarrow (\mathrm{id}/\iota)^2 & & & & \\
 & & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & = & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}
$$

of sheaves of abelian groups on $X_{\text{ét}}$. In particular, the compositions $j \circ \tilde{\imath}_{\boldsymbol{\kappa}}$, $j \circ \tilde{\imath}_{\mathbf{G}_{\mathrm{m}}}$, and $j \circ \tilde{\imath}_{\mathbf{z}}$ are identified with the canonical inclusions of the respective subgroups into $\mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}}$. As a consequence, we have identifications

$$j^2 \circ \tilde{\imath}^2_{\boldsymbol{\kappa}} = \phi^2, \quad j^2 \circ \tilde{\imath}^2_{\mathbf{G}_{\mathrm{m}}} = f^*, \quad j^2 \circ \tilde{\imath}^2_{\mathbf{z}} = \phi^2_{\mathbf{z}}$$

of the respective maps after taking $H^2_{\text{ét}}$. Here, $\phi^2_{\mathbf{z}}$ is the composition of maps $H^2_{\text{ét}}(X, \mathbf{z}_n^Z) \cong H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4) \to H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}})$ from Proposition 3.16, which satisfies $\phi^2_{\mathbf{z}}\, t(\mathscr{E}, q, \mathscr{L}) = [\widetilde{\mathscr{C}_0}(\mathscr{E}, q, \mathscr{L}), \tau_0]$ in $H^2_{\text{ét}}(Z, \mathbf{G}_{\mathrm{m}})$. Finally, applying j^2 to (35) yields the statement of the theorem. □

4. Relationship to the Clifford bimodule

As usual, let X be a noetherian separated scheme with 2 invertible and let $\mathscr{L}$ be a fixed invertible $\mathscr{O}_X$-module. While the results in §3 comparing the (refined) Tits algebra to (involutive Brauer classes associated to) the even Clifford algebra were quite neat, the results comparing the similarity Clifford invariant to the even Clifford algebra were less so. In this section, we shall see that the similarity Clifford invariant much more elegantly reflects involutive structures arising from the Clifford bimodule.

4.1. **Clifford bimodule.** As in the case of central simple algebras with orthogonal involution, line bundle-valued quadratic forms do not generally enjoy a "full" Clifford algebra, of which the even Clifford algebra (see §1.8) is the even degree part. As a replacement, algebras with orthogonal involution have a Clifford bimodule (see [69, II §9]) while line bundle-valued quadratic forms enjoy a refinement of this notion of Clifford bimodule. If $(\mathscr{E}, q, \mathscr{L})$ is a regular line bundle-valued quadratic form, then there exists a locally free $\mathscr{O}_X$-module $\mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$ with the structure of an invertible $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$-bimodule. The corresponding Clifford bimodule defined in [69] is then isomorphic to $\mathscr{E} \otimes \mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$.

Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form on X and $\mathscr{C}'(\mathscr{E}, q, \mathscr{L})$ its generalized Clifford algebra (see §1.8). Define the *Clifford bimodule* $\mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$ of $(\mathscr{E}, q, \mathscr{L})$ to be the degree 1 submodule of $\mathscr{C}'(\mathscr{E}, q, \mathscr{L})$. It's also possible to construct the Clifford bimodule via a direct tensorial construction or by gluing the odd degree parts of Clifford algebras over a Zariski cover trivializing $\mathscr{L}$, as in §1.8.

Functorial properties. The Clifford bimodule has the following functorial properties (assuming that $(\mathscr{E}, q, \mathscr{L})$ is a regular quadratic form of rank n on X):

a) There is a canonical embedding of locally free $\mathscr{O}_X$-modules

$$i : \mathscr{E} \to \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}).$$

b) Via multiplication in the generalized Clifford algebra, $\mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$ is an invertible $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$-bimodule and there's a canonical isomorphism

$$m : \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \otimes_{\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})} \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \xrightarrow{\sim} \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \otimes_{\mathscr{O}_X} \mathscr{L}$$

of $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$-bimodules (with trivial action on $\mathscr{L}$) satisfying

$$m(i(v) \otimes i(v)) = 1 \otimes q(v)$$

for a section v of $\mathscr{E}$. See [18, §2] or [17, Lemma 3.1].

c) Any similarity transformation $(\varphi, \lambda) : (\mathscr{E}, q, \mathscr{L}) \to (\mathscr{E}', q', \mathscr{L}')$ induces an $\mathscr{O}_X$-module isomorphism
$$\mathscr{C}_1(\varphi, \lambda) : \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \to \mathscr{C}_1(\mathscr{E}', q', \mathscr{L}').$$
that is $\mathscr{C}_0(\varphi, \lambda)$-semilinear according to the diagram
$$\begin{array}{ccc} \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \otimes_{\mathscr{C}_0(\mathscr{E},q,\mathscr{L})} \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) & \xrightarrow{m} & \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \otimes_{\mathscr{O}_X} \mathscr{L} \\ \downarrow {\scriptstyle \mathscr{C}_1(\varphi,\lambda)\otimes\mathscr{C}_1(\varphi,\lambda)} & & \downarrow {\scriptstyle \mathscr{C}_0(\varphi,\lambda)\otimes\lambda} \\ \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \otimes_{\mathscr{C}_0(\mathscr{E},q,\mathscr{L})} \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) & \xrightarrow{m} & \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \otimes_{\mathscr{O}_X} \mathscr{L} \end{array}$$
See [18, Prop. 2.6].

d) If $n = 2m$ is even, and $\mathscr{Z}$ is the center of $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ then $\mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$ is a $\mathscr{Z}$-bimodule satisfying $x \cdot z = \iota(z) \cdot x$ for sections z of $\mathscr{Z}$ and x of $\mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$, where ι is the nontrivial element of $\mathrm{Gal}(\mathscr{Z}/\mathscr{O}_X)$. See [68, IV Prop. 4.3.1(4)].

e) There is an $\mathscr{O}_X$-module isomorphism $\tau_1 : \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \to \mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$ of order 2 satisfying
$$\tau_1(axb) = \tau_0(b)\tau_1(x)\tau_0(a), \quad (\tau_0 \otimes \mathrm{id}_{\mathscr{L}})(m(x \otimes y)) = m(\tau_1(y) \otimes \tau_1(x)), \quad \tau_1(i(v)) = v$$
for sections a, b of $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$, x, y of $\mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$, and v of $\mathscr{E}$.

f) Any regular bilinear form $(\mathscr{N}, n, \mathscr{N}^{\otimes 2})$ of rank 1, induces an $\mathscr{O}_X$-algebra isomorphism
$$\mathscr{C}_1(n \otimes \mathrm{id}_{\mathscr{E}}) : \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \otimes \mathscr{N} \to \mathscr{C}_1(\mathscr{N} \otimes \mathscr{E}, n \otimes q, \mathscr{N}^{\otimes 2} \otimes \mathscr{L}).$$

g) For any morphism of schemes $g : X' \to X$, there's a canonical $\mathscr{O}_X$-module isomorphism
$$g^*\mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \xrightarrow{\sim} \mathscr{C}_1(g^*(\mathscr{E}, q, \mathscr{L})).$$

When a regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank n is fixed, we will write $\mathscr{C}_0$ for the even Clifford algebra (as an $\mathscr{O}_X$-algebra) and $\mathscr{C}_1$ for the Clifford bimodule (as an $\mathscr{O}_X$-module). When n is even, write $Z = \mathbf{Spec}\, \mathscr{Z}$ with structure morphism $f : Z \to X$, and denote by $\widetilde{\mathscr{C}}_0$ and $\widetilde{\mathscr{C}}_1$ the associated $\mathscr{O}_Z$-algebra (see §1.8) and $\widetilde{\mathscr{C}}_0$-bimodule, respectively. Note that $\widetilde{\mathscr{C}}_1$ has a left and right $\mathscr{O}_Z$-module structure, which are interchanged by twisting by ι.

Given $\mathscr{O}_Z$-bimodules $\mathscr{P}$ and $\mathscr{P}'$, take care that $\mathscr{P} \otimes_{\mathscr{O}_Z} \mathscr{P}'$ stands for the tensor product with respect to the right $\mathscr{O}_Z$-action of $\mathscr{P}$ and the left $\mathscr{O}_Z$-action of $\mathscr{P}'$, and is again an $\mathscr{O}_Z$-bimodule.

Given an étale quadratic $f : Z \to X$ with Galois group generated by ι and an $\mathscr{O}_Z$-module $\mathscr{B}$, consider the naïve "switch" morphism $s_{\mathscr{B}} : \mathscr{B} \otimes_{\mathscr{O}_Z} \mathscr{B} \to \mathscr{B} \otimes_{\mathscr{O}_Z} \mathscr{B}$ and the ι-semilinear "switch" morphism
$$\begin{aligned} s^{\iota}_{\mathscr{B}} : \mathscr{B} \otimes_{\mathscr{O}_Z} \iota_*\mathscr{B} &\to \iota_*(\mathscr{B} \otimes_{\mathscr{O}_Z} \iota_*\mathscr{B}) \\ a \otimes \iota_* b &\mapsto \iota_*(b \otimes \iota_* a). \end{aligned}$$
Furthermore, if $\beta : \mathscr{B} \to \mathscr{B}$ is an $\mathscr{O}_Z$-module isomorphism, denote by $\mathrm{Ad}_\beta : \mathscr{E}nd(\mathscr{B}) \to \mathscr{E}nd(\mathscr{B})$ the induced $\mathscr{O}_Z$-algebra isomorphism $\varphi \mapsto \tau \circ \varphi \circ \tau^{-1}$; if β is a ι-semilinear isomorphism $\mathscr{B} \to \iota_*\mathscr{B}'$, denote by
$$\mathrm{Ad}^{\iota}_\beta : \mathscr{E}nd_{\mathscr{O}_Z}(\mathscr{B}) \xrightarrow{\mathrm{Ad}_\beta} \mathscr{E}nd_{\mathscr{O}_Z}(\iota_*\mathscr{B}') \cong \iota_*\, \mathscr{E}nd(\mathscr{B}')$$
the induced ι-semilinear $\mathscr{O}_Z$-algebra automorphism.

4.2. **Torsion data.** The notion of a torsion datum on an Azumaya algebra is a way of rigidifying it's realizability as a torsion class in the Brauer group. This has many benefits when working in the groups $H^2_{\text{ét}}(X, \boldsymbol{\mu}_n)$ as opposed to ${}_n\mathrm{Br}(X)$. A torsion datum should also be thought of as a "preinvolution."

2-torsion data. Consider the cokernel of the exact sequence of sheaves of groups
$$1 \to \boldsymbol{\mu}_2 \to \mathbf{GL}_n \to \mathbf{GL}_n/\boldsymbol{\mu}_2 \to 1, \tag{36}$$
on $X_{\text{ét}}$. Since 2 is invertible on X, the sheaf of groups $\mathbf{GL}_n/\boldsymbol{\mu}_2$ is representable by a smooth affine reductive group scheme on X. Note that $\mathbf{GL}_1/\boldsymbol{\mu}_2 \cong \mathbf{G}_{\mathrm{m}}$ via the Kummer sequence, but that in general, $\mathbf{GL}_n/\boldsymbol{\mu}_2$ is not isomorphic to $\mathbf{GL}_n$ for $n \geq 2$. The category of $\mathbf{GL}_n/\boldsymbol{\mu}_2$-torsors on $X_{\text{ét}}$ is equivalent to the category of *2-torsion data*, see Knus [68, III §9.3].

Definition 4.1. A *2-torsion datum* of degree n on X is a triple $(\mathscr{A}, \mathscr{P}, \varphi)$, consisting of an Azumaya $\mathscr{O}_X$-algebra $\mathscr{A}$ of degree n, a locally free $\mathscr{O}_X$-module $\mathscr{P}$ of rank n^2, and an $\mathscr{O}_X$-algebra isomorphism $\varphi : \mathscr{A} \otimes \mathscr{A} \xrightarrow{\sim} \mathscr{E}nd(\mathscr{P})$. In particular, the class of $\mathscr{A}$ in the Brauer group has period ≤ 2. A morphism of 2-torsion data is a pair $(\psi, \beta) : (\mathscr{A}, \mathscr{P}, \varphi) \to (\mathscr{A}', \mathscr{P}', \varphi')$, where $\psi : \mathscr{A} \to \mathscr{A}'$ is an $\mathscr{O}_X$-algebra isomorphism and $\beta : \mathscr{P} \to \mathscr{P}'$ is an $\mathscr{O}_X$-module isomorphism satisfying $\varphi' \circ (\psi \otimes \psi) = \mathrm{Ad}_\beta \circ \varphi$.

To every locally free $\mathscr{O}_X$-module $\mathscr{V}$ of rank n, we associate a *split datum* $(\mathscr{E}nd(\mathscr{V}), \mathscr{V}\otimes\mathscr{V}, \varphi_{\mathscr{V}})$ of degree n, where

$$\varphi_{\mathscr{V}} : \mathscr{E}nd(\mathscr{V})\otimes\mathscr{E}nd(\mathscr{V}) \xrightarrow{\sim} \mathscr{E}nd(\mathscr{V}\otimes\mathscr{V})$$

is the canonical $\mathscr{O}_X$-algebra isomorphism.

Proposition 4.2. *Let X be a scheme with 2 invertible. The category of 2-torsion data of degree n on X is equivalent to the category of $\mathbf{GL}_n/\boldsymbol{\mu}_2$-torsors for the étale topology.*

Proof. The proof involves completing the exercise [69, VII Exer. 5] on $X_{\text{ét}}$. See Auel [4]. □

The interpretation of exact sequence (36) on cohomology

$$H^1_{\text{ét}}(X, \mathbf{GL}_n) \to H^1_{\text{ét}}(X, \mathbf{GL}_n/\boldsymbol{\mu}_2) \to H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$$

is as follows: the class associated to a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of rank n maps to the class of the associated split datum $(\mathscr{E}nd(\mathscr{V}), \mathscr{V}\otimes\mathscr{V}, \varphi_{\mathscr{V}})$; the class of 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ maps (by the 2nd coboundary) to a class $[\mathscr{A}, \mathscr{P}, \varphi] \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$, which we call the *involutive class* associated to $(\mathscr{A}, \mathscr{P}, \varphi)$, and which we shall now describe.

Given a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$, Knus–Parimala–Srinivas [67, Thm. 4.1] construct a canonical involution σ_φ of the first kind (in fact, of orthogonal type) on the $\mathscr{O}_X$-algebra $\mathscr{E}nd_{\mathscr{A}}(\mathscr{A}\oplus\mathscr{P})$, hence an involutive Brauer class.

Lemma 4.3. *We have an equality of classes $[\mathscr{A}, \mathscr{P}, \varphi] = \Psi[\mathscr{E}nd_{\mathscr{A}}(\mathscr{A}\oplus\mathscr{P}), \sigma_\varphi]$ in $H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$.*

Proof. Restricting the homomorphism H from the proof of Theorem 2.11 to $\mathbf{GL}_n$ yields $\mathbf{GL}_n \to \mathbf{O}_{n,n}$ fitting into a morphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{GL}_n & \longrightarrow & \mathbf{GL}_n/\boldsymbol{\mu}_2 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{O}_{n,n} & \longrightarrow & \mathbf{PO}_{n,n} & \longrightarrow & 1 \end{array}$$

of sheaves of groups on $X_{\text{ét}}$. The map $H^1_{\text{ét}}(X, \mathbf{GL}_n/\boldsymbol{\mu}_2) \to H^1_{\text{ét}}(X, \mathbf{PO}_{n,n})$ has the following interpretation: the class associated to a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of rank n is mapped to the class of the $\mathscr{O}_X$-algebra with orthogonal involution $(\mathscr{E}nd(\mathscr{A}\oplus\mathscr{P}), \sigma_\varphi)$. This is stated, for example, in [69, VII Exer. 6]. A comparison of the coboundary maps finishes the proof. □

Z/X-torsion data. Now, let $f : Z \to X$ be étale quadratic. Via the central inclusion $\mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} \to \mathbf{R}_{Z/X}\mathbf{GL}_n$, consider the cokernel of the exact sequence of sheaves of groups

$$(37) \qquad 1 \to \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} \to \mathbf{R}_{Z/X}\mathbf{GL}_n \to \mathbf{R}_{Z/X}\mathbf{GL}_n/\mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} \to 1,$$

on $X_{\text{ét}}$. Note that $\mathbf{R}_{Z/X}\mathbf{GL}_1/\mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} \cong \mathbf{G}_{\mathrm{m}}$ via the norm map.

Recall the definition of the *norm* or *corestriction* $\mathscr{O}_X$-module (resp. algebra) $N_{Z/X}(\mathscr{A})$ of an $\mathscr{O}_Z$-module (resp. algebra) $\mathscr{A}$. The norm is a functor from the category of $\mathscr{O}_Z$-modules (resp. algebras) to the category of $\mathscr{O}_X$-modules (resp. algebras).

Definition 4.4 ([69, VII Exer. 7])**.** A *Z/X-torsion datum* of degree n on X is a triple $(\mathscr{A}, \mathscr{P}, \varphi)$, consisting of an Azumaya $\mathscr{O}_Z$-algebra $\mathscr{A}$ of degree n, a locally free $\mathscr{O}_X$-module $\mathscr{P}$ of rank n^2, and an $\mathscr{O}_X$-algebra isomorphism $\varphi : N_{Z/X}(\mathscr{A}) \xrightarrow{\sim} \mathscr{E}nd(\mathscr{P})$. A morphism of Z/X-torsion data is a pair $(\psi, \beta) : (\mathscr{A}, \mathscr{P}, \varphi) \to (\mathscr{A}', \mathscr{P}', \varphi')$, where $\psi : \mathscr{A} \to \mathscr{A}'$ is an $\mathscr{O}_Z$-algebra isomorphism and $\beta : \mathscr{P} \to \mathscr{P}'$ is an $\mathscr{O}_X$-module isomorphism satisfying $\varphi' \circ N_{Z/X}(\psi) = \mathrm{Ad}_\beta \circ \varphi$.

To every locally free $\mathscr{O}_Z$-module $\mathscr{V}$ of rank n, we associate a *split datum* $(\mathscr{E}nd(\mathscr{V}), N_{Z/X}(\mathscr{V}), \varphi_{\mathscr{V}})$, where

$$\varphi_{\mathscr{V}} : N_{Z/X}(\mathscr{E}nd(\mathscr{V})) \xrightarrow{\sim} \mathscr{E}nd(N_{Z/X}(\mathscr{V}))$$

is the canonical $\mathscr{O}_X$-algebra isomorphism.

Proposition 4.5. *Let X be a scheme with 2 invertible and $f : Z \to X$ étale quadratic. The category of Z/X-torsion data of degree n is equivalent to the category of $\mathbf{R}_{Z/X}\mathbf{GL}_n/\mathbf{R}^1_{Z/X}\mathbf{G}_m$-torsors on $X_{\text{ét}}$.*

Proof. The proof involves completing the exercise [69, VII Exer. 7] on $X_{\text{ét}}$. See Auel [4]. □

The interpretation of exact sequence (37) on cohomology

$$H^1_{\text{ét}}(Z, \mathbf{GL}_n) \to H^1_{\text{ét}}(X, \mathbf{R}_{Z/X}\mathbf{GL}_n/\mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}) \to H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}})$$

is as follows: the class associated to a locally free $\mathscr{O}_X$-module $\mathscr{V}$ of rank n maps to the class of the associated split datum $(\mathscr{E}nd(\mathscr{V}), \mathscr{V}\otimes\mathscr{V}, \varphi_{\mathscr{V}})$; the class of 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ maps (by the 2nd coboundary) to a class $[\mathscr{A},\mathscr{P},\varphi] \in H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}})$, which we call the *involutive class* associated to $(\mathscr{A},\mathscr{P},\varphi)$, and which we shall now describe.

Given a Z/X-torsion datum $(\mathscr{A},\mathscr{P},\varphi)$, Knus–Parimala–Srinivas [67, Thm. 4.2] construct a canonical Z/X-unitary involution σ_φ on the $\mathscr{O}_Z$-algebra $\mathscr{E}nd_{\mathscr{A}}(\mathscr{A}\oplus\mathscr{P})$, hence a Z/X-unitary involutive Brauer class.

Lemma 4.6. *We have an equality of classes* $[\mathscr{A},\mathscr{P},\varphi] = \Psi_{Z/X}[\mathscr{E}nd_{\mathscr{A}}(\mathscr{A}\oplus\mathscr{P}),\sigma_\varphi]$ *in* $H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m)$.

Proof. A Z/X-hermitian variant of the homomorphism H from the proof of Theorem 2.11 gives rise to $\mathbf{R}_{Z/X}\mathbf{GL}_n \to \mathbf{U}_{n,n}$, which fits into a morphism of short exact sequences

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{GL}_n & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{GL}_n/\mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}} & \longrightarrow & \mathbf{U}_{n,n} & \longrightarrow & \mathbf{PU}_{n,n} & \longrightarrow & 1
\end{array}$$

of sheaves of groups on $X_{\text{ét}}$, where $\mathbf{U}_{n,n}$ denotes the unitary group of the rank $2n$ hyperbolic Z/X-hermitian form. The map $H^1_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{GL}_n/\mathbf{R}^1_{Z/X}\mathbf{G}_{\text{m}}) \to H^1_{\text{ét}}(X, \mathbf{PU}_{n,n})$ has the following interpretation: the class associated to a Z/X-torsion datum $(\mathscr{A},\mathscr{P},\varphi)$ of rank n is mapped to the class of the $\mathscr{O}_Z$-algebra with Z/X-unitary involution $(\mathscr{E}nd(\mathscr{A}\oplus\mathscr{P}),\sigma_\varphi)$. This is stated, for example, in [69, VII Exer. 7]. A comparison of the coboundary maps finishes the proof. □

Remark 4.7. Any 2-torsion datum $(\mathscr{A}_0,\mathscr{P}_0,\varphi_0)$ on X defines a Z/X-torsion datum $(f^*\mathscr{A}_0,\mathscr{P}_0,\varphi_0)$ with respect to an isomorphism $N_{Z/X}(f^*\mathscr{A}_0) \cong \mathscr{A}_0\otimes\mathscr{A}_0$. Conversely, it is *not* true that a Z/X-torsion datum $(\mathscr{A},\mathscr{P},\varphi)$ with $\mathscr{A} \cong f^*\mathscr{A}_0$ is equivalent to a 2-torsion datum on X, see Knus–Parimala–Srinivas [67].

4.3. **Clifford data.** Axiomatizing the properties of the even Clifford algebra and Clifford bimodule of a line bundle-valued quadratic form lead to the notion of a *Clifford datum*, a mixture of having an involution and a torsion datum.

Definition 4.8. Let $n = 2m$ be even. A *Clifford datum* of rank n consists of a tuple $(Z/X,\mathscr{A},\sigma,\mathscr{P},\varphi)$, where:

- *(if m is odd)* $Z \to X$ is étale quadratic, $(\mathscr{A},\sigma)$ is an Azumaya $\mathscr{O}_Z$-algebra of degree 2^{m-1} with Z/X-unitary involution, and $(\mathscr{A},\mathscr{P},\varphi)$ is a 2-torsion datum of degree 2^{m-1} on Z
- *(if m is even)* $Z \to X$ is étale quadratic, $(\mathscr{A},\sigma)$ is an Azumaya $\mathscr{O}_Z$-algebra of degree 2^{m-1} with involution of the first kind of type $(-1)^{m/2}$, and $(\mathscr{A},\mathscr{P},\varphi)$ is a Z/X-torsion datum of degree 2^{m-1}.

A morphism of Clifford data of rank $n = 2m$ is a tuple $(g,\psi,\beta) : (Z/X,\mathscr{A},\sigma,\mathscr{P},\varphi) \to (Z'/X,\mathscr{A}',\sigma',\mathscr{P}',\varphi')$, where $g : Z \to Z'$ is an X-isomorphism, $\psi : (\mathscr{A},\sigma) \to g^*(\mathscr{A}',\sigma')$ is an $\mathscr{O}_Z$-algebra isomorphism preserving the involutions, and $(\psi,\beta) : (\mathscr{A},\mathscr{P},\varphi) \to g^*(\mathscr{A}',\mathscr{P}',\varphi')$ is a morphism of torsion data.

Given a regular line bundle-valued quadratic form $(\mathscr{E},q,\mathscr{L})$ of rank $n = 2m$ and Arf covering $f : Z \to X$, we construct an associated Clifford datum $(Z/X,\widetilde{\mathscr{C}}_0,\tau_0,\widetilde{\mathscr{B}}_1,\nu)$ of rank n as follows:

For m odd, by Proposition 3.11, $(\widetilde{\mathscr{C}}_0,\tau_0)$ is an Azumaya $\mathscr{O}_Z$-algebra with Z/X-unitary Clifford involution; let $\widetilde{\mathscr{B}}_1 = \widetilde{\mathscr{C}}_1$, then by the functorial properties in §4.1, $\widetilde{\mathscr{B}}_1$ is an invertible $\widetilde{\mathscr{C}}_0$-bimodule (in particular is locally free of rank 2^{n-2}) with an $\mathscr{O}_Z$-module automorphism $\tau_1 : \widetilde{\mathscr{B}}_1 \to \widetilde{\mathscr{B}}_1$. The $\widetilde{\mathscr{C}}_0$-bimodule structure on $\widetilde{\mathscr{B}}_1$ yields a canonical map

$$\begin{aligned}
\nu : \widetilde{\mathscr{C}}_0 \otimes_{\mathscr{O}_Z} \widetilde{\mathscr{C}}_0 &\to \mathscr{E}nd_{\mathscr{O}_Z}(\widetilde{\mathscr{B}}_1) \\
a\otimes b &\mapsto x \mapsto \tau_1(b\,\tau_1(ax)) = \text{“}a\,x\,\tau_0(b)\text{”}
\end{aligned}$$

which is an $\mathscr{O}_Z$-algebra isomorphism (since its a nontrivial homomorphism between Azumaya algebras of the same degree). Thus $(Z/X,\widetilde{\mathscr{C}}_0,\tau_0,\widetilde{\mathscr{B}}_1,\nu)$ is a 2-torsion datum on Z of degree n.

For m even, by Proposition 3.11, $(\widetilde{\mathscr{C}}_0,\tau_0)$ is an Azumaya $\mathscr{O}_Z$-algebra with Clifford involution of the first kind of type $(-1)^{m/2}$; by the functorial properties in §4.1, $\widetilde{\mathscr{C}}_1$ is an invertible $\widetilde{\mathscr{C}}_0$-bimodule (in particular is locally free of rank 2^{n-2}) with an (ι-semilinear) $\mathscr{O}_Z$-module automorphism $\tau_1 : \widetilde{\mathscr{C}}_1 \to \iota_*\widetilde{\mathscr{C}}_1$. The $\widetilde{\mathscr{C}}_0$-bimodule structure on $\widetilde{\mathscr{C}}_1$ yields a canonical map

$$\begin{aligned}
\widetilde{\nu} : \widetilde{\mathscr{C}}_0 \otimes_{\mathscr{O}_Z} \iota_*\widetilde{\mathscr{C}}_0 &\to \mathscr{E}nd_{\mathscr{O}_Z}(\widetilde{\mathscr{C}}_1) \\
a\otimes \iota_* b &\mapsto x \mapsto \tau_1(\iota_* b\,\tau_1(ax)) = \text{“}a\,x\,\tau_0(b)\text{”}
\end{aligned}$$

which is an $\mathscr{O}_Z$-algebra isomorphism (since its a nontrivial homomorphism between Azumaya algebras of the same degree). The following compatibility condition

$$\iota_*\nu \circ s^{\iota}_{\widetilde{\mathscr{C}}_0} = \mathrm{Ad}^{\iota}_{\tau_1} \circ \widetilde{\nu}$$

holds, ensuring that $\widetilde{\nu}$ descends to an $\mathscr{O}_X$-algebra isomorphism

$$\nu : N_{Z/X}(\widetilde{\mathscr{C}}_0) \to \mathscr{E}nd_{\mathscr{O}_X}(\mathscr{S}ym(\widetilde{\mathscr{C}}_1, \tau_1))$$

where $\mathscr{S}ym(\widetilde{\mathscr{C}}_1, \tau_1))$ is the descent of the $\mathscr{O}_Z$-submodule of $\widetilde{\mathscr{C}}_1$ of elements fixed by τ_1, see [68, V Lemma 4.2.4] (and its preceding discussion) or (the Claim contained in the proof of) [69, II Thm. 9.12]. Thus, letting $\widetilde{\mathscr{B}}_1 = \mathscr{S}ym(\widetilde{\mathscr{C}}_1, \tau_1)$, the tuple $(Z/X, \widetilde{\mathscr{C}}_0, \tau_0, \widetilde{\mathscr{B}}_1, \nu)$ is a Z/X-torsion datum of degree n.

4.4. **Clifford invariants and Clifford data.** In this section, we provide some general interpretations of the similarity Clifford invariant in terms of Clifford data. Our main result of this section shows that the similarity Clifford invariant naturally captures the "torsion data" part of the Clifford data of a regular line bundle-valued quadratic form, while the results of §3.4 show that (the refinement of) the Tits algebra naturally captures the "involution" part. Recall from §4.2 that a torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ has an associated involutive class $[\mathscr{A}, \mathscr{P}, \varphi]$.

Theorem 4.9. *Let X be a scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a regular line bundle-valued quadratic form of even rank $n = 2m$ and with Arf cover $f : Z \to X$. We have:*

a) if m is odd then $gc(\mathscr{E}, q, \mathscr{L}) = [\widetilde{\mathscr{C}}_0, \widetilde{\mathscr{B}}_1, \nu]$ in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}^Z_n) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)$

b) if m is even then $i^2 gc(\mathscr{E}, q, \mathscr{L}) = [\widetilde{\mathscr{C}}_0, \widetilde{\mathscr{B}}_1, \nu]$ in $H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m)$

where $i : \boldsymbol{\kappa}^Z_n \cong \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 \to \mathbf{R}^1_{Z/X}\mathbf{G}_m$.

Proof. We have a graded isomorphism of (full) Clifford $\mathscr{O}_X$-algebras

$$\mathscr{C}(h^Z_n) \cong \mathscr{C}(h_{n-2}) \widehat{\otimes} \mathscr{C}(h^Z_2) \cong \mathscr{M}_{2^{m-1}}(\mathscr{O}_X) \widehat{\otimes}\, \mathscr{E}nd_{\mathscr{O}_X}(f_*\mathscr{O}_Z)$$

see [68, IV Prop. 2.11, V §2.3], in particular, we have an $\mathscr{O}_Z$-algebra isomorphism $\widetilde{\mathscr{C}}_0(h^Z_n) \cong \mathscr{M}_{2^{m-1}}(\mathscr{O}_Z)$. After pulling back to Z, a choice of half-spin representation

$$\rho^+ : f^{-1}\boldsymbol{\Gamma}^Z_n \cong \boldsymbol{\Gamma}_{m,m} \to \mathscr{E}nd_{\mathscr{O}_Z}(f^*\widetilde{\mathscr{C}}_0(h^Z_n)) \cong \mathbf{GL}_{2^{m-1}}$$

induces, by the universal property of Weil restriction, a homomorphism $\rho : \boldsymbol{\Gamma}^Z_n \to \mathbf{R}_{Z/X}\mathbf{GL}_{2^{m-1}}$.

For m odd, ρ fits into a commutative diagram with exact rows (but not exact columns)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\kappa}^Z_n & \longrightarrow & \boldsymbol{\Gamma}^Z_n & \longrightarrow & \mathbf{GSO}^Z_n & \longrightarrow & 1 \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{GL}_{2^{m-1}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{GL}_{2^{m-1}}/\boldsymbol{\mu}_2 & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{O}_{2^{m-1},2^{m-1}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{PO}_{2^{m-1},2^{m-1}} & \longrightarrow & 1
\end{array}$$

on $X_{\text{ét}}$. The interpretation of the right vertical maps on cohomology

$$H^1_{\text{ét}}(X, \mathbf{GSO}^Z_n) \to H^1_{\text{ét}}(Z, \mathbf{GL}_{2^{m-1}}/\boldsymbol{\mu}_2) \to H^1_{\text{ét}}(Z, \mathbf{PO}_{2^{m-1},2^{m-1}})$$

is as follows: the class of a regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank n and Arf invariant $[Z/X]$ maps to the 2-torsion datum $(\widetilde{\mathscr{C}}_0, \widetilde{\mathscr{B}}_1, \nu)$ on Z; a 2-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of degree n on Z maps to the involutive Brauer class of $(\mathscr{E}nd_{\mathscr{A}}(\mathscr{A} \oplus \mathscr{P}), \sigma_\varphi)$. Finally, chasing the class of $(\mathscr{E}, q, \mathscr{L})$ around the diagram of coboundary maps yields the statement in *a)*.

Similarly for m even, ρ fits into a commutative diagram with exact rows (but not exact columns)

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \boldsymbol{\kappa}^Z_n & \longrightarrow & \boldsymbol{\Gamma}^Z_n & \longrightarrow & \mathbf{GSO}^Z_n & \longrightarrow & 1 \\
 & & \downarrow\wr & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{GL}_{2^{m-1}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{GL}_{2^{m-1}}/\mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{U}_{2^{m-1},2^{m-1}} & \longrightarrow & \mathbf{PU}_{2^{m-1},2^{m-1}} & \longrightarrow & 1
\end{array}$$

on $X_{\text{ét}}$. The interpretation of the right vertical maps on cohomology

$$H^1_{\text{ét}}(X, \mathbf{GSO}^Z_n) \to H^1_{\text{ét}}(X, \mathbf{R}_{Z/X}\mathbf{GL}_{2^{m-1}}/\mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}) \to H^1_{\text{ét}}(X, \mathbf{PU}_{2^{m-1},2^{m-1}})$$

is as follows: the class of a regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank n and Arf invariant $[Z/X]$ maps to the Z/X-torsion datum $(\widetilde{\mathscr{C}}_0, \widetilde{\mathscr{B}}_1, \nu)$; a Z/X-torsion datum $(\mathscr{A}, \mathscr{P}, \varphi)$ of degree n maps to the Z/X-unitary involutive Brauer class of $(\mathscr{E}nd_{\mathscr{A}}(\mathscr{A} \oplus \mathscr{P}), \sigma_\varphi)$. Finally, chasing the class of $(\mathscr{E}, q, \mathscr{L})$ around the diagram of coboundary maps yields the statement in *b)*. □

In the case of where m is even, we can further describe the similarity Clifford invariant. We will first need a presentation of the group $H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4)$ analogous to the involutive Brauer group(s). For such a description when X is the spectrum of a field, see [69, VII Prop. 30.13] or Colliot-Thélène–Gille-Parimala [28, Prop. 2.10]. Given an Azumaya $\mathscr{O}_X$-algebra $(\mathscr{B}, \sigma)$ with orthogonal involution, the Azumaya $\mathscr{O}_Z$-algebra $f^*\mathscr{B} = \mathscr{O}_Z \otimes_{f^{-1}\mathscr{O}_X} f^*\mathscr{A}$ has a natural Z/X-unitary involution $\sigma_{Z/X} = \iota^\sharp \otimes \sigma$ (see §3.2).

Proposition 4.10. *Let X be a scheme with 2 invertible and $f : Z \to X$ étale quadratic. Denote by $\mathrm{Br}^*_4(Z/X)$ the abelian group of pairs $\big([\mathscr{A}, \tau], [\mathscr{B}, \sigma]\big) \in \mathrm{Br}^*(Z/X) \times \mathrm{Br}^*(X)$ such that $[\mathscr{A}, \tau]^{\otimes 2} = [f^*\mathscr{B}, \sigma_{Z/X}]$ Then there is a canonical homomorphism*

$$\Psi_{Z/X,4} : \mathrm{Br}^*_4(Z/X) \to H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4)$$

which fits into a commutative diagram with exact rows

$$\begin{array}{ccccccc}
H^1_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m) & \longrightarrow & \mathrm{Br}^*_4(Z/X) & \longrightarrow & \mathrm{Br}^*(Z/X) & \xrightarrow{4} & \mathrm{Br}^*(Z/X) \\
\| & & \downarrow{\scriptstyle \Psi_{Z/X,4}} & & \downarrow{\scriptstyle \Psi_{Z/X}} & & \downarrow{\scriptstyle \Psi_{Z/X}} \\
H^1_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m) & \xrightarrow{4} & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_m)
\end{array}$$

of abelian groups. Finally, $\Psi_{Z/X,4}$ is surjective if and only if every 4-torsion element of $\ker\big(N_{Z/X} : H^2_{\text{ét}}(Z, \mathbf{G}_m) \to H^2_{\text{ét}}(X, \mathbf{G}_m)\big)$ is represented by an Azumaya $\mathscr{O}_Z$-algebra and every 2-torsion element of $H^2_{\text{ét}}(X, \mathbf{G}_m)$ is represented by an Azumaya $\mathscr{O}_X$-algebra.

Proof. Applying étale cohomology to the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m} & \xrightarrow{4} & \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle 2} & & \| & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m} & \xrightarrow{2} & \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m} & \longrightarrow & 1
\end{array}$$

yields a commutative a diagram of abelian groups with exact rows

$$\begin{array}{ccccccc}
H^1_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) & \xrightarrow{4} & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) \\
\downarrow & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 2} & & \downarrow \\
H^1_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) & \longrightarrow & H^2_{\text{ét}}(X, \boldsymbol{\mu}_2) & \longrightarrow & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m}) & \xrightarrow{2} & H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\mathbf{G}_\mathrm{m})
\end{array}$$

of which the central square is cartesian by a simple diagram chase. Composing the projections from $\mathrm{Br}^*_4(Z/X)$ with the maps Ψ and $\Psi_{Z/X}$ (from Theorem 3.9), the universal property of cartesian squares provides the homomorphism $\Psi_{Z/X,4}$. Moreover, since $\mathrm{Br}^*_4(Z/X)$ is defined as a fiber product, the maps Ψ, $\Psi_{Z/X}$, and $\Psi_{Z/X,4}$ define a morphism of cartesian squares. In particular, this extends to the sought after commutative diagram. The final statement of the Proposition follows from considerations similar to Remark 3.10. □

Remark 4.11. Similar to Remark 3.10, if we consider the fiber $\mathrm{Br}^*_4(Z/X)^+$ of $\mathrm{Br}^*_4(Z/X)$ over $\mathrm{Br}^+(X)$ via the projection, then the restriction $\Psi^+_{Z/X,4}$ is an isomorphism whenever the conditions stated at the end of Proposition 4.10 hold.

By Theorem 3.9*a)*, $\Psi[\mathscr{E}nd(\mathscr{E}), \sigma_b] = c_1(\mathscr{L}, \boldsymbol{\mu}_2) \in H^2_{\text{ét}}(X, \boldsymbol{\mu}_2)$ for any regular $\mathscr{L}$-valued ϵ-symmetric bilinear form. By abuse of notation, we will write $c_1(\mathscr{L}, \boldsymbol{\mu}_2) \in \mathrm{Br}^*(X)$.

Theorem 4.12. *Let X be a scheme with 2 invertible and $(\mathscr{E}, q, \mathscr{L})$ a regular line bundle-valued quadratic form of rank $n = 2m \equiv 0 \bmod 4$ and Arf cover $f : Z \to X$. Then*

$$gc(\mathscr{E}, q, \mathscr{L}) = \Psi_{Z/X,4}\big([\widetilde{\mathscr{C}}_0, \widetilde{\mathscr{B}}_1, \nu], c_1(\mathscr{L}, \boldsymbol{\mu}_2)\big)$$

in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}^Z_4) \cong H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4)$.

Proof. This follows by a combination of Theorem 4.9*b)*, Theorem 2.10*a)*, and the construction of $\mathrm{Br}^*_4(Z/X)$ as a cartesian square in Proposition 4.10. □

5. Low dimensional exceptional isomorphisms

In the context of quadratic forms over schemes, *low rank* usually means of rank ≤ 6. In this interval, the isomorphisms of Dynkin diagrams $\mathsf{A}_1 = \mathsf{B}_1 = \mathsf{C}_1$, $\mathsf{D}_2 = \mathsf{A}_1^2$, $\mathsf{B}_2 = \mathsf{C}_2$, and $\mathsf{A}_3 = \mathsf{D}_3$, have beautiful reverberations in the theory of quadratic forms of rank 3, 4, 5, and 6, respectively. Over rings, these isomorphisms were deeply investigated by Knus, Ojanguren, Parimala, Paques, and Sridharan in the 1980s and 1990s. Now, a standard reference on this work is Knus [68, Ch. V]. Over fields, a wonderful reference is [69, IV §15]. Over general schemes, much of the theory over rings can be globalized, but a unified treatment did not yet exist in the literature until Auel [4].

As usual, let X be a noetherian separated scheme with 2 invertible and let $\mathscr{L}$ be a fixed invertible $\mathscr{O}_X$-module.

5.1. **Norm forms.** A *normed algebra* or *composition algebra* $(\mathscr{A}, n)$ on X is a locally free $\mathscr{O}_X$-algebra $\mathscr{A}$ of finite rank together with a multiplicative quadratic form $n : \mathscr{A} \to \mathscr{O}_X$. A globalization of Hurwitz's theorem (see Petersson [86, Prop. 1.7d]) states that if the norm n is regular (as a quadratic form) then $\mathscr{A}$ is either $\mathscr{O}_X$, an étale quadratic $\mathscr{O}_X$-algebra, or an Azumaya quaternion $\mathscr{O}_X$-algebra (or a generalized octonion algebra if $\mathscr{A}$ is not assumed to be associative). An involution σ of the first kind on an $\mathscr{O}_X$-algebra $\mathscr{A}$ is called a *standard involution* if the multiplicative map

$$n_\sigma : \mathscr{A} \xrightarrow{\Delta} \mathscr{A} \times \mathscr{A} \xrightarrow{\mathrm{id}\times\sigma} \mathscr{A} \times \mathscr{A}^{\mathrm{op}} \xrightarrow{m} \mathscr{A}$$

has image in the identity subalgebra $\mathscr{O}_X \hookrightarrow \mathscr{A}$. In this case, $(\mathscr{A}, n_\sigma)$ is a normed algebra. For results about the existence and uniqueness of standard involutions in the affine setting, see Voight [95].

For a fixed normed algebra $(\mathscr{A}, n)$, a *norm form* of *type* $(\mathscr{A}, n_{\mathscr{A}})$ is a tuple $(\mathscr{P}, q, \mathscr{M})$ consisting of a right $\mathscr{A}$-module $\mathscr{P}$, an $\mathscr{O}_X$-module $\mathscr{M}$, and an $\mathscr{M}$-valued quadratic form $q : \mathscr{P} \to \mathscr{M}$ that is n-semilinear for the action of $\mathscr{A}$ on $\mathscr{P}$, i.e. there's a commutative diagram of maps

$$\begin{array}{ccc} \mathscr{P} \otimes \mathscr{A} & \xrightarrow{q\otimes n} & \mathscr{M} \otimes \mathscr{O}_X \\ \big\downarrow{\scriptstyle \cdot} & & \big\downarrow{\scriptstyle \cdot} \\ \mathscr{P} & \xrightarrow{\quad q \quad} & \mathscr{M} \end{array}$$

or equivalently, that $q(xa) = q(x)\,n(a)$ locally on sections. A morphism $(\psi, \lambda) : (\mathscr{P}, q, \mathscr{M}) \to (\mathscr{P}', q', \mathscr{M}')$ between norm forms of type $(\mathscr{A}, \sigma)$ consists of $\mathscr{O}_X$-module morphisms $\psi : \mathscr{P} \to \mathscr{P}'$ and $\lambda : \mathscr{M} \to \mathscr{M}'$ such that $q' \circ \psi = \lambda \circ q$.

Theorem 5.1. *Let $(\mathscr{A}, n)$ be a normed algebra. Then for any right $\mathscr{A}$-module $\mathscr{P}$, there exists an $\mathscr{O}_X$-module $\mathscr{N}(\mathscr{P})$ and a norm form $(\mathscr{P}, n_{\mathscr{P}}, \mathscr{N}(\mathscr{P}))$, universal for the property that given any norm form $(\mathscr{P}, q, \mathscr{M})$ there exists an $\mathscr{O}_X$-module morphism $\psi : \mathscr{N}(\mathscr{P}) \to \mathscr{M}$ such that $q = \psi \circ n_{\mathscr{P}}$. Furthermore:*

a) if $\mathscr{P} = \mathscr{A}$ with the standard right action, then then there's a canonical isomorphism $(\mathscr{P}, n_{\mathscr{P}}, \mathscr{N}(\mathscr{P})) \cong (\mathscr{A}, n, \mathscr{O}_X)$,
b) the construction of the triple $(\mathscr{P}, n_{\mathscr{P}}, \mathscr{N}(\mathscr{P}))$ commutes with arbitrary base change,
c) if $\mathscr{P}$ is an invertible $\mathscr{A}$-module then $\mathscr{N}(\mathscr{P})$ is an invertible $\mathscr{O}_X$-module.

Proof. Bichsel [18, 3.1] and Knus [68, III.7] give a tensorial construction in the affine setting, which can be globalized. Knus–Ojanguren–Sridharan [62] give a construction by faithfully flat descent in the affine setting, which can also be globalized. □

5.2. **Forms of rank 2.** In this section, we will review the classification of regular line bundle-valued quadratic forms of rank 2 in terms of norm forms associated to quadratic normed algebras and relate the Clifford invariant to this classification.

Any Clifford datum of rank 2 is isomorphic to $(Z/X, \mathscr{O}_Z, \iota^\sharp, \mathscr{P}, <1>)$, where $f : Z \to X$ is étale quadratic, $\mathscr{P}$ is an invertible $\mathscr{O}_Z$-module, and where we consider $<1>: \mathscr{O}_Z \otimes_{\mathscr{O}_Z} \mathscr{O}_Z \to \mathscr{O}_Z \cong \mathscr{E}nd_{\mathscr{O}_Z}(\mathscr{P})$. We will denote this Clifford datum simply by $(Z/X, \mathscr{P})$.

Conversely, given $(Z/X, \mathscr{P})$, the norm form construction produces a unique similarity class of regular line bundle-valued quadratic form of rank 2. Indeed, the $\mathscr{O}_X$-algebra $f_*\mathscr{O}_Z$ has a unique standard involution induced from ι, with corresponding norm n_f. The locally free $\mathscr{O}_X$-module $f_*\mathscr{M}$ is an invertible $f_*\mathscr{O}_Z$-module, and so defines a universal norm form $(f_*\mathscr{M}, n_{f_*\mathscr{M}}, \mathscr{N}(f_*\mathscr{M}))$ of type $(f_*\mathscr{O}_Z, n_f)$.

Remark 5.2. This recaptures (in the case of locally free $\mathscr{O}_X$-algebras of rank 2) a general "norm functor" construction due to Ferrand [36, §5.3].

Theorem 5.3 (Kneser [60, §6 Prop. 2], Ferrand [37, §5.4], Bichsel–Knus [17, §4.2])**.** *Let X be a scheme with 2 invertible. The norm form and Clifford datum define inverse equivalence functors between the following categories:*

- *objects are Clifford data $(Z/X, \mathscr{P})$ of rank 2 and morphisms are isomorphisms of Clifford data*
- *objects are regular line bundle-valued quadratic forms $(\mathscr{E}, q, \mathscr{L})$ of rank 2 and morphisms are similarity transformations,*

Under this equivalence, $[Z/X]$ maps to $d_\pm(\mathscr{E}, q, \mathscr{L})$ under $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \to H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ from Proposition 1.13.

Remark 5.4. The generalization of Theorem 5.3 to flat quadratic coverings and nondegenerate forms is established by Ferrand [37] over an arbitrary scheme with 2 invertible.

The norm for associated to $(Z/X, \mathscr{O}_Z)$ is our standard form h_2^Z and there's a canonical isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \xrightarrow{N} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{SO}_2^Z & \longrightarrow & \mathbf{GSO}_2^Z & \xrightarrow{\mu} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \end{array}$$

given by the right multiplication map $\rho : \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} \to \mathbf{GSO}_2^Z$. Then since h_2^Z is of type $(f_*\mathscr{O}_Z, n_f)$, we verify that $n_{\mathscr{O}_Z}(\rho_z(v)) = n_{\mathscr{O}_Z}(v \cdot z) = n_{\mathscr{O}_Z}(v) n_f(z)$, so that indeed ρ_z is a similarity transformation with multiplier $n_f(z)$, and the diagram commutes. The fact that ρ is an isomorphism may be checked locally, the affine case being verified in [17, §6.1].

Also, there's a canonical isomorphism of short exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \xrightarrow{2} & \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\kappa}_2^Z & \longrightarrow & \mathbf{\Gamma}_2^Z & \xrightarrow{s} & \mathbf{GSO}_2^Z & \longrightarrow & 1 \end{array} \tag{38}$$

which follows from the above, and from [68, V Lemma 2.5.2].

Proposition 5.5. *Let X be a scheme with 2 invertible, $f : Z \to X$ étale quadratic, $\mathscr{P}$ and invertible $\mathscr{O}_Z$-module, and $(\mathscr{E}, q, \mathscr{L})$ the corresponding norm form. Then we have*

$$gc(\mathscr{E}, q, \mathscr{L}) = c_1(\mathscr{P}, \boldsymbol{\mu}_2)$$

in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}_2^Z) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)$.

Proof. This follows from Theorem 4.9, or directly by chasing a class associated to a regular line bundle-valued quadratic form $(\mathscr{E}, q, \mathscr{L})$ around the commutative diagram of coboundary maps induced from diagram 38. □

Note that Theorem 2.10*b)* is then a consequence of the isomorphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \boldsymbol{\kappa}_2^Z & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & 1 \end{array}$$

and Proposition 5.5. Using the classification of regular line bundle-valued forms of rank 2, we can immediately give a generalization of [68, V Ex. 9.2.3].

Corollary 5.6. *If a regular quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank 2 has trivial signed discriminant, then it's similar to an $\mathscr{L}$-valued hyperbolic form $H_{\mathscr{L}}(\mathscr{V})$ for an invertible $\mathscr{O}_X$-module $\mathscr{V}$. Furthermore, $gc(q)$ is trivial if and only if $\mathscr{L}$ and $\mathscr{V}$ are both squares in* $\mathrm{Pic}(X)$.

Proof. If $(\mathscr{E}, q, \mathscr{L})$ has trivial discriminant, then by Theorem 5.3, it's a norm form $(Z/X, \mathscr{P})$ with Z/X split and $\mathscr{P} \cong (\mathscr{P}_1, \mathscr{P}_2)$ in $\mathrm{Pic}(Z) \cong \mathrm{Pic}(X) \times \mathrm{Pic}(X)$. Then $\mathscr{E} \cong f_*\mathscr{P} \cong \mathscr{P}_1 \oplus \mathscr{P}_2$ and $\mathscr{L} \cong \mathscr{N}_f(\mathscr{P}) \cong \det(f_*\mathscr{P}) \cong \mathscr{P}_1 \otimes \mathscr{P}_2$. It's then easy to verify, using Remark 5.2, that $(\mathscr{E}, q, \mathscr{L})$ is similar to the form $(\mathscr{P}_1 \oplus \mathscr{P}_2, \otimes, \mathscr{P}_1 \otimes \mathscr{P}_2)$, which in turn is similar to the hyperbolic form $H_{\mathscr{P}_1 \otimes \mathscr{P}_2}(\mathscr{P}_1)$. This proves the first claim, which is a generalization of [68, IV.9.2.3 Ex.].

By Proposition 5.5, $gc(q)$ is trivial if and only $\mathscr{P}$ is a square in $\mathrm{Pic}(Z)$ if and only if $\mathscr{P}_1$ and $\mathscr{P}_2$ are squares in $\mathrm{Pic}(X)$. But this is also equivalent to both $\mathscr{V} \cong \mathscr{P}_1$ and $\mathscr{L} \cong \mathscr{P}_1 \otimes \mathscr{P}_2$ being squares. □

5.3. **Forms of rank 4.** In this section, we will review the classification of regular line bundle-valued quadratic forms of rank 4 in terms of norm forms associated to Azumaya quaternion algebras and relate the Clifford invariant to this classification.

Any symplectic involution on an Azumaya quaternion $\mathscr{O}_Z$-algebra $\mathscr{A}$ is isomorphic to the standard involution $\sigma_{\mathscr{A}}$ (this follows from Pumplün [87, Theorem 2.6], for example), and thus any Clifford datum of rank 4 is isomorphic to $(Z/X, \mathscr{A}, \sigma_{\mathscr{A}}, \mathscr{P}, \varphi)$, which we will denote simply by $(Z/X, \mathscr{A}, \mathscr{P}, \varphi)$. Thus a Clifford datum of rank 4 is equivalent to a 2-torsion datum on Z.

Every regular line bundle-valued quadratic form gives rise to a Clifford datum of rank 4. Conversely, a "twisted" version of the universal norm form functor can reconstruct the similarity class of a quadratic form of rank 4 from its associated Clifford datum. This functor is essentially described in [65, §10] or [68, V §4.2] and can be viewed as a refinement of [69, IV.15.B]. For a precise definition of the twisted norm form, see Auel [4].

Theorem 5.7. *Let X be a scheme with 2 invertible. The twisted norm form and Clifford datum define inverse equivalence functors between the following categories:*

- *objects are Clifford data $(Z/X, \mathscr{A}, \mathscr{P}, \varphi)$ of rank 4 and morphisms are isomorphisms of Clifford data*
- *objects are regular line bundle-valued quadratic forms $(\mathscr{E}, q, \mathscr{L})$ of rank 4 and morphisms are similarity transformations.*

Under this equivalence, $[Z/X]$ is mapped to $d_{\pm}(\mathscr{E}, q, \mathscr{L})$ under $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \cong H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ from Proposition 1.13.

To compute the Clifford invariant in terms of the classification in Theorem 5.7, we use the following torsorial reinterpretation. Recall the notation $h_4^Z = H_{\mathscr{O}_X}(\mathscr{O}_X) \perp (f_*\mathscr{O}_Z, h^Z)$. There's a canonical isomorphism $\mathbf{Spin}_4^Z \cong \mathbf{R}_{Z/X}\mathbf{SL}_1(\widetilde{\mathscr{C}_0}(h_4^Z))$ by [68, V.4.4.1] (see also [69, IV Prop. 15.10]), which generalizes immediately from the affine case. In particular, we have an isomorphism $\mathbf{Spin}_4^Z \cong \mathbf{R}_{Z/X}\mathbf{SL}_2$ for any choice of $\mathscr{O}_Z$-algebra isomorphism $\widetilde{\mathscr{C}_0} \cong M_2(\mathscr{O}_Z)$ (a canonical choice is furnished from the graded $\mathscr{O}_X$-algebra isomorphism $\mathscr{C}(h_2^Z) \cong \mathscr{C}(h_2)\widehat{\otimes}\mathscr{C}(h_4^Z)$ of (full) Clifford algebras).

The Weil restriction functor induces the following exact sequence of groups schemes

$$1 \to \mathbf{R}_{Z/X}\mathbf{SL}_2 \to \mathbf{R}_{Z/X}\mathbf{GL}_2 \xrightarrow{\mathbf{R}_{Z/X}\det} \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} \to 1$$

on $X_{\text{ét}}$. Set $\mathbf{G} = \mathbf{R}_{Z/X}\mathbf{GL}_2/\mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}$ (see §4.2) and denote by $\boldsymbol{\Gamma}$ the sheaf kernel of the composite epimorphism $\mathbf{R}_{Z/X}\mathbf{GL}_2 \to \mathbf{R}_{Z/X}\mathbf{G}_{\mathrm{m}} \xrightarrow{\mathrm{id}/\iota} \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}}$. Hence we have exact sequences

$$1 \to \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} \to \mathbf{R}_{Z/X}\mathbf{GL}_2 \to \mathbf{G} \to 1 \qquad 1 \to \boldsymbol{\Gamma} \to \mathbf{R}_{Z/X}\mathbf{GL}_2 \xrightarrow{\mathrm{id}/\iota} \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} \to 1$$

of sheaves of groups on $X_{\text{ét}}$.

Finally, restricting the quotient map $\mathbf{R}_{Z/X}\mathbf{GL}_2 \to \mathbf{G}$ to $\boldsymbol{\Gamma}$ yields a homomorphism $s : \boldsymbol{\Gamma} \to \mathbf{G}$, restricting $\mathbf{R}_{Z/X}\det$ to $\boldsymbol{\Gamma}$ yields a homomorphism $\boldsymbol{\Gamma} \to \mathbf{G}_{\mathrm{m}}$, and restricting $N_{Z/X} \circ \mathbf{R}_{Z/X}\det$ to $\mathbf{G}$ yields a homomorphism $\mathbf{G} \to \mathbf{G}_{\mathrm{m}}$.

Theorem 5.8. *There are canonical isomorphisms*

$$\boldsymbol{\Gamma} \cong \mathbf{S\Gamma}_4^Z, \qquad \mathbf{R}_{Z/X}\mathbf{SL}_2 \cong \mathbf{Spin}_4^Z$$

and

$$\mathbf{G} \cong \mathbf{GSO}_4^Z, \qquad (\mathbf{R}_{Z/X}\mathbf{SL}_2)/\boldsymbol{\mu}_2 \cong \mathbf{SO}(h^2_{Z/X})$$

making the even fundamental diagram (15) associated to the form h_4^Z isomorphic to

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \boldsymbol{\mu}_2 & \to & \mathbf{R}_{Z/X}\mathbf{SL}_2 & \to & \mathbf{R}_{Z/X}\mathbf{SL}_2/\boldsymbol{\mu}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \to & \boldsymbol{\Gamma} & \to & \mathbf{G} & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \boldsymbol{\mu}_2 & \to & \mathbf{G}_m & \xrightarrow{2} & \mathbf{G}_m & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Proof. See Auel [4]. We have the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & \boldsymbol{\Gamma} & \longrightarrow & \mathbf{R}_{Z/X}\mathbf{GL}_2 & \longrightarrow & \mathbf{R}^1_{Z/X}\mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \boldsymbol{\Gamma}/\mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 & \xrightarrow{\sim} & \mathbf{G} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{39}
$$

of sheaves of groups on $X_{\text{ét}}$. In particular, the morphism $\boldsymbol{\Gamma} \to \mathbf{G}$ can be identified with the cokernel of the inclusion $\mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4 \to \boldsymbol{\Gamma}$. □

Proposition 5.9. *Let X be a scheme with 2 invertible, $f : Z \to X$ étale quadratic, $(Z/X, \mathscr{A}, \mathscr{P}, \varphi)$ a Clifford datum of rank 4, and $(\mathscr{E}, q, \mathscr{L})$ the associated twisted norm form. Then*

$$gc(\mathscr{E}, q, \mathscr{L}) = \Psi_{Z/X,4}\big([\mathscr{A}, \mathscr{P}, \varphi], c_1(\mathscr{L}, \boldsymbol{\mu}_2)\big)$$

in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}_4^Z) \cong H^2_{\text{ét}}(X, \mathbf{R}^1_{Z/X}\boldsymbol{\mu}_4)$.

Proof. This is a consequence of Theorem 4.12. □

To every locally free $\mathscr{O}_Z$-module $\mathscr{V}$ of rank 2, we can associate the *split* Clifford datum $(Z/X, \mathscr{E}nd(\mathscr{V}), \mathscr{V} \otimes \iota_*\mathscr{V}, \varphi_{\mathscr{V}})$ of rank 4, where $\varphi_{\mathscr{V}}$ is the composition of canonical $\mathscr{O}_Z$-algebra isomorphisms

$$\mathscr{E}nd(\mathscr{V}) \otimes_{\mathscr{O}_Z} \iota_*\,\mathscr{E}nd(\mathscr{V}) \xrightarrow{\text{can}} \mathscr{E}nd(\mathscr{V}) \otimes_{\mathscr{O}_Z} \mathscr{E}nd(\iota_*\mathscr{V}) \to \mathscr{E}nd(\mathscr{V} \otimes \iota_*\mathscr{V}).$$

The map

$$H^1_{\text{ét}}(Z, \mathbf{GL}_2) \cong H^1_{\text{ét}}(X, \mathbf{R}_{Z/X}\mathbf{GL}_2) \to H^1_{\text{ét}}(X, \mathbf{G}) \cong H^1_{\text{ét}}(X, \mathbf{GSO}_4^Z)$$

has the following interpretation: a locally free $\mathscr{O}_Z$-module of rank 2 is sent to the twisted norm form of its split Clifford datum.

The twisted norm form of the split Clifford datum associated to $\mathscr{V}$ is isometric to $N_{Z/X}(\mathscr{V}, \wedge, \det \mathscr{V}) = (N_{Z/X}(\mathscr{V}), N_{Z/X}(\wedge), N_{Z/X}(\det \mathscr{V}))$, which we can think of as the "bilinear form norm" applied the canonical skew-symmetric form $\wedge : \mathscr{V} \otimes \mathscr{V} \to \det \mathscr{V}$ on Z. When $f : Z \to X$ is split, then a locally free $\mathscr{O}_Z$-module of rank 2 can be though of as a pair of locally free $\mathscr{O}_X$-modules $(\mathscr{V}, \mathscr{W})$ of rank 2, and the twisted norm for of the associated Clifford datum is similar to the tensor product $\mathscr{V} \otimes \mathscr{W} \to \det \mathscr{V} \otimes \det \mathscr{W}$ of the canonical skew-symmetric wedging forms.

Corollary 5.10. *With the notations of Proposition 5.9, the class $gc(\mathscr{E}, q, \mathscr{L})$ vanishes if and only if $(\mathscr{E}, q, \mathscr{L})$ is similar to $N_{Z/X}(\mathscr{V}, \wedge, \det \mathscr{V})$ for some locally free $\mathscr{O}_Z$-module of rank 2 such that* $\det \mathscr{V} \cong \iota_* \det \mathscr{V}$.

Proof. This follows from the above description of twisted norm forms of split Clifford data of rank 4, along with a diagram chase involving the cohomology of diagram (39). □

When $f : Z \to X$ is split, then $\mathbf{R}_{Z/X}\mathbf{GL}_2 = \mathbf{GL}_2 \times \mathbf{GL}_2$ and $\boldsymbol{\Gamma} = \mathbf{GL}_2 \times_{\mathbf{G}_{\mathrm{m}}} \mathbf{GL}_2$ (pairs of invertible transformations with equal determinant), making the even fundamental diagram is isomorphic to

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{SL}_2 \times \mathbf{SL}_2 & \longrightarrow & \mathbf{SO}_{2,2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_4 & \longrightarrow & \mathbf{GL}_2 \times_{\mathbf{G}_{\mathrm{m}}} \mathbf{GL}_2 & \longrightarrow & \mathbf{GSO}_{2,2} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_{\mathrm{m}} & \xrightarrow{2} & \mathbf{G}_{\mathrm{m}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

compare with Knus [68, V §4.5–4.6]. Furthermore, Clifford data of rank 4 (with Z/X split) can be viewed as tuples $(\mathscr{A}, \mathscr{B}, \mathscr{P}, \varphi)$ of degree 2, where $\mathscr{A}$ and $\mathscr{B}$ are Azumaya quaternion $\mathscr{O}_X$-algebras, $\mathscr{P}$ is a locally free $\mathscr{O}_X$-module of rank 4, and $\varphi : \mathscr{A} \otimes \mathscr{B} \to \mathscr{E}nd(\mathscr{P})$ is an $\mathscr{O}_X$-algebra isomorphism.

The category of such tuples are $(\mathbf{GL}_2 \times \mathbf{GL}_2)/\mathbf{G}_{\mathrm{m}}$-torsors on $X_{\text{ét}}$. In this case, the twisted norm form is just the universal or reduced norm form, and is studied in [18], [65], [64], and [17].

5.4. **Forms of rank 6.** In this section, we will review the classification of regular line bundle-valued quadratic forms of rank 6 in terms of the pfaffian forms (or discriminant algebras) associated to Azumaya algebras of degree 4 relate the Clifford invariant to this classification.

Any regular line bundle-valued quadratic form of rank 6 determines a Clifford datum of rank 6. Conversely, a "twisted" version of the reduced pfaffian form functor can reconstruct the quadratic form from the associated Clifford datum. This functor is essentially described in [61], [65, §7], [66, §5], [68, V §5.5], and can be viewed as a refinement of [69, IV.15.D]. For a precise definition of the twisted pfaffian form, see Auel [4].

Theorem 5.11. *Let X be a scheme with 2 invertible. The twisted pfaffian form and Clifford datum define inverse equivalence functors between the following categories:*

- *objects are Clifford data $(Z/X, \mathscr{A}, \sigma, \mathscr{P}, \varphi)$ of rank 6 and morphisms are isomorphisms of Clifford data*
- *objects are regular line bundle-valued quadratic forms $(\mathscr{E}, q, \mathscr{L})$ of rank 6 and morphisms are similarity transformations.*

Under this equivalence, $[Z/X]$ maps to $d_\pm(\mathscr{E}, q, \mathscr{L})$ under $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}) \cong H^1_{\text{ét}}(X, \boldsymbol{\mu}_2)$ from Proposition 1.13.

To compute the Clifford invariant in terms of the classification in Theorem 5.11, we use the following torsorial reinterpretation. Recall the notation $h_6^Z = H_{\mathscr{O}_X}(\mathscr{O}_X^2) \perp (f_*\mathscr{O}_Z, h^Z)$. There are canonical isomorphisms $\mathbf{Spin}_6^Z \cong \mathbf{SU}_{2,2}$ and $\boldsymbol{\Gamma}_6^Z \cong \mathbf{SGU}_{2,2}$, by [68, V §5.6] (see also [69, IV Prop. 15.27]), where $\mathbf{GU}_{2,2}$ is the general unitary group of the hyperbolic Z/X-hermitian form of rank 4.

Theorem 5.12. *The above isomorphisms make the even fundamental diagram (15) associated to the form h_6^Z isomorphic to*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{SU}_{2,2} & \longrightarrow & \mathbf{SU}_{2,2}/\boldsymbol{\mu}_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & \mathbf{SGU}_{2,2} & \longrightarrow & \mathbf{SGU}_{2,2}/\mathbf{R}_{Z/X}\boldsymbol{\mu}_2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_2 & \longrightarrow & \mathbf{G}_m & \xrightarrow{2} & \mathbf{G}_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Proposition 5.13. *Let X be a scheme with 2 invertible, $f : Z \to X$ étale quadratic, $(Z/X, \mathscr{A}, \tau, \mathscr{P}, \varphi)$ a Clifford datum of rank 6, and $(\mathscr{E}, q, \mathscr{L})$ the associated twisted pfaffian form. Then*

$$gc(\mathscr{E}, q, \mathscr{L}) = [\mathscr{A}, \mathscr{P}, \varphi]$$

in $H^2_{\text{ét}}(X, \boldsymbol{\kappa}_6^Z) \cong H^2_{\text{ét}}(X, \mathbf{R}_{Z/X}\boldsymbol{\mu}_2) \cong H^2_{\text{ét}}(Z, \boldsymbol{\mu}_2)$.

Proof. This is a consequence of Theorem 4.9*a)*. □

To every regular line bundle-valued Z/X-hermitian form $(\mathscr{H}, h, \mathscr{L})$ of rank 4, we can associate a *split* Clifford datum $(Z/X, \mathscr{E}nd(\mathscr{H}), \tau_h, \mathscr{H} \otimes \mathscr{H}, \varphi_{\mathscr{H}})$ of rank 6, where $(\mathscr{E}nd(\mathscr{H}), \tau_h)$ is the associated Z/X-unitary adjoint involution (see §3.2), and $\varphi_{\mathscr{H}}$ is the canonical $\mathscr{O}_Z$-algebra isomorphism

$$\mathscr{E}nd(\mathscr{H}) \otimes \mathscr{E}nd(\mathscr{H}) \to \mathscr{E}nd(\mathscr{H} \otimes \mathscr{H}).$$

The map

$$H^1_{\text{ét}}(X, \mathbf{GU}_{2,2}) \to H^1_{\text{ét}}(X, \mathbf{SGU}_{2,2}/\mathbf{R}_{Z/X}\boldsymbol{\mu}_2) \cong H^1_{\text{ét}}(X, \mathbf{GSO}(h_6^Z))$$

has the following interpretation: a regular line bundle-valued Z/X-hermitian form is sent to the twisted pfaffian form of its split Clifford datum.

Corollary 5.14. *With the notations of Proposition 5.13, the class $gc(\mathscr{E}, q, \mathscr{L})$ vanishes if and only if there exists a regular line bundle-valued Z/X-hermitian form $(\mathscr{H}, h, \mathscr{M})$ of rank 4 with trivial hermitian discriminant such that $f^*(\mathscr{E}, q, \mathscr{L})$ is similar (with the pullback Z/X-hermitian structure) to the canonical quadratic form $(\bigwedge^2 \mathscr{H}, \wedge, \det \mathscr{H})$.*

When $f : Z \to X$ is split, then $\mathbf{GU}_{2,2} \cong \mathbf{G}_{\mathrm{m}} \times \mathbf{GL}_4$, $\mathbf{SGU}_{2,2} \cong \mathbf{G}_{\mathrm{m}} \times_{\mathbf{G}_{\mathrm{m}}} \mathbf{GL}_{2,2}$ (pairs (a, A) with $\det A = a^2$), and $\mathbf{SU}_{2,2} \cong \mathbf{SL}_4$. We also have $\mathbf{SGU}_{2,2}/(\boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2) \cong \mathbf{GL}_4/\boldsymbol{\mu}_2$. Hence the even fundamental diagram is isomorphic to

$$\begin{CD}
@. 1 @. 1 @. 1 @. \\
@. @VVV @VVV @VVV @. \\
1 @>>> \boldsymbol{\mu}_2 @>>> \mathbf{SL}_4 @>>> \mathbf{SL}_4/\boldsymbol{\mu}_2 @>>> 1 \\
@. @VVV @VVV @VVV @. \\
1 @>>> \boldsymbol{\mu}_2 \times \boldsymbol{\mu}_2 @>>> \mathbf{G}_{\mathrm{m}} \times_{\mathbf{G}_{\mathrm{m}}} \mathbf{GL}_4 @>>> \mathbf{GL}_{2,2}/\boldsymbol{\mu}_2 @>>> 1 \\
@. @VVV @VVV @VVV @. \\
1 @>>> \boldsymbol{\mu}_2 @>>> \mathbf{G}_{\mathrm{m}} @>{2}>> \mathbf{G}_{\mathrm{m}} @>>> 1 \\
@. @VVV @VVV @VVV @. \\
@. 1 @. 1 @. 1 @.
\end{CD}$$

and diagram (21), which defines the oriented invariants, is isomorphic to

$$\begin{CD}
@. @. 1 @. 1 @. \\
@. @. @VVV @VVV @. \\
1 @>>> \boldsymbol{\mu}_2 @>>> \mathbf{SL}_4 @>>> \mathbf{SL}_4/\boldsymbol{\mu}_2 @>>> 1 \\
@. @| @VVV @VVV @. \\
1 @>>> \boldsymbol{\mu}_2 @>>> \mathbf{GL}_4 @>>> \mathbf{GL}_4/\boldsymbol{\mu}_2 @>>> 1 \\
@. @. @V{\det}VV @VV{\mu}V @. \\
@. @. \mathbf{G}_{\mathrm{m}} @= \mathbf{G}_{\mathrm{m}} @. \\
@. @. @VVV @VVV @. \\
@. @. 1 @. 1 @.
\end{CD}$$

In particular the oriented invariants $gc^{\pm}$ are described as in Example 2.20. Furthermore, Clifford data of rank 6 (with Z/X split) can be viewed simply as 2-torsion data $(\mathscr{A}, \mathscr{P}, \varphi)$ of degree 4 on X. In this case, the twisted pfaffian form is just the reduced pfaffian form, which is studied in [61], [65], and [17]. Actually, not only can the similarity class of a quadratic form be recovered from a Clifford datum, but also a choice of orientation.

Appendix

Lemma 5.15. *Consider a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \boldsymbol{\kappa} & \longrightarrow & \mathbf{\Gamma}^+ & \longrightarrow & \mathbf{G}^+ & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \boldsymbol{\kappa} & \longrightarrow & \mathbf{\Gamma} & \longrightarrow & \mathbf{G} & \longrightarrow & 1 \end{array}$$

of sheaves of groups on a site X. Suppose that $\boldsymbol{\kappa} \to \mathbf{\Gamma}^+$ is central and $\mathbf{G}^+ \to \mathbf{G}$ is a normal monomorphism. Then the coboundary map $H^1(X, \mathbf{G}^+) \to H^2(X, \boldsymbol{\kappa})$ is constant on the fibers of the map $H^1(X, \mathbf{G}^+) \to H^1(X, \mathbf{G})$.

Proof. If $g : \mathbf{G}^+ \to \mathbf{G}$ is a normal monomorphism, then so is $\mathbf{\Gamma}^+ \to \mathbf{\Gamma}$ and there's a natural homomorphism $\mathbf{\Gamma}/\mathbf{\Gamma}^+ \to \mathbf{G}/\mathbf{G}^+$ of sheaves of groups, which is an isomorphism by the Nine Lemma 5.16.

First we show that the coboundary map is constant on the kernel of g^1. We have a commutative diagram

$$\begin{array}{ccccc} H^0(X, \mathbf{\Gamma}/\mathbf{\Gamma}^+) & \xrightarrow{\sim} & H^0(X, \mathbf{G}/\mathbf{G}^+) & & \\ \downarrow & & \downarrow{\scriptstyle \delta_{\mathbf{G}}} & & \\ H^1(X, \mathbf{\Gamma}^+) & \longrightarrow & H^1(\mathbf{G}^+) & \xrightarrow{\delta} & H^2(\mathbf{k}) \\ \downarrow & & \downarrow{\scriptstyle g^1} & & \| \\ H^1(X, \mathbf{\Gamma}) & \longrightarrow & H^1(\mathbf{G}) & \longrightarrow & H^2(\mathbf{k}) \end{array}$$

of pointed sets. In particular, $\ker(g^1)$ equals the image of $\delta_{\mathbf{G}}$. By the commutativity of the top square in the diagram, $\ker(g^1)$ can be lifted to $H^1(X, \mathbf{\Gamma}^+)$, i.e. the coboundary map δ is trivial (hence constant) on $\ker(g^1)$.

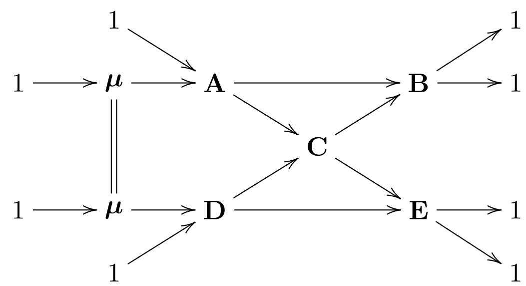
Now, for each fixed $\xi \in H^1(X, \mathbf{G})$ we can reduce to the previous case, where ξ is the neutral element. Indeed, if ξ is not in the image of g^1 the statement of the lemma is vacuous, otherwise, let $\xi^+ \in H^1(X, \mathbf{G}^+)$ map to ξ. If $\xi^+_{ij} \in Z^1(\mathscr{U}, \mathbf{G}^+)$ is a Čech 1-cocycle representing ξ^+ for a cover $\mathscr{U}$ of X, then $\xi_{ij} \in Z^1(\mathscr{U}, \mathbf{G})$, where $\xi_{ij} = g(\xi^+_{ij})$, is a Čech 1-cocycle representing ξ. Finally, twisting the exact sequences by ξ^+_{ij} and ξ_{ij} accomplishes the reduction. □

Lemma 5.16 (Nine Lemma)**.** *Consider a commutative diagram*

$$\begin{array}{ccccccccc} & & 1 & & 1 & & 1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{A}_1 & \longrightarrow & \mathbf{B}_1 & \longrightarrow & \mathbf{C}_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{A}_2 & \longrightarrow & \mathbf{B}_2 & \longrightarrow & \mathbf{C}_2 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathbf{A}_3 & \longrightarrow & \mathbf{B}_3 & \longrightarrow & \mathbf{C}_3 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 1 & & 1 & & 1 & & \end{array}$$

of sheaves of groups on a site X. If all columns and the two bottom rows are exact then the top row is exact; if all columns and the two top rows are exact then the bottom row is exact.

Lemma 5.17 (Roman IX Lemma)**.** *Consider a commutative diagram with exact rows and diagonals*



of sheaves of groups on a site X. Assume that $\mu \to \mathbf{A}$ and $\mu \to \mathbf{C}$ are central. Then the induced diagram on cohomology

$$\begin{array}{ccccccccc}
H^1(X,\boldsymbol{\mu}) & \longrightarrow & H^1(X,\mathbf{A}) & & \longrightarrow & & H^1(X,\mathbf{B}) & \longrightarrow & H^2(X,\boldsymbol{\mu}) \\
\| & & & \searrow & & \nearrow & & & \| \\
 & & & & H^1(X,\mathbf{C}) & & & & \\
\| & & & \nearrow & & \searrow & & & \| \\
H^1(X,\boldsymbol{\mu}) & \longrightarrow & H^1(X,\mathbf{D}) & & \longrightarrow & & H^1(X,\mathbf{E}) & \longrightarrow & H^2(X,\boldsymbol{\mu})
\end{array}$$

is commutative, except for the right hand pentagon, which is anticommutative.

Remark 5.18 (Roman IX rearrangement). Given a commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & \boldsymbol{\mu} & \longrightarrow & \mathbf{A} & \xrightarrow{\alpha} & \mathbf{B}' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \beta & & \\
1 & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{B} & \longrightarrow & 1 \\
 & & \delta \downarrow & & \downarrow & & & & \\
 & & \mathbf{E}' & \xrightarrow{\gamma} & \mathbf{E} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array}$$

of sheaves of groups on a site X, whose top two rows and left two columns are exact, then by the Nine Lemma 5.16, β and γ are isomorphisms, and there's a rearranged commutative diagram with exact rows and diagonals

$$\begin{array}{ccccccccccc}
 & & 1 & & & & & & & & 1 \\
 & & & \searrow & & & & & & \nearrow & \\
1 & \longrightarrow & \boldsymbol{\mu} & \longrightarrow & \mathbf{A} & & \xrightarrow{f} & & \mathbf{B} & \longrightarrow & 1 \\
 & & \| & & & \searrow & & \nearrow & & & \\
 & & \| & & & & \mathbf{C} & & & & \\
 & & \| & & & \nearrow & & \searrow & & & \\
1 & \longrightarrow & \boldsymbol{\mu} & \longrightarrow & \mathbf{D} & & \xrightarrow{g} & & \mathbf{E} & \longrightarrow & 1 \\
 & & & \nearrow & & & & & & \searrow & \\
 & & 1 & & & & & & & & 1
\end{array}$$

where $f = \beta \circ \alpha$ and $g = \gamma \circ \delta$. Hence the Roman IX Lemma 5.17 applies.

Lemma 5.19. *Let*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbf{k} & \longrightarrow & \boldsymbol{\Gamma} & \longrightarrow & \mathbf{G} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathbf{k}' & \longrightarrow & \boldsymbol{\Gamma}' & \longrightarrow & \mathbf{G}' & \longrightarrow & 1
\end{array}$$

be a commutative diagram of sheaves of groups on a site X with central and exact rows and central columns. Let δ and δ' be the second coboundary maps associated to the first and second rows, respectively. Then for each $\xi \in H^1(X,\mathbf{G})$ and $\xi' \in H^1(X,\mathbf{G}')$, we have

$$\delta'(\xi \cdot \xi') = \delta(\xi) + \delta'(\xi')$$

where $\cdot$ and $+$ represent the action of $H^1(X,\mathbf{G})$ on $H^1(X,\mathbf{G}')$ and of $H^2(X,\mathbf{k})$ on $H^2(X,\mathbf{k}')$, respectively.

Proof. Let $\mathscr{U} = \{U_i \to X\}_{i\in I}$ be a cover of X. Let $\xi_{ij} \in Z^1(\mathscr{U},\mathbf{G})$ and $\xi'_{ij} \in Z^1(\mathscr{U},\mathbf{G}')$ be 1-cocycles, and $\gamma_{ij} \in C^1(\mathscr{U},\boldsymbol{\Gamma})$ and $\gamma'_{ij} \in C^1(\mathscr{U},\boldsymbol{\Gamma}')$ be 1-cochains lifting ξ_{ij} and ξ'_{ij}, respectively. Then we have

$$\begin{aligned}
\delta'(\xi_{ij} \cdot \xi'_{ij}) &= (\gamma_{ij\underline{k}} \cdot \gamma'_{ij\underline{k}})\,(\gamma_{\underline{i}jk} \cdot \gamma'_{\underline{i}jk})\,(\gamma_{i\underline{j}k} \cdot \gamma'_{i\underline{j}k})^{-1} \\
&= \gamma_{ij\underline{k}}\gamma_{\underline{i}jk}\gamma^{-1}_{i\underline{j}k} + \gamma'_{ij\underline{k}}\gamma'_{\underline{i}jk}\gamma'^{-1}_{i\underline{j}k} \\
&= \delta(\xi_{ij}) + \delta'(\xi'_{ij})
\end{aligned}$$

since $\boldsymbol{\Gamma} \to \boldsymbol{\Gamma}'$ is central. Also see Giraud [47, IV Cor. 3.3.4(ii)]. ☐

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Emory University, Department of Mathematics & CS, Emory University, 400 Dowman Drive NE W401, Atlanta, GA 30322

E-mail address: auel@mathcs.emory.edu