Brill-Noether theory for Cubic Fourfolds pretalk

Asher Auel (Dartmouth)

Cubic fourfolds $\xrightarrow{\text{Hodge theory}}$ K3 surfaces

Cubic fourfolds $\xrightarrow{\text{Birational Geometry}}$ K3 surfaces

Cubic fourfolds $\xrightarrow{\text{Derived category}}$ K3 surfaces

K3 surfaces $\xrightarrow{\text{projective models}}$ Curves

Cubic Fourfolds

$X \subseteq \mathbb{P}^5$ smooth cubic hypersurface

$\mathcal{C} \subseteq \mathbb{P}(H^0(\mathbb{P}^5, \mathcal{O}(3)))/PGL_6$ coarse moduli space

Hodge diamond

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
0 & & 1 & & 21 & & 1 & & 0 \\
 & 0 & & 0 & & 0 & & 0 & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}$$

$$H^4(X,\mathbb{Z}) \cong \langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}$$

Integral polarized weight 4 Hodge structure with distinguished class $h^2 \in H^{2,2}(X)$

Torelli theorem (Voisin)

IPHS on $H^4(X,\mathbb{Z})$ recovers X up to isomorphism

$$\mathcal{C} \hookrightarrow \Gamma \backslash \mathcal{D}$$

Algebraic Cohomology

$$CH^i(X) \longrightarrow H^{2i}(X,\mathbb{Z}) \cap H^{i,i}(X) \quad \text{cycle class map}$$

$$[Z] \longmapsto cl([Z])$$

Hodge Conjecture

X smooth projective $/\mathbb{C}$

$$CH^i(X)\otimes\mathbb{Q} \twoheadrightarrow H^{2i}(X,\mathbb{Q}) \cap H^{i,i}(X)$$

(Conte/Murre/Zucker) True for rationally connected 4folds

Integral Hodge Conjecture

$$CH^i(X) \longrightarrow H^{2i}(X,\mathbb{Z}) \cap H^{i,i}(X)$$

(Atiyah-Hirzebruch, Totaro, Kollár, ...) counterexamples

(Voisin) True for cubic fourfolds!

$$CH^2(X) \xrightarrow{\sim} H^4(X,\mathbb{Z}) \cap H^{2,2}(X) =: A(X)$$

$A(X)$ is positive definite lattice with $h^2 \in A(X)$

Fact $A(X) = \mathbb{Z}h^2$ for X very general

K3 Surfaces

S smooth projective surface with $\omega_S \cong \mathcal{O}_S$
and $H^1(X, \mathcal{O}_X) = 0$ i.e. S not abelian surface

Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$\mathrm{Pic}(S) = CH^1(S) \xrightarrow{\sim} H^2(S, \mathbb{Z}) \cap H^{1,1}(S)$ Lefschetz (1,1)

Fact $\mathrm{Pic}(S) = \mathbb{Z}H$, H ample, for very general S

$H^2(S,\mathbb{Z})$ signature $(3,19)$ unimodular even lattice

Integral polarized weight 2 Hodge structure

Torelli theorem

IPHS on $H^2(S,\mathbb{Z})$ recovers S up to isomorphism

Moduli Spaces

$d \geq 2$ even

$\mathcal{K}_d$ moduli space of polarized K3 surfaces of degree d i.e. objects are

(S, H), S K3, $H \in \operatorname{Pic} S$ primitive, ample, $H.H = d$

From Cubic Fourfolds to K3s

X cubic fourfold with $\mathrm{rk}_{\mathbb{Z}} A(X) > 1$

then $\langle h^2, T \rangle \subseteq A(X)$ primitive sublattice rk 2

(Hassett) Let $d = \operatorname{disc} \langle h^2, T \rangle$. If $4 \nmid d$, $9 \nmid d$, $p \nmid d$ for $p \equiv -1\ (3)$

then

$$\langle h^2, T \rangle^{\perp} \cong \langle H \rangle^{\perp}(-1) \quad \text{for some polarized K3 } (S, H) \text{ of degree } d$$

$$\langle h^2, T \rangle \subset H^4(X, \mathbb{Z}), \qquad \langle H \rangle \subset H^2(S, \mathbb{Z})$$

Get embedding of moduli spaces

$$\mathcal{C}_d^{mar} \hookrightarrow \mathcal{K}_d$$

From K3s to Curves

(S, H) polarized K3 of degree $d = 2g-2$

$$\begin{array}{ccc} S & \xhookrightarrow{|H|} & \mathbb{P}^g \\ \cup\!\!\mid & & \cup\!\!\mid \\ C = S \cap \mathbb{P}^{g-1} & \xhookrightarrow{|K_C|} & \mathbb{P}^{g-1} \end{array}$$

Hyperplane sections of projective models of K3 surfaces are canonical curves.

$C \in |H|$ general is smooth proj. curve genus g.

and $H|_C = K_C$

Example

$X \subseteq \mathbb{P}^5$ cubic fourfold containing disjoint planes $P_1, P_2 \subseteq \mathbb{P}^5$

$P_1 = \{x_0 = x_1 = x_2 = 0\}$, $P_2 = \{x_3 = x_4 = x_5 = 0\}$

Then we can write

$X = \{F_1 + F_2 = 0\}$ F_1, F_2 homogeneous forms on $P_1 \times P_2$ of bidegree $(1,2)$ and $(2,1)$

Have a birational map

$$\mathbb{P}_1 \times \mathbb{P}_2 \dashrightarrow X$$

$$p_1, p_2 \longmapsto \text{third point on line through } p_1, p_2$$

Indeterminacy locus

$$S = \{F_1 = F_2 = 0\} \subseteq \mathbb{P}_1 \times \mathbb{P}_2$$

Complete intersection of bidegree $(1,2)$ and $(2,1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$

Adjunction $\Rightarrow$ S is a K3 surface

$S \subseteq \mathbb{P}^2 \times \mathbb{P}^2 \xrightarrow[\text{Segre}]{} \mathbb{P}^8$ defines degree 14 polarization $H \in \operatorname{Pic} S$

X very general $\Longrightarrow$ $\mathrm{rk}_{\mathbb{Z}} A(X) = 3$ and

$\mathrm{rk}_{\mathbb{Z}} \operatorname{Pic} S = 2$

$A(X)$	h^2	P_1	P_2
h^2	3	1	1
P_1	1	3	0
P_2	1	0	3

$\operatorname{Pic} S$	H	$L = \mathcal{O}(0,1)\|_S$
H	14	7
L	7	2

and $\langle h^2, P_1 \rangle^{\perp} \cong \langle H \rangle^{\perp}(-1)$

Finally $C \in |H|$ is genus 8 curve and $L|_C = g^2_7$

"Brill-Noether special"

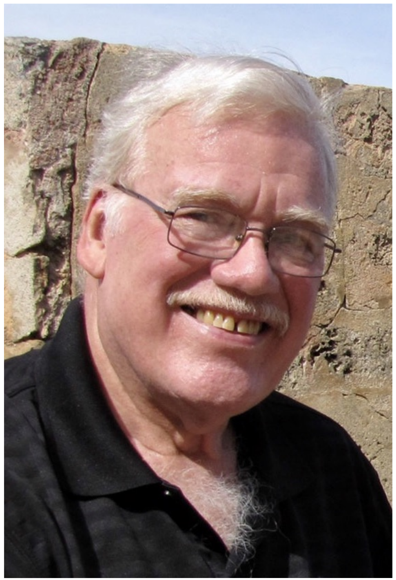
Brill-Noether theory for Cubic Fourfolds

Asher Auel (Dartmouth)

Edinburgh HODGE Seminar

Wednesday April 21, 2021

Dedicated to the memory of



Ray Hoobler

DIED 29 APR 2020

James Humphreys

DIED 27 AUG 2020

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IPHS on $H^4(X, \mathbb{Z})$ recovers X up to isomorphism

Integral Hodge Conjecture (Voisin)

$CH^2(X) \longrightarrow H^4(X, \mathbb{Z}) \cap H^{2,2}(X) =: A(X)$ isomorphism

Fact $A(X) = \mathbb{Z}h^2$ for X very general

The Noether-Lefschetz locus

$$\{ X \in \mathcal{C} : \operatorname{rk}_{\mathbb{Z}} A(X) > 1 \} = \bigcup_d \mathcal{C}_d$$

$X \in \mathcal{C}_d \iff \exists\, T \in A(X)$ with $\langle h^2, T \rangle \subseteq A(X)$ primitive sublattice of discriminant d

X special cubic fourfold of disc. d

Thm (Hassett) $\mathcal{C}_d \neq \emptyset$ irreducible divisor

$\iff d > 6$ and $d \equiv 0, 2 \pmod 6$

Hassett divisors

Description of Special Cubic Fourfolds

d	general $X \in \mathcal{C}_d$ contains
8	plane
12	cubic scroll
14	quartic scroll
18	sextic elliptic ruled surface (Addington, Hassett, Tschinkel, Várilly-Alvarado)
20	Veronese surface
$12 \leq d \leq 38$	blow-ups of $\mathbb{P}^2$ (Nuer)
42	nonic scroll (Lai)
44	Fano model of an Enriques surface (Nuer)
$d \geq 158$	$\mathcal{C}_d$ is of general type (Tanimoto, Várilly-Alvarado)

Associated K3 surface

(X, K_d) marked cubic fourfold $\quad K_d \leftrightarrow A(X) \quad d(K_d) = d$

(S, H) polarized K3 surface $\quad H \in \mathrm{Pic}(S) \quad H^2 = d$

$H^2(S,\mathbb{Z})$	wt 2	sig	$(3,19)$	1 20 1
$H^4(X,\mathbb{Z})$	wt 4	sig	$(21,2)$	0 1 21 1 0
$H^2(S,\mathbb{Z})_0 = H^\perp$	wt 2	sig	$(2,19)$	1 19 1
$H^4(X,\mathbb{Z})_0 = K_d^\perp$	wt 4	sig	$(19,2)$	0 1 19 1 0

Thm (Hassett) There exists (S, H) of degree d with

$H^4(X,\mathbb{Z})_0 \cong H^2(S,\mathbb{Z})_0(-1) \iff 4 \nmid d,\ 9 \nmid d,\ p \nmid d$ for $p \equiv 2\ (3)$

admissible $d = 14, 26, 38, 42, 62, 74, \ldots$

Rationality

Conjecture The very general X is not rational.

Some rational cubic fourfolds

- $P_1, P_2 \subseteq X$ disjoint planes $P_1 \times P_2 \overset{\sim}{\dashrightarrow} X$

  indeterminacy locus $S \subseteq P_1 \times P_2$ deg 14 K3

  lattice polarization

	H	L
H	14	7
L	7	2

  $H = \mathcal{O}(1,1)|_S \quad L = \mathcal{O}(1,0)|_S$

- (Beauville, Donagi) general $X \in \mathcal{C}_{14}$ is pfaffian

  $X = Pf \cap \mathbb{P}^5 \subseteq \mathbb{P}^{14} = \mathbb{P}(\Lambda^2 \mathbb{C}^6)$

  $S = G(2,6) \cap \mathbb{P}^8 \subseteq \mathbb{P}^{14}$ associated deg 14 K3

Brill-Noether theory

C curve genus g

How "special" is $C \in \mathcal{M}_g$?

Measured by existence of unexpected $C \xrightarrow{|g^r_e|} \mathbb{P}^r$

Brill-Noether Thm The general $C \in \mathcal{M}_g$ does not admit a g^r_e whenever

$$\rho(g,r,e) = g - (r+1)(g-e+r) < 0$$

Otherwise, we call C Brill-Noether special

Example C has gonality $e \iff C$ admits g^1_e and no g^1_{e-1}

Brill-Noether special for $g > 2e-2$

Def Clifford index $\gamma(g^r_e) = e - 2r$

$$\gamma(C) = \min\{\gamma(g^r_e) \mid r \geq 1,\ d \leq g-1\}$$

Theorem $0 \leq \gamma(C) \leq \lfloor \frac{g-1}{2} \rfloor$

Example In $g=8$ the Brill-Noether special linear systems are (up to $A \mapsto \omega_C \otimes A^\vee$)

$$\underbrace{g^1_2 \quad g^2_4 \quad g^3_6}_{\gamma = 0} \qquad \underbrace{g^1_3 \quad g^2_5 \quad g^3_7}_{\gamma = 1} \qquad \underbrace{g^1_4 \quad g^2_6}_{\gamma = 2} \qquad \underbrace{g^2_7}_{\gamma = 3}$$

Brill-Noether for K3s (Mukai)

Def A polarized K3 surface (S, H) of degree $d = 2g-2$ is Brill-Noether general if

$$h^0(S,N)\cdot h^0(S,M) < h^0(S,H) = g+1$$

for any nontrivial decomposition $H = N \otimes M$ in $\mathrm{Pic}(S)$

Analogy C is Brill-Noether general if

$$h^0(C,L)\cdot h^0(C, L^\vee \otimes \omega_X) \leq h^0(C,\omega_X) = g$$

for every $L \in \mathrm{Pic}(C)$.

Example $\mathrm{Pic}(S) = \mathbb{Z}H \implies (S,H)$ is BN-general

Theorem (Lazarsfeld)
$\mathrm{Pic}(S) = \mathbb{Z}H \implies$ general $C \in |H|$ is BN-general

Observation
$C \in |H|$ is BN-general $\implies (S,H)$ is BN-general

Theorem (Mukai) "$\Leftarrow$" holds for $g \leq 10$ or $g = 12$
(R. Haburcak) "$\Leftarrow$" holds for $g \leq 14$
(Saint-Donat, Reid, Green–Lazarsfeld, Mertens, Knutsen, Lelli-Chiesa)

The Noether-Lefschetz locus in $\mathcal{K}_d$

$$\{ (S,H) \in \mathcal{K}_d \mid \mathrm{rk}_{\mathbb{Z}} \operatorname{Pic}(S) > 1 \} = \bigcup_{e,n} \mathcal{K}^d_{e,n}$$

where $(S,H) \in \mathcal{K}^d_{e,n}$ admit primitive $\begin{array}{c|cc} & H & L \\ \hline H & d & e \\ L & e & n \end{array} \subseteq \operatorname{Pic}(S)$

Thm (Greer, Li, Tian)

The Brill-Noether special locus $\mathcal{K}^{BN}_d \subseteq \mathcal{K}_d$ is the union of $\mathcal{K}^d_{e,n}$ where

$$\sqrt{dn} < e < \min\left\{ \frac{d}{2},\ 1 + \frac{d+n}{2} - \frac{d+4}{n+4} \right\}$$

Example $\mathcal{K}_{14}^{BN}$ has 6 components

$\mathcal{K}_{1,0}^{14}$

	H	L
H	14	1
L	1	0

elliptic w/ section

H has nontrivial fixed part

$\mathcal{K}_{2,0}^{14}$

	H	L
H	14	2
L	2	0

$L|_C$ is g^1_2

$\gamma = 0$

$\mathcal{K}_{3,0}^{14}$

	H	L
H	14	3
L	3	0

$L|_C$ is g^1_3

$\gamma = 1$

$\mathcal{K}_{4,0}^{14}$

	H	L
H	14	4
L	4	0

$L|_C$ is g^1_4

$\mathcal{K}_{6,2}^{14}$

	H	L
H	14	6
L	6	2

$L|_C$ is g^2_6

$\gamma = 2$

$\mathcal{K}_{7,2}^{14}$

	H	L
H	14	7
L	7	2

$L|_C$ is g^2_7

$\gamma = 3$

H base point free $\Rightarrow$ general $C \in |H|$ smooth

Brill-Noether Special Cubic Fourfolds

A cubic fourfold X is Brill-Noether special if it admits a Brill-Noether special associated K3.

$\mathcal{C}_d^{BN}$ marked cubic fourfolds (X, K_d) with $H^4(X,\mathbb{Z})_0 \cong H^2(S,\mathbb{Z})_0(-1)$ and (S,H) BN-special.

$$\begin{array}{ccc} \mathcal{C}_d^{mar} & \hookrightarrow & \mathcal{K}_d \\ \uparrow & & \uparrow \\ \mathcal{C}_d^{BN} & \hookrightarrow & \mathcal{K}_d^{BN} \end{array}$$

Program Relate the geometry of $X \in \mathcal{C}_d^{BN}$ with (S,H)

Question Which BN-special K3s are associated to (smooth) cubic fourfolds?

Theorem The image of $\mathcal{C}_{14}^{BN} \hookrightarrow \mathcal{K}_{14}^{BN}$ is $\mathcal{K}_{7,2}^{14}$ and $\mathcal{C}_{14}^{BN}$ coincides with the locus of cubic fourfolds containing disjoint planes.

Consequences

- If (S,H) is associated to $X \in \mathcal{C}_{14}$ then $C \in |H|$ has gonality $= 5$ $(\gamma(C) = 3)$
- $X \in \mathcal{C}_{14}$ is BN-general $\Leftrightarrow$ X is pfaffian

Why are (smooth) cubic fourfolds not very Brill-Noether special?

$$\mathcal{C}_d^{mar} \longrightarrow \mathcal{K}_d$$

$$(X, K_d) \longmapsto (S, H)$$

lattice polarization

$K_d \subseteq \Lambda \subseteq A(X)$

BN-special lattice polarization

$H \in \Sigma \subseteq \operatorname{Pic} S$

Accident? $\exists$ roots in $\Lambda \Leftarrow \gamma$ "low"

$\Updownarrow$

X singular

Questions

- Define BN-special cubic fourfolds intrinsically? In terms of $\mathcal{A}_X \subseteq D^b(X)$?

- Lazarsfeld–Mukai bundles $E_{C,L}$ on S. Can we see them naturally in $D^b(S) \subseteq D^b(X)$?

- Lower bounds on the gonality (or Clifford index) of BN-special $X \in \mathcal{C}_d$ for $d \to \infty$?