NOETHER–LEFSCHETZ GENERAL COMPLETE INTERSECTION K3 SURFACES OVER $\mathbb{Q}$

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Abstract. We prove that the locus of Noether–Lefschetz general polarized K3 surfaces of degree d defined over $\mathbb{Q}$ is Zariski dense in the moduli space for $d \leq 8$. Previously, this was proved by van Luijk in the quartic case, and it follows from work of Elsenhans and Jahnel in the degree 2 case. Innovations on their methods, and employing Mukai's Hodge isometry, suffices to handle the degree 8 case. New methods allow us to deal with the case of degree 6.

Introduction

For each even $d \geq 2$, the moduli space $\mathcal{K}_d$ of polarized K3 surfaces of degree d is a 19 dimensional quasiprojective variety. Away from a countably infinite union of closed divisors in $\mathcal{K}_d$, a polarized K3 surface of degree d has Picard rank 1 over $\mathbb{C}$. These are the Noether–Lefschetz general K3 surfaces. We consider the following open question, cf. [25, Ch. 17, p. 408-409], [46, Remark 1.3.7], [8, Problems 1.4–1.5].

Question 1. *For a given even $d \geq 2$, does there exist a K3 surface S of degree d defined over $\mathbb{Q}$ such that $\mathrm{Pic}(S_{\mathbb{C}}) \cong \mathbb{Z}$?*

In other words, are there $\mathbb{Q}$-points on the Noether–Lefschetz general locus in $\mathcal{K}_d$? Since this locus is the complement of infinitely many divisors, it is not *a priori* clear whether it contains rational points over any given countable field. However, Ellenberg [15] proved that the Noether–Lefschetz general locus in $\mathcal{K}_d$ does admit $\overline{\mathbb{Q}}$-points for every $d \geq 2$. Earlier, results of Terasoma [48] gave a positive answer to Question 1 for $d = 4, 6, 8$, namely, when the general K3 surface of degree d is a complete intersection. We remark that a conjecture of Shafarevich [45], which predicts that only finitely many isomorphisms classes of lattices arise as geometric Picard lattices of K3 surfaces defined over a fixed number field, would imply that Question 1 has a positive answer for only finitely many d. Shafarevich proved the conjecture for K3 surfaces over a number field of geometric Picard rank 20. Orr and Skorobogatov [42], and Orr [43], proved Shafarevich's conjecture for K3 surfaces of CM type. However, the conjecture seems wide open for K3 surfaces of geometric Picard rank one.

In a similar vein, since $\mathcal{K}_d$ becomes of general type for $d > 122$ by the work of Gritsenko, Hulek, and Sankaran [21, Theorem 6.1], the Bombieri–Lang conjecture would predict that the set of rational points is not Zariski dense, making rational points on the Noether–Lefschetz general locus increasingly rare as d grows. In contrast, when d is small, $\mathcal{K}_d$ is unirational and we expect many rational points. Our main result completes the picture for complete intersection K3 surfaces.

Theorem 1. *The set of Noether–Lefschetz general K3 surfaces defined over $\mathbb{Q}$ is Zariski dense in $\mathcal{K}_d$ for $d \leq 8$.*

Van Luijk [50] proved this in degree 4 by a pioneering method of leveraging the Weil and Tate conjectures, together with properties of the specialization homomorphism for the Picard group, to construct specific quartic K3 surfaces over $\mathbb{F}_p$ and $\mathbb{F}_{p'}$ with geometric Picard rank 2 and incompatible Picard lattices, forcing a common lift to $\mathbb{Q}$ to have geometric Picard rank 1. Kloosterman [30] showed that in van Luijk's method, the specific K3 surface modulo the second prime p' could be traded for information coming from the Artin–Tate formula. Elsenhans and Jahnel [16, 17] adapted van Luijk's strategy for K3 surface of degree 2, and then developed a new general technique [18] for K3 surfaces (and applied is specifically in degree 2) that only required working modulo a single prime, see also [24, Proposition 5.3]. These authors were able to present the first explicit examples of Noether–Lefschetz general K3 surfaces over $\mathbb{Q}$; the case of quartic K3s had been a long-standing challenge due to Mumford, with van Luijk's work yielding the first known explicit examples, see [51, § 2.4-2.6]. As already utilized in [50, p. 12], the nature of the explicit constructions, by lifting specific K3 surfaces over a finite field, yields the Zariski density in Theorem 1 for $d = 2, 4$, see Section 4 for further details. For $d = 8$, an argument (see Section 2) involving Mukai's degree 8 to degree 2 isogeny, together with examples first due to Elsenhans and Jahnel [18, § 8], yields the result. Finally, we must further develop the techniques of van Luijk and Elsenhans–Jahnel for the application to K3 surfaces of degree 6.

A K3 surface of degree 6 is a complete intersection $X = X_{2,3} \subset \mathbb{P}^4$. We specifically consider such X whose reduction X_p over $\mathbb{F}_p$ contains a line, and then projection from the line yields a double cover $X_p \to \mathbb{P}^2_{\mathbb{F}_p}$. Generically, this degree 2 model does not contain a tritangent line, so we cannot use the techniques of [18] directly. Instead, we prove a generalization of [24, Proposition 5.3] to lines contained in amply embedded K3 surfaces. After checking that $\rho(X_p) = 2$, we lift to a surface $X \subset \mathbb{P}^4$ over $\mathbb{Q}$ that does not contain any lines by a verification using Groebner bases.

In degree 8, we consider complete intersections $X \subset \mathbb{P}^5$ of three quadrics over $\mathbb{Q}$ that are Hodge-isogenous to degree 2 K3 surfaces Y over $\mathbb{Q}$. When the reduction Y_p over $\mathbb{F}_p$ admits a tritangent line and has geometric Picard rank two, yet Y does not admit a tritangent line, then Y (and hence X) will have geometric Picard rank one. For further details and our examples, see Section 2; an earlier example, working with two different primes, is supplied in [18, § 8].

While this work increases the degrees of known Noether–Lefschetz general K3 surfaces, it remains open whether there are any of even degree greater than 8.

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1. Prerequisites

A *K3 surface* is a smooth, projective, geometrically integral surface X over a field k whose canonical bundle ω_X on X is trivial and $H^1(X, \mathcal{O}_X) = 0$. A *polarized K3 surface* is a pair (X, H) where X is a K3 surface and $H \in \mathrm{Pic}(X)$ is an ample divisor class.

We say that the polarization is of *degree* d if $H^2 = d$. For each even $d \geq 2$, the moduli space $\mathcal{K}_d$ of polarized K3 surfaces of degree d is a 19-dimensional quasiprojective variety. For integers r, e, we define the *Noether-Lefschetz divisor* $\mathcal{K}^r_{d,e} \subset \mathcal{K}_d$ to be the locus of polarized K3 surfaces (X, H) of degree d such that $\mathrm{NS}(X)$ contains a primitive rank 2 sublattice spanned by H and L with Gram matrix

X	H	L
H	d	e
L	e	r

When r is even and $dr - e^2 < 0$ then $\mathcal{K}^r_{d,e} \subset \mathcal{K}_d$ is a nonempty irreducible divisor by [41], cf. [13], [20], [2].

A subvariety $X \subseteq \mathbb{P}^n$ is a *complete intersection* if the ideal of X is generated by exactly $\operatorname{codim} X$ elements. The adjunction formula and the Lefschetz hyperplane theorem imply that the only complete intersection K3 surfaces are a quartic in $\mathbb{P}^3$, an intersection of a quadric and a cubic in $\mathbb{P}^4$, and the intersection of three quadrics in $\mathbb{P}^5$.

The *Picard rank* of X is the rank $\rho(X) = \mathrm{rk}_{\mathbb{Z}} \mathrm{NS}(X)$ of the Néron–Severi group $\mathrm{NS}(X)$ and the *geometric Picard rank* of X is $\mathrm{rk}_{\mathbb{Z}} \mathrm{NS}(\overline{X})$, where $\overline{X} = X \times_{\operatorname{Spec} k} \operatorname{Spec} \overline{k}$ for a choice of algebraic closure $\overline{k}$ of k. For a K3 surface X over k, we have that $\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong \mathbb{Z}^{\rho(X)}$ is an even lattice of signature $(1, \rho(X) - 1)$ with respect to the intersection product, with $1 \leq \rho(X) \leq 22$, see [25, Section 17.2].

Now, let X be a K3 surface over $\mathbb{Q}$ and p be a prime. If there exists a proper flat model of X over $\mathbb{Z}$ with a smooth fiber X_p over $\mathbb{F}_p$, we say that p is a prime of *good reduction* for X. We write $\overline{X}_p = X_p \times_{\operatorname{Spec} \mathbb{F}_p} \operatorname{Spec} \overline{\mathbb{F}}_p$. For a prime p of good reduction, there exist *specialization homomorphisms*

$$\mathrm{sp} : \mathrm{NS}(X) \to \mathrm{NS}(X_p) \quad \text{and} \quad \overline{\mathrm{sp}} : \mathrm{NS}(\overline{X}) \to \mathrm{NS}(\overline{X}_p)$$

which are injective and respect the intersection product, cf. [19, §20.3], [49, Proposition 6.2], [35, §3.2]. Moreover, we have the following.

Lemma 1.1 (Elsenhans and Jahnel [18, Corollary 3.7])**.** *Let X be a proper scheme over $\mathbb{Q}$ and $p \neq 2$ be a prime of good reduction. Then the cokernel of the specialization homomorphism $\overline{sp} : \mathrm{NS}(\overline{X}) \to \mathrm{NS}(\overline{X}_p)$ is torsion free.*

For a variety X over $\mathbb{F}_p$, we have the *absolute Frobenius* morphism $F : X \to X$ which is the identity on the topological space of X and acts by $u \mapsto u^p$ on the structure sheaf. Let F^* be its pullback on $H^2_{\text{ét}}(\overline{X}_p, \mathbb{Q}_\ell)$. We call the characteristic polynomial of F^* the *Weil polynomial* of X_p. By the Weil conjectures, it is a polynomial of degree 22 whose roots are algebraic integers with absolute value p.

For a K3 surface X_p over $\mathbb{F}_p$, the cycle class map $\mathrm{NS}(\overline{X}_p) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell \to H^2_{\text{ét}}(\overline{X}_p, \mathbb{Q}_\ell(1))$ is a Frobenius equivariant injection, cf. [35], [49, Proposition 6.2]. By observing that each divisor class over $\overline{\mathbb{F}}_p$ is defined over some finite extension $\mathbb{F}_{p^m}$ of $\mathbb{F}_p$, each divisor is fixed by F^m for some positive integer m. Thus, if we let $F^*(1)$ be the induced map (of $\mathbb{Q}_\ell$-vector spaces) on $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(1))$, under the cycle class map, each divisor class becomes an eigenvector of $F^*(1)$ with eigenvalue a root of unity. Thus, by the injectivity of the cycle class map, the $\mathbb{Q}_\ell$-dimension of $\mathrm{Pic}(\overline{X}_p) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$, and hence the $\mathbb{Z}$-rank of $\mathrm{NS}(\overline{X}_p)$, is bounded above by the number of eigenvalues of $F^*(1)$ that are roots of unity, counted with multiplicity. Eigenvalues of $F^*(1)$ differ from eigenvalues of F^* by a factor of p, so the geometric Picard rank $\rho(\overline{X})$ is bounded above by the number of eigenvalues λ of F^* for which λ/p is a root of unity, counted with multiplicity, cf. [50, Corollary 2.3]. In fact, a consequence of the Tate conjecture for K3 surfaces, proved by Nygaard–Ogus [40], Maulik [34], Charles [9,10], Madapusi Pera [32], Kim–Madapusi Pera [29,33], and Ito–Ito–Koshikawa [27], gives the following.

Theorem 1.2. *For a K3 surface X defined over $\mathbb{F}_q$, the geometric Picard rank $\rho(\overline{X})$ is equal to the number of eigenvalues λ of F^* for which λ/q is a root of unity, counted with multiplicity.*

There are algorithms to compute the Weil polynomial of a K3 surface over $\mathbb{F}_p$ given a degree 2 model (due to Elsenhans and Jahnel [18] and implemented in `Magma` [7]) or given a model as a hypersurface in toric varieties (due to Kedlaya [28], Abbott–Kedlaya–Roe [1], and Costa–Harvey–Kedlaya [11]). Over small finite fields, it can be faster to use point counting techniques, whereas over larger finite field, p-adic methods are usually faster.

2. Degree 8 Case

A general polarized K3 surface (X, H) of degree 8 is a complete intersection of three quadrics in $\mathbb{P}^5$. In this case, the space of quadrics $\mathbb{P}^2 = \mathbb{P}(H^0(\mathbb{P}^5, \mathscr{I}_X(2)))$ containing X, called the net of quadrics associated to X, is spanned by the three quadrics. The locus $C \subseteq \mathbb{P}^2$ of degenerate quadrics in the net is a sextic curve. Let $f : Y \to \mathbb{P}^2$ be the discriminant double cover of the net, which is branched over C. The discriminant cover can be defined either in terms of the center of the even Clifford algebra, cf. [31, §3], [4, §1.6, Appendix B], or the Stein factorization of the relative moduli space of maximal isotropic subspaces, cf. [23, §3], [36, §3.2], associated to the net. When C is smooth, $(Y, f^*\mathscr{O}_{\mathbb{P}^2}(1))$ is a polarized K3 surface of degree 2, called the *discriminant* K3 surface associated to X.

Explicitly, over a field k of characteristic $\neq 2$, if $X = V(q_0, q_1, q_2) \subset \mathbb{P}^5 = \mathbb{P}(V)$ for quadratic forms q_0, q_1, q_2 on V, then we consider the *linear span* quadratic form $q = x_0q_0 + x_1q_1 + x_2q_2$ on $V \otimes_k k[x_0, x_1, x_2]$, see [4, Definition 1.2.3]. The signed discriminant $\mathrm{disc}(q) = -\det(x_0Q_0 + x_1Q_1 + x_2Q_2) \in k[x_0, x_1, x_2]$, where Q_i the symmetric Gram matrix associated to the quadratic form q_i, is then a homogeneous form of degree 6 and the discriminant K3 surface $Y \subset \mathbb{P}(1, 1, 1, 3)$ is the variety defined by $y^2 = \mathrm{disc}(q)$, with the double cover $Y \to \mathbb{P}^2$ given by projection away from the last coordinate, see [36, §5].

Proposition 2.1. *The formation of the discriminant K3 surface determines a dominant rational map $\Phi : \mathcal{K}_8 \dashrightarrow \mathcal{K}_2$.*

Proof. To obtain a rational map, the only question is whether the above explicit construction is defined on an open in $\mathcal{K}_8$. The construction is well defined on the locus in $\mathcal{K}_8$ where the polarization H embeds X as a smooth complete intersection of 3 quadrics whose discriminant K3 is smooth. This locus is the complement of finitely many Noether–Lefschetz divisors. Indeed, the failure of smoothness occurs when the polarization fails to be very ample, which occurs on the unigonal locus $\mathcal{K}_{8,1}^2$ or the hyperelliptic locus $\mathcal{K}_{8,2}^0$, see [44, §5, §8]; further, the failure of being a complete intersection occurs on the trigonal locus $\mathcal{K}_{8,3}^0$, see [44, Theorem 7.2]; when X is smooth, the discriminant K3 can acquire at most ordinary double points, see [5, §6.2], and this occurs on $\mathcal{K}_{8,1}^{-2}$ (containing a line) or $\mathcal{K}_{8,2}^{-2}$ (containing a conic). Finally, we remark that the rational map is dominant by [5, Proposition 6.23]. $\square$

In particular, there exist open sets $\mathcal{K}_8^\circ \subset \mathcal{K}_8$ and $\mathcal{K}_2^\circ \subset \mathcal{K}_2$ such that $\Phi : \mathcal{K}_8^\circ \to \mathcal{K}_2^\circ$ is a dominant morphism.

When a complete intersection K3 surface $X \subset \mathbb{P}^5$ of degree 8 is defined over $\mathbb{Q}$, the explicit construction of the discriminant K3 shows that $Y \to \mathbb{P}^2$ is also defined over $\mathbb{Q}$. Over $\mathbb{C}$, Mukai [38, Example 0.9], [39, Example 2.2] shows that $Y = M_X(v)$ is a moduli space of stable sheaves $\mathscr{E}$ on X with Mukai vector $v = (2, H, 2)$, i.e., sheaves on X with rank 2, first Chern class H, and second Chern class 4. This implies, cf. [37, Corollary 6.5], the existence of an algebraic cycle that induces an inclusion $T_X \hookrightarrow T_Y$ with cokernel $\mathbb{Z}/2\mathbb{Z}$. In particular, if X (and hence Y) is defined over $\mathbb{Q}$, then X and Y have the same geometric Picard rank.

This can also be seen by observing that Y is a *twisted Fourier–Mukai partner* of X. We briefly review this theory and relevant results. A more complete introduction can be found in [25, Section 16.4]. The *Mukai lattice* of a K3 surface X over $\mathbb{C}$ is

$$\widetilde{H}(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

and admits a pure Hodge structure of weight 2 with

$$\begin{aligned} \widetilde{H}^{1,1}(X) &= H^0(X) \oplus H^{1,1}(X) \oplus H^4(X) \\ \widetilde{H}^{2,0}(X) &= H^{2,0}(X). \end{aligned}$$

For $v \in \widetilde{H}(X, \mathbb{Z})$, write $M_X(v)$ for the moduli space of stable sheaves $\mathscr{E}$ on X with Mukai vector $v(\mathscr{E}) = \mathrm{ch}(\mathscr{E})\mathrm{Td}_X^{1/2} = (\mathrm{rk}(\mathscr{E}), c_1(\mathscr{E}), c_1(\mathscr{E})^2/2 - c_2(\mathscr{E}) + \mathrm{rk}(\mathscr{E})) = v$. If the moduli space is fine and $v^2 = 0$, then $M_X(v)$ is also a K3 surface, see [37, Theorem 1.4], and X and $M_X(v)$ are *Fourier–Mukai partners*, where the universal sheaf determines a Fourier–Mukai equivalence $\mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(M_X(v))$ of bounded derived categories.

In general, $Y = M_X(v)$ is not a fine moduli space, and X and Y are *twisted Fourier–Mukai partners*. Indeed, the universal sheaf on $X \times Y$ is $p^*\alpha$-twisted for $\alpha \in \mathrm{Br}(Y)$ where $p : X \times Y \to Y$, see [36, Section 3.2, Lemma 15], and hence determines a twisted Fourier–Mukai equivalence $\mathsf{D}^{\mathrm{b}}(X) \cong \mathsf{D}^{\mathrm{b}}(Y, \alpha)$, see [12, Theorem 1.3]. Thus, X and Y have Hodge isometric *twisted Mukai lattices*, see [25, Prop 16.4.2]. Indeed, choosing a B-field lift $B \in H^2(Y, \mathbb{Q})$ of a Brauer class $\alpha \in \mathrm{Br}(Y)$, see [25, Definition 16.4.1], the twisted Mukai lattice $\widetilde{H}(Y, B, \mathbb{Z})$ is defined to be the same abelian group as $\widetilde{H}(Y, \mathbb{Z})$, but with a different Hodge decomposition

$$\widetilde{H}^{p,q}(Y, B) = \exp(B) \cdot H^{p,q}(Y).$$

The integral (1,1)-part of this decomposition, $\widetilde{H}^{1,1}(Y, B, \mathbb{Z})$, is called the generalized Picard group $\mathrm{Pic}(Y, B)$ in [26, p. 912], and it always has the same rank as $\mathrm{Pic}(Y \oplus H^0(Y, \mathbb{Z}) \oplus H^4(Y, \mathbb{Z})$, see [26, p. 913]. Thus two K3 surfaces defined over $\mathbb{Q}$ that are twisted Fourier–Mukai partners over $\mathbb{C}$ have the same geometric Picard rank. Either way of looking at it, we have the following.

Proposition 2.2. *For a smooth complete intersection $X \subset \mathbb{P}^5$ over $\mathbb{Q}$ whose discriminant cover $Y \to \mathbb{P}^2$ of the associated net of quadrics is smooth, we have that $\rho(\overline{X}) = \rho(\overline{Y})$.*

Proof. We remark that if X is defined over $\mathbb{Q}$ then so is Y. We have shown through two methods above that $\rho(X_{\mathbb{C}}) = \rho(Y_{\mathbb{C}})$, which gives our result over $\overline{\mathbb{Q}}$ in view of rigidity for the Picard group, cf. [35, Proposition 3.1]. □

Finally, we give a condition to ensure that a K3 surface of degree 8 over $\mathbb{Q}$ is Noether–Lefschetz general.

Proposition 2.3. *Let $X \subset \mathbb{P}^5_{\mathbb{Q}}$ be a smooth complete intersection of three quadrics whose associated discriminant $Y \to \mathbb{P}^2$ is smooth, and let $Y_p \to \mathbb{P}^2_{\mathbb{F}_p}$ be the reduction modulo p for a prime $p > 2$ of good reduction of Y. If $\rho(\overline{Y}_p) = 2$, the branch locus of $\overline{Y}_p \to \mathbb{P}^2_{\mathbb{F}_p}$ has a tritangent line, and the branch locus of $\overline{Y}$ has no tritangent line, then $\rho(\overline{X}) = \rho(\overline{Y}) = 1$.*

Proof. We follow the strategy outlined in [18, Example 1.7], [24, Proposition 5.3], and [36, Proposition 26] to prove that $\rho(\overline{Y}) = 1$, and then appeal to Proposition 2.2. Indeed, by Proposition 2.2 and the injectivity of the specialization map, we have that $\rho(\overline{X}) = \rho(\overline{Y}) \leq 2$. Assume, to get a contradiction, that $\rho(\overline{X}) = \rho(\overline{Y}) = 2$. In this case, the image of $\overline{\mathrm{sp}} : \mathrm{Pic}(\overline{Y}) \to \mathrm{Pic}(\overline{Y}_p)$ has rank 2, so has torsion cokernel. By Lemma 1.1, the cokernel is torsion free, so it is 0, hence $\overline{\mathrm{sp}}$ is surjective.

Now, the pullback of the tritangent line to $\overline{Y}_p$ splits into two irreducible components. Let L be the divisor class of one of these components, and let $L' \in \mathrm{Pic}(\overline{Y})$ be a lift of L under $\overline{\mathrm{sp}}$. As in [18, Example 1.7], L then must arise from a tritangent line to the branch locus of $Y \to \mathbb{P}^2$, but there are none by hypothesis, giving a contradiction. □

To find explicit X satisfying the hypotheses of Proposition 2.3 for a fixed prime p, one can first iterate over complete intersections $X_p \subset \mathbb{P}^5_{\mathbb{F}_p}$ and check whether the associated discriminant double cover Y_p admits a tritangent. We then use the techniques of Section 1 to check whether $\rho(X_p) = 2$. One can find examples after a number of iterations:

Example 2.4. Let $X_{47} = V(q_1, q_2, q_3) \subset \mathbb{P}^5_{\mathbb{F}_{47}}$, where

$$\begin{aligned}
q_1 = {} & 5x_0^2 + 6x_0x_1 + x_1^2 + 10x_0x_2 + 45x_1x_2 + x_2^2 + 6x_0x_3 \\
& + 45x_1x_3 + 45x_2x_3 + 37x_0x_4 + 39x_1x_4 + 39x_2x_4 \\
& + 2x_4^2 + 10x_0x_5 + 10x_1x_5 + 45x_2x_5 + 8x_4x_5 + 2x_5^2 \\
q_2 = {} & 43x_0^2 + 41x_0x_1 + 41x_0x_2 + 8x_1x_2 + 44x_2^2 + 2x_0x_3 \\
& + 39x_1x_3 + 5x_3^2 + 45x_0x_4 + 2x_1x_4 + 45x_2x_4 + 10x_3x_4 + 3x_4^2 \\
& + 43x_0x_5 + 6x_1x_5 + 2x_2x_5 + 2x_3x_5 + 10x_4x_5 + 3x_5^2 \\
q_3 = {} & 5x_0^2 + 45x_0x_1 + 46x_1^2 + 37x_0x_2 + 2x_1x_2 + 4x_2^2 + 2x_0x_3 \\
& + 8x_1x_3 + 6x_2x_3 + 42x_3^2 + 8x_0x_4 + 39x_1x_4 + 43x_2x_4 + 4x_3x_4 \\
& + 2x_1x_5 + 43x_2x_5 + 43x_3x_5 + 39x_4x_5 + 44x_5^2
\end{aligned}$$

and $(x_0 : \ldots : x_5)$ are homogeneous coordinates on $\mathbb{P}^5$. Then X_{47} is a smooth complete intersection whose discriminant double cover $Y_{47} \to \mathbb{P}^2$ is smooth with branch locus defined by the vanishing of

$$\begin{aligned}
& 13u^6 + 43u^5v + 19u^4v^2 + 7u^3v^3 + 46u^2v^4 + 11uv^5 + 21v^6 + 43u^5w + 22u^4vw \\
& + u^3v^2w + 27u^2v^3w + 17uv^4w + 41v^5w + 26u^4w^2 + 42u^3vw^2 + 33u^2v^2w^2 \\
& + 46uv^3w^2 + 19v^4w^2 + 42u^3w^3 + 8u^2vw^3 + 34uv^2w^3 \\
& + 17v^3w^3 + 41u^2w^4 + 32uvw^4 + 21v^2w^4 + 46uw^5 + 33vw^5 + 17w^6,
\end{aligned}$$

where u, v, w are homogenous coordinates for $\mathbb{P}^2$. The discriminant double cover Y_{47} admits a tritangent line and $\rho(\overline{X}_{47}) = \rho(\overline{Y}_{47}) = 2$, which we verify by computing the Weil polynomial for Y_{47} to be

$$\begin{aligned}(t-47)^2(t^{20} - 15t^{19} - 2491t^{18} + 92778t^{17} + 2387929t^{16} - 34157767t^{15} - 6421660196t^{14}\\ - 53896076645t^{13} + 27357648505002t^{12} + 95245146647044t^{11} - 49241740816521748t^{10}\\ + 2103965289433320196t^9 + 13349659761453666 4362t^8 - 580957415544742891205t^7\\ - 15290799177137632896515 6t^6 - 17966689033136718653405 83t^5\\ + 277457012068868469490452889t^4 + 23813049644519406903224087082t^3\\ - 14123406348689962522840762124 11t^2 - 187867952374083463747221458443 35t\\ + 276666871196233580945074801134 2401).\end{aligned}$$

With such an example in hand, we can choose random lifts X to $\mathbb{Z}$, then using a Gröbner basis algorithm, cf. [16, Algorithm 8], we can quickly check whether the discriminant double cover Y admits a tritangent line over $\overline{\mathbb{Q}}$.

Corollary 2.5. *Let* $X = V(q_1, q_2, q_3) \subset \mathbb{P}^5_{\mathbb{Q}}$, *where*

$$\begin{aligned}q_1 &= -136x_0^2 - 464x_0x_1 - 140x_1^2 - 272x_0x_2 + 374x_1x_2 + x_2^2 + 288x_0x_3 + 186x_1x_3\\ &\quad + 468x_2x_3 + 47x_3^2 - 292x_0x_4 - 196x_1x_4 + 274x_2x_4 - 188x_3x_4 + 237x_4^2 - 84x_0x_5\\ &\quad + 386x_1x_5 + 562x_2x_5 - 282x_3x_5 - 274x_4x_5 - 139x_5^2\\ q_2 &= 43x_0^2 + 88x_0x_1 + 141x_1^2 - 100x_0x_2 + 384x_1x_2 + 185x_2^2 - 280x_0x_3 - 8x_1x_3\\ &\quad - 376x_2x_3 - 89x_3^2 + 562x_0x_4 + 190x_1x_4 + 562x_2x_4 + 104x_3x_4 + 144x_4^2 - 98x_0x_5\\ &\quad - 182x_1x_5 - 468x_2x_5 + 190x_3x_5 - 84x_4x_5 - 44x_5^2\\ q_3 &= 193x_0^2 - 2x_0x_1 + 234x_1^2 - 292x_0x_2 + 190x_1x_2 + 51x_2^2 + 2x_0x_3 - 180x_1x_3\\ &\quad + 6x_2x_3 + 183x_3^2 + 8x_0x_4 + 274x_1x_4 + 184x_2x_4 + 286x_3x_4 + 470x_0x_5 - 280x_1x_5\\ &\quad + 560x_2x_5 + 90x_3x_5 - 196x_4x_5 + 185x_5^2\end{aligned}$$

Then $X \subset \mathbb{P}^5$ *is a smooth complete intersection K3 surface of degree 8 over* $\mathbb{Q}$ *with geometric Picard rank 1.*

We note that Elsenhans and Jahnel [17, §8] give an example of a K3 surface $Y \to \mathbb{P}^2$ defined over $\mathbb{Q}$ with geometric Picard rank 1, whose sextic branch curve $C \subset \mathbb{P}^2$ is the determinant of a 6×6 matrix of linear forms, and whose reductions modulo p are smooth and admit a tritangent line. From these examples, and using Proposition 2.3, one can also extract K3 surfaces of degree 8 over $\mathbb{Q}$ with geometric Picard rank 1. A similar example can be found in [36, §5.4].

Remark 2.6. Note that the locus of polarized K3 surfaces of degree 2 admitting a line tritangent to its branch locus is a Noether–Lefschetz divisor $\mathcal{K}_{2,1}^{-2} \subset \mathcal{K}_2$. In particular, by Proposition 2.1, we see that the locus in $\mathcal{K}_8$, consisting of polarized K3 surfaces X of degree 8 whose projective model is a smooth complete intersection of three quadrics in $\mathbb{P}^5$ and whose discriminant $Y \to \mathbb{P}^2$ is smooth and admits a tritangent line, is Zariski closed in $\mathcal{K}_8^\circ$. We will use this in Section 4 in the proof that the set of Noether–Lefschetz general K3 surfaces is Zariski dense in $\mathcal{K}_8$.

3. Degree 6 Case

Every basepoint free nonhyperelliptic polarized K3 surface (X, H) of degree 6 is a complete intersection of type $(2, 3)$ in $\mathbb{P}^4$, i.e., a complete intersection of a quadric and cubic hypersurface, see [44, Theorem 6.1]. To bound the geometric Picard rank of X, one could follow the strategy of van Luijk [50], computing the Weil polynomial of a reduction X_p at a prime of good reduction and applying Theorem 1.2 in view of the injectivity of the specialization map. To compute the Weil polynomial, one could hope to count points over $\mathbb{F}_{p^n}$ for $n = 1, \ldots, 12$. However, a naive point counting algorithm that enumerates over all points in $\mathbb{P}^4$ would be on the border of computational feasibility. In theory, p-adic algorithms to compute the Weil polynomial of a complete intersection exist, but are not publicly implemented. Instead, in the spirit of [3, §3], we use the geometry of linear projections to design a faster point counting algorithm.

We first recall that any ample divisor L on a K3 surface with L^2 a big and nef divisor L on a K3 surface with $L^2 = 2$ determines a morphism $X \to \mathbb{P}^2$, which is a blow-down followed by a double cover as long as (X, H) is not unigonal, i.e., there exists no divisor E with $L.E = 1$ and $E^2 = 0$. In the later case, $(L - 2E)^2 = -2$ and $L.(L - 2E) = 0$, hence L has Zariski decomposition $L = 2E + (L - 2E)$ and moving part $2E$. In particular, the map $X \to \mathbb{P}^2$ induced by L factors through the elliptic fibration $X \to \mathbb{P}^1$ induced by E and a Veronese embedding $\mathbb{P}^1 \subset \mathbb{P}^2$, so that $X \to \mathbb{P}^2$ is not dominant. When the map $X \to \mathbb{P}^2$ induced by L is dominant, then any (-2)-curve $C \subset X$ with $L.C = 0$ is blown down, and the map $X \to \mathbb{P}^2$ factors as a sequence of blow-downs $X \to X_0$ followed by a double cover $X_0 \to \mathbb{P}^2$. By Riemann–Hurwitz (and assuming we are in characteristic $\neq 2$), the branch locus of this double cover must be a sextic curve, which is smooth if and only if there are no blown-down curves.

With this in mind, we recall the following simplification of the geometry of a surface in $\mathbb{P}^4$ that contains a line.

Lemma 3.1. *Let $X \subset \mathbb{P}^4$ be a nondegenerate (a, b) complete intersection surface with hyperplane section H. If the smooth locus of X contains a line L, then projection from L restricts to a dominant generically finite morphism $X \to \mathbb{P}^2$ of generic degree $(a - 1)(b - 1)$ induced by the linear system of $H - L$.*

Proof. Resolving the projection from L yields a morphism $\phi : \mathrm{Bl}_L \mathbb{P}^4 \to \mathbb{P}^2$. As $L \subset X$ is a smooth divisor in the smooth locus, the strict transform coincides with X, giving an embedding $X \subset \mathrm{Bl}_L \mathbb{P}^4$ and we consider the restriction $\phi|_X : X \to \mathbb{P}^2$. By the construction of the projection map in terms of planes through L, the projective morphism $\phi|_X$ corresponds to the the linear system of $H - L$. The fibers of the projection are the residual intersections of X with planes containing L. Writing $X = X_a \cap X_b$ as a complete intersection of hypersurfaces in $\mathbb{P}^4$ of degree a and b, the residual intersections of X_a and X_b with a plane P containing L are plane curves $C_{a-1}, C_{b-1} \subset P$ of degree $a - 1$ and $b - 1$, which are both positive since X is nondegenerate. Hence the fiber of $\phi|_X : X \to \mathbb{P}^2$ corresponding to P is the intersection $C_{a-1} \cap C_{b-1}$ of these plane curves. Over the generic fiber, this intersection of plane curves must be finite (otherwise they would share a component and hence X would be birational to a curve bundle over $\mathbb{P}^2$, which is impossible since X is a surface), and hence by Bézout's theorem is a finite scheme of degree $(a - 1)(b - 1)$. □

In the case of a sextic K3 surface, we can say more.

Lemma 3.2. *Let $X \subset \mathbb{P}^4$ be a smooth sextic K3 surface containing a line L. The projection $X \to \mathbb{P}^2$ from L is the composition of a blow-down of exceptional curves and a finite flat double cover branched over a sextic curve.*

Proof. By Lemma 3.1, projection $X \to \mathbb{P}^2$ from L is dominant and induced by the linear system of $H - L$. Since $(H-L)^2 = 6-2\cdot 1-2 = 2$ and $H-L$ induces a dominant map $X \to \mathbb{P}^2$, it must be a blow-down of (-2)-curves C such that $(H-L).C = 0$ followed by a double cover, see [44, § 5], which must be branched over a sextic curve. □

Remark 3.3. We remark that in the moduli space $\mathcal{K}_6$, the locus of polarized K3 surfaces (X, H) of degree 6 "containing a line" i.e., an irreducible curve $L \subset X$ with $H.L = 1$ and $L^2 = -2$, is the Noether–Lefschetz divisor $\mathcal{K}^2_{6,1}$. Then $H - L$ is effective and $(H - L)^2 = 2$. Moreover, $H - L$ is base point free and the induced map $X \to \mathbb{P}^2$ is a double cover if and only if (X, H) is neither unigonal nor hyperelliptic, i.e., is not contained in the Noether–Lefschetz divisors $\mathcal{K}^0_{6,1}$ nor $\mathcal{K}^0_{6,2}$.

As a consequence, a degree 6 K3 surface containing a line admits a degree 2 model, on which we can use highly optimized Weil polynomial calculation algorithms implemented in `Magma` [7] by Elsenhans and Jahnel.

However, to employ this algorithm, we need to be able to computationally verify that projection from the line is a double cover $X \to \mathbb{P}^2$ and we need to compute an explicit model as a hypersurface of weighted degree 6 in $\mathbb{P}(1, 1, 1, 3)$, or, what will be enough for our purposes (see Remark 3.11), at least explicitly compute the sextic branch curve in $\mathbb{P}^2$. To this end, up to a linear change of variables, we can fix $L = V(x_0, x_1, x_2)$ where $(x_0 : x_1 : x_2 : y_0 : y_1)$ are homogenous coordinates for $\mathbb{P}^4$. Assuming that $X = V(f_2, f_3) \subset \mathbb{P}^4$ contains L, where $f_2, f_3 \in k[x_0, x_1, x_2, y_1, y_2]$ are homogenous forms of degree 2 and 3, respectively, then f_2, f_3 can be expressed as

$$\begin{aligned} f_2 &= l_0y_0 + l_1y_1 + q \\ f_3 &= l_{00}y_0^2 + l_{01}y_0y_1 + l_{11}y_1^2 + q_0y_0 + q_1y_1 + c \end{aligned} \tag{1}$$

where l_0, l_1, l_{00}, l_{01}, and l_{11} are homogenous linear forms, q, q_0, q_1 are homogenous quadratic forms, and c is a homogenous cubic form, all in $k[x_0, x_1, x_2]$.

Theorem 3.4. *Let k be a field of characteristic $\neq 2$ and let $X \subset \mathbb{P}^4$ be a smooth sextic K3 surface over k containing a line $L \subset \mathbb{P}^4$. In the notation of (1), consider the following symmetric matrix*

$$A = \begin{pmatrix} 0 & v^t \\ v & M \end{pmatrix} = \begin{pmatrix} 0 & l_0 & l_1 & q \\ l_0 & 2l_{00} & l_{01} & q_0 \\ l_1 & l_{01} & 2l_{11} & q_1 \\ q & q_0 & q_1 & 2c \end{pmatrix} \tag{2}$$

of homogeneous forms on $\mathbb{P}^2$, write $g_6 = \det(A)$ and $g_3 = \det(M)$, and let $D = V(g_6)$ and $C = V(g_3)$ in $\mathbb{P}^2$. If D is smooth, then projection from L determines a finite flat double cover $\pi : X \to \mathbb{P}^2$ with branch locus D that is tangent to the nodal cubic $C \subset \mathbb{P}^2$.

For a generic choice of coefficients, $X \subset \mathbb{P}^4$ is smooth, $D \subset \mathbb{P}^2$ is smooth, $C \subset \mathbb{P}^2$ has a single node away from D, and C intersects D tangentially in 9 distinct points.

Remark 3.5. Before we give the proof, we remark that Hodge theory predicts the existence of a 9-tangent rational cubic, i.e., a rational cubic $C \subset \mathbb{P}^2$ tangent to the branch sextic D in 9 points, in the double cover model of X. Indeed, the existence of the line L in a polarized sextic K3 surface (X, H) determines a lattice polarization of type

X	H	L
H	6	1
L	1	-2

and by Lemma 3.1, the double plane model (assuming it exists, equivalently, in the non-unigonal case) is determined by the linear system $H - L$, giving a lattice polarization of type

X	$H - L$	L
$H - L$	2	3
L	3	-2

Thus we see that projection from L determines a morphism of Noether–Lefschetz divisors $\mathcal{K}_{6,1}^{-2} \to \mathcal{K}_{2,3}^{-2}$. Finally, we see that the image of L in $\mathbb{P}^2$ is a rational cubic curve that splits into two rational cubic curve components L and $3(H - L) - L = 3H - 4L$ meeting in $L.(3H - 4L) = 11$ points. The 11 points comprise the preimages of the 9 points of intersection of C and D together with the two distinct preimages of the node on C.

To prove Theorem 3.4, we'll need the following consequence of projective duality.

Lemma 3.6. *Let k be a field of characteristic $\neq 2$ and $L \subset \mathbb{P}^n$ be a hyperplane dual to $v \in k^{n+1}$. For a smooth quadric $Q \subset \mathbb{P}^n$ with Gram matrix M, we have that L is tangent to Q if and only if $v^t \operatorname{adj}(M) v = 0$.*

If Q is a cone over a smooth quadric of dimension $n - 2$, then L is tangent to Q along a ruling of the cone if and only if $v^t \operatorname{adj}(M) v = 0$ and $v^t M v = 0$. In particular, when $n = 2$ and Q is a union of two distinct lines in $\mathbb{P}^2$, then $v^t \operatorname{adj}(M) v = 0$ and $v^t M v = 0$ if and only if L is one of the two lines.

And we'll also need the following linear algebra identity.

Lemma 3.7. *Let M be an $n \times n$ matrix and v a column vector of length n over an integral domain. Then we have*

$$v^t \operatorname{adj}(M) v = -\det \begin{pmatrix} 0 & v^t \\ v & M \end{pmatrix}$$

where $\operatorname{adj}(M)$ is the matrix adjugate of M.

Proof. Note that $\operatorname{adj}(M)$ is given by $\operatorname{adj}(M)_{ij} = (-1)^{i+j} M_{ji}$, where M_{ij} is the (i, j)-cofactor of M, i.e., the determinant of M with the ith row and jth column removed. Note that we can compute the determinant on the right hand side using cofactor expansion along the first row and then the first column. This becomes

$$\begin{aligned} -\det \begin{pmatrix} 0 & v^t \\ v & M \end{pmatrix} &= -\sum_{i=1}^{n} (-1)^i v_i \sum_{j=1}^{n} (-1)^{j+1} v_j M_{ji} \\ &= \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} v_i M_{ji} v_j = \sum_{i=1}^{n} \sum_{j=1}^{n} v_i \operatorname{adj}(M)_{ij} v_j = v^t \operatorname{adj}(M) v \end{aligned}$$

where $v^t = (v_1, \ldots, v_n)$. ∎

Proof of Theorem 3.4. Resolving the projection from L yields a morphism $\phi : \widetilde{\mathbb{P}}^2 = \mathrm{Bl}_L \mathbb{P}^4 \to \mathbb{P}^2$. Writing $X = X_2 \cap X_3$ for hypersurfaces $X_2, X_3 \subset \mathbb{P}^4$ of degrees 2 and 3, respectively, we consider the strict transforms $\widetilde{X}_2, \widetilde{X}_3 \subset \widetilde{\mathbb{P}}^4$. We have that $\widetilde{\mathbb{P}}^4 \cong \mathbb{P}(\mathscr{E})$ where $\mathscr{E} = \mathscr{O}_{\mathbb{P}^2}^{\oplus 2} \oplus \mathscr{O}_{\mathbb{P}^2}(-1)$, cf. [14, Section 9.3.2], and $\phi : \mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$ coincides with the projective bundle projection map. The homogeneous coordinates $(y_0 : y_1)$ correspond, via the identification $\widetilde{\mathbb{P}}^4 \cong \mathbb{P}(\mathscr{E})$, to a basis of global sections of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$. Let z be a nonzero global section of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \phi^* \mathscr{O}_{\mathbb{P}^2}(-1)$. Then z is unique up to scaling, as

$$\begin{aligned} h^0(\mathbb{P}(\mathscr{E}), \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \phi^* \mathscr{O}_{\mathbb{P}^2}(-1)) &= h^0(\mathbb{P}^2, \phi_* \mathscr{O}_{\mathbb{P}(\mathscr{E})}(1) \otimes \mathscr{O}_{\mathbb{P}^2}(-1))) \\ &= h^0(\mathbb{P}^2, \mathscr{E}^\vee \otimes \mathscr{O}_{\mathbb{P}^2}(-1)) = 1 \end{aligned}$$

by the projection formula. Thus $(y_0 : y_1 : z)$ forms a relative system of homogeneous coordinates on $\mathbb{P}(\mathscr{E})$ over $\mathbb{P}^2$. Then $\widetilde{X}_2$ and $\widetilde{X}_3$ can be identified with the subschemes of $\mathbb{P}(\mathscr{E})$ defined by the vanishing of global sections

$$l_0 y_0 + l_1 y_1 + qz \quad \text{and} \quad l_{00} y_0^2 + l_{01} y_0 y_1 + l_{11} y_1^2 + q_0 y_0 z + q_1 y_1 z + c z^2,$$

of $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(2) \otimes \phi^* \mathscr{O}_{\mathbb{P}^2}(1)$, respectively.

This gives an explicit was of considering the proof of Lemma 3.1. Indeed, restricting $\phi|_{\widetilde{X}_i} : \widetilde{X}_i \to \mathbb{P}^2$ yields a relative plane curve of degree $i-1$ in $\mathbb{P}(\mathscr{E})$ determined by one of the above equations. Because L is a smooth divisor in the smooth locus of X, we know that X coincides with its strict transform in $\widetilde{\mathbb{P}}^4$, and hence that $\widetilde{X}_2 \cap \widetilde{X}_3 \cong X$. Thus the fiber of $\phi|_X : X \to \mathbb{P}^2$ over a point $a \in \mathbb{P}^2$ is the intersection of the fibers over a of $\phi_{\widetilde{X}_2} : \widetilde{X}_2 \to \mathbb{P}^2$ and $\phi_{\widetilde{X}_2} : \widetilde{X}_2 \to \mathbb{P}^2$. When both relative hypersurfaces are flat at $a \in \mathbb{P}^2$, equivalently, the homogeneous forms

$$l_0(a) y_0 + l_1(a) y_1 + q(a) z \quad \text{and} \quad l_{00}(a) y_0^2 + l_{01}(a) y_0 y_1 + l_{11}(a) y_1^2 + q_0(a) y_0 z + q_1(a) y_1 z + c(a) z^2,$$

in $(y_0 : y_1 : z)$ are not identically zero, then the fiber is the intersection of a line and a conic in the fiber over a of $\mathbb{P}(\mathscr{E}) \to \mathbb{P}^2$. Even when one relative hypersurface (or both) is not flat, the fiber above a is still nonempty. In particular, the morphism $\phi|_X : X \to \mathbb{P}^2$ is surjective, and is a double cover away from the non-flat fibers of the individual relative hypersurfaces.

We note that $\phi_{\widetilde{X}_2} : \widetilde{X}_2 \to \mathbb{P}^2$ is not flat over points in $V(l_0, l_1, q) \subset \mathbb{P}^2$ and $\phi_{\widetilde{X}_3} : \widetilde{X}_3 \to \mathbb{P}^2$ is not flat over points in $V(l_{00}, l_{01}, l_{11}, q_0, q_1, c) \subset \mathbb{P}^2$. Clearly, for generic coefficients, both nonflat loci are empty.

Over the locus where both $\widetilde{X}_2$ and $\widetilde{X}_3$ are flat, the double cover $\phi|_X : X \to \mathbb{P}^2$ is ramified precisely when the fiber of $\widetilde{X}_2$ (which is a line in the fibral plane) is tangent to the fiber of $\widetilde{X}_3$ (which is a conic). Lemma 3.6 thus shows that this occurs over the vanishing locus of $v^t \operatorname{adj}(M) v$, where M is the bottom right 3×3 submatrix in (2) and v is the vector with $v^t = (l_0, l_1, q)$. By Lemma 3.7, this is the same as the vanishing locus $D \subset \mathbb{P}^2$ of the determinant of the matrix A in (2).

Now, we proceed to show that if D is smooth then the map $\phi|_X : X \to \mathbb{P}^2$ is flat, hence a finite flat double cover of $\mathbb{P}^2$ branched over D. By Jacobi's formula, we have that

$$\frac{\partial}{\partial x_i} \det(A) = \operatorname{tr}\left(\operatorname{adj}(A) \frac{\partial A}{\partial x_i} \right)$$

and we'll proceed to use the jacobian criterion to show that D is singular at points over which $\phi|_X : X \to \mathbb{P}^2$ is not flat, equivalently, either one of the $\widetilde{X}_i \to \mathbb{P}^2$ is not flat or the fiber of $\widetilde{X}_2$ (the line) is a component of the fiber of $\widetilde{X}_3$ (which is hence a union of lines). Over a point $a \in \mathbb{P}^2$ where $\widetilde{X}_2$ is not flat, the first rown and column of A is uniformly zero, hence $\operatorname{adj}(A)$ is at most nonzero only in the $(1,1)$-entry. But this multiplied by $\frac{\partial A}{\partial x_i}$ is the zero matrix. Similary, where $\widetilde{X}_3$ is not flat, then the entire lower right 3×3 block of A is zero, meaning that already $\operatorname{adj}(A)$ is the zero matrix. Over a point $a \in \mathbb{P}^2$ where $\widetilde{X}_3$ is a union of distinct lines, one of which is $\widetilde{X}_2$, we can change variables in the fibral plane with homogeneous coordinate $(x : y : z)$ so that $\widetilde{X}_3 = V(xy)$ and $\widetilde{X}_2 = V(x)$. In these variables, Lemma 3.6 says that $v^t = (1,0,0)$ or $v^t = (0,1,0)$, in which case $\operatorname{adj}(A)$ is the zero matrix. Where $\widetilde{X}_3$ is the double line $\widetilde{X}_2$, we can change variables in the fibral plane with homogeneous coordinate $(x : y : z)$ so that $\widetilde{X}_3 = V(x^2)$ and $\widetilde{X}_2 = V(x)$ and $v^t = (1,0,0)$, in which case $\operatorname{adj}(A)$ is again the zero matrix. Hence in all cases, D is singular at a. This shows that the smoothness of D implies the flatness of the double cover, and it has already been noted that the branch locus will then be D.

As for the tangency of the intersection of D and C, this is a standard fact about the resolution of 9 points in $\mathbb{P}^2$ and the shape of the matrix A in (2), cf. [6, Section 4.1]. The fact that C has a node can be checked directly.

Finally, in the space of all coefficients, the required smoothness and intersection conditions are open, so it suffices to find a single case where they all occur simultaneously, e.g., the example appearing in Corollary 3.10. □

Remark 3.8. The fact that the branch locus of the degree 2 model of the sextic K3 surface containing a line has a symmetric determinental presentation of the form

$$\det \begin{pmatrix} 0 & l_0 & l_1 & q \\ l_0 & 2l_{00} & l_{01} & q_0 \\ l_1 & l_{01} & 2l_{11} & q_1 \\ q & q_0 & q_1 & 2c \end{pmatrix}$$

is indicative that there should be a relationship with cubic fourfolds containing a plane. Indeed, the above matrix determines a cubic fourfold $Y \subset \mathbb{P}^5$ containing a plane, which has equation $f_3 + y_3 f_2 = 0$ if $(x_0 : x_1 : x_2 : y_1 : y_2 : y_3)$ are homogeneous coordinates on $\mathbb{P}^5$. Hence Y is singular at $(0:0:0:0:0:1)$ and generically this singularity is an ordinary double point. As in [22, Section 4.2], the sextic K3 surface X appears in the resolution of the projection $Y \dashrightarrow \mathbb{P}^4$ from the node as the locus in $\mathbb{P}^4$ of lines through the node. In this case, the tangent cone of Y at the node coincides with the projective cone in $\mathbb{P}^5$ over the quadric $V(f_2) \subset \mathbb{P}^4$. Note that Y also contains the plane $V(x_0, x_1, x_2) \subset \mathbb{P}^5$. Hence the locus of these cubic fourfolds forms a component of $\mathcal{C}_6 \cap \mathcal{C}_8$, where $\mathcal{C}_6, \mathcal{C}_8 \subset \mathcal{C}$ are the loci of cubic fourfolds admitting a node and containing a plane, respectively, in the moduli space of (semi-stable) cubic fourfolds. However, this is a different component from the one considered in [47], since the sextic branch divisor of the associated K3 surface of degree 2 is actually smooth for the very general element. We record that the intersection form on the lattice of integral type $(2,2)$ Hodge classes in the limiting Hodge structure associated to the nodal cubic fourfold Y has a lattice polarization by

Y	h^2	P	X
h^2	3	1	6
P	1	3	1
X	6	1	14

where h^2 is the square of the hyperplane class, P is the class of the plane, and X is the class of the sextic K3 surface.

The following criteria for a projective K3 surface to have geometric Picard rank 1 is inspired by [18, Example 1.7], [24, Proposition 5.3], and [36, Proposition 26].

Theorem 3.9. *Let $X \subset \mathbb{P}^n_{\mathbb{Q}}$ be a K3 surface. Assume that $\overline{X}$ contains no lines over $\overline{\mathbb{Q}}$, and that for some prime $p > 2$ of good reduction X_p contains a line and $\rho(\overline{X}_p) = 2$. Then $\rho(\overline{X}) = 1$.*

Proof. Let $H \in \mathrm{NS}(\overline{X})$ be the hyperplane section. Assume, to get a contradiction, that X has geometric Picard rank at least 2. If $\operatorname{rk}\mathrm{NS}(\overline{X}) > 2$, we have a contradiction because $\overline{\mathrm{sp}} : \mathrm{NS}(\overline{X}) \to \mathrm{NS}(\overline{X}_p)$ is injective and we are assuming $\operatorname{rk}\mathrm{Pic}(\overline{X}_p) = 2$. Thus, we can assume that $\operatorname{rk}\mathrm{NS}(\overline{X}) = 2$. Because $\overline{\mathrm{sp}}$ is injective, the image of $\overline{\mathrm{sp}}$ has rank 2 as well, so the cokernel has rank 0. However, by Lemma 1.1, the cokernel is torsion free, so it must be 0. Thus $\overline{\mathrm{sp}}$ is surjective.

Finally, let L be a line on X_p. Because $\overline{\mathrm{sp}}$ is surjective, we have a divisor class $L' \in \mathrm{NS}(\overline{X})$ such that $\overline{\mathrm{sp}}(L') = L$, but $\overline{\mathrm{sp}}$ preserves degree (i.e., intersection with H), so L' is the class of a line as well, contradicting our hypothesis. □

We remark that for a K3 surface that is a double cover $X \to \mathbb{P}^2$, a "line" on X, i.e., a divisor class $L \in \mathrm{Pic}(X)$ with $H.L = 1$ and $L^2 = -2$, is precisely a component of the preimage of a tritangent line to the branch divisor of the cover. Hence, one can view Theorem 3.9 as an amply polarized generalization to any degree of the strategy initiated in [18, Example 1.7], cf. [24, Proposition 5.3], for degree 2 K3 surfaces.

To construct an explicit example, we randomly generate sextic K3 surfaces X_p over $\mathbb{F}_p$ containing a fixed line $L \subset \mathbb{P}^4$ until we find one with $\rho(\overline{X}_p) = 2$, which we verify by calculating the Weil polynomial using projection from the line. Then, we lift X_p to X over $\mathbb{Q}$ in such a way that $\overline{X}$ contains no lines, and then apply Theorem 3.9. We implemented this strategy to find the following.

Corollary 3.10. *Let $X = V(f_2, f_3) \subset \mathbb{P}^4_{\mathbb{Q}}$ where*

$$\begin{aligned}
f_2 &= x_0^2 - 3x_0x_1 + 3x_1^2 + 5x_0x_2 + 4x_1x_2 + 5x_2^2 - x_0x_3 - 2x_1x_3 \\
&\quad - 3x_2x_3 - 5x_0x_4 + 5x_1x_4 + 47x_3^2 + 47x_4^2 \\
f_3 &= 2x_0^3 + 3x_0^2x_1 + 3x_0x_1^2 + x_1^3 - x_0x_1x_2 - 3x_1^2x_2 + 4x_0x_2^2 - 4x_1x_2^2 + 5x_2^3 \\
&\quad + 4x_0^2x_3 + x_0x_1x_3 + 5x_1^2x_3 + 4x_0x_2x_3 + 4x_1x_2x_3 - 3x_2^2x_3 + 4x_1x_3^2 - x_2x_3^2 \\
&\quad + 5x_0^2x_4 - 4x_0x_1x_4 + 2x_1^2x_4 + x_0x_2x_4 + 4x_1x_2x_4 - 2x_2^2x_4 \\
&\quad + 4x_0x_3x_4 - 3x_2x_3x_4 - x_0x_4^2 - x_1x_4^2 + 5x_2x_4^2,
\end{aligned}$$

Then X is a sextic K3 surface with geometric Picard rank 1.

Proof. We constructed the example so that X_{47} contains the line $V(x_0, x_1, x_2)$. Projecting from this line, and using Theorem 3.4, we find that the double cover $X \to \mathbb{P}^2$

has branch divisor the vanishing of $g_6(u,v,w)$ given by

$$\begin{aligned}
&14u^6+36u^5v+40u^4v^2+2u^3v^3+38u^2v^4+40uv^5+26v^6+7u^5w+29u^4vw\\
&\quad+12u^3v^2w+29u^2v^3w+2uv^4w+28v^5w+29u^4w^2+15u^3vw^2+12u^2v^2w^2\\
&\quad+16uv^3w^2+11v^4w^2+40u^3w^3+31u^2vw^3+38uv^2w^3+26v^3w^3+35u^2w^4\\
&\qquad+10uvw^4+18v^2w^4+2uw^5+43vw^5+w^6
\end{aligned}$$

where $(u:v:w)$ are homogenous coordinates for $\mathbb{P}^2$. Using `Magma`'s implemented algorithm, we computed the Weil polynomial of the double cover model $s^2=g_6(u,v,w)$ as

$$\begin{aligned}
&(t-47)^2(t^{20}+35t^{19}+1410t^{18}+79524t^{17}-311469t^{16}+39037448t^{15}+5504280168t^{14}\\
&-86233722632t^{13}-1013246240926t^{12}-666716026529308t^{11}-78339133117193690t^{10}\\
&-1472775702603241372t^9-49443184301680246066t^8-929531864871588625928t^7\\
&+13106399294689399625584 8t^6+2053335889501339274674952t^5\\
&-36190045052461104716146029t^4+2041118540958806305990636 0356t^3\\
&+799438095208865803179665780610t^2+43835855553952808207685006970115t\\
&+2766668711962335809450748011342401)
\end{aligned}$$

As the Weil polynomial contains no cyclotomic roots besides $(t-47)^2$ coming from the hyperplane section and the line, Lemma 1.2 implies that X_p has geometric Picard rank 2.

Next, we use a Gröbner basis calculation on each Schubert cell of the Grassmannian $\mathrm{Gr}(2,5)$ of lines in $\mathbb{P}^4$ to verify that X contains no lines over $\overline{\mathbb{Q}}$. Finally, we conclude that so $\rho(X)=1$ by Theorem 3.9, cf. [18]. $\square$

Remark 3.11. Knowing the branch locus $D\subseteq\mathbb{P}^2$ only specifies the polynomial g_6 up to multiplication by a unit. This is insufficient to specify the isomorphism class of the double cover $X\to\mathbb{P}^2$ with branch locus $V(g_6)$, as there can be nontrivial quadratic twists. Explicitly, the double cover models $s^2=g_6$ and $s^2=\lambda g_6$ might be different for λ a nonsquare in the base field. In fact, one easily finds examples where these quadratic twists have different point counts over a finite field, hence are be nonisomorphic. Of course, over a finite field of characteristic $\neq 2$, and for a generic choice of g_6, there will be at most one quadratic twist. However, since any two quadratic twists are isomorphic over the algebraic closure, they have the same geometric Picard rank. So, while we do not know which is so this is not an issue for our argument. After some experimentation, we believe that the correct model for the double cover is $s^2=g_6$, where g_6 is as in Theorem 3.4.

4. Zariski density

In this section, we will prove Theorem 1, and along the way streamline the argument that van Luijk employs [50, Proof of Theorem 1.1] to prove Zariski density in degree 4. Note that, unlike in van Luijk, we are not concerned with whether our Noether–Lefschetz general K3 surfaces have infinitely many $\mathbb{Q}$-points or not, though this could make for an interesting follow up.

In degrees $d\leq 8$, one constructs a subset of $\mathcal{K}_d(\mathbb{Q})$ consisting of Noether–Lefschetz general K3 surfaces. In each case, this subset has the form $T\cap U$, where T is the set of all K3 surfaces with an integral model that reduces to a fixed model mod p, and $U \subset \mathcal{K}_d$ is a Zariski open dense subvariety consisting of the complement of a finite union of Noether–Lefschetz divisors.

We first explain the Zariski open $U \subset \mathcal{K}_d$. For $d = 2$, Elsenhans and Jahnel [16, 17] take $U = \mathcal{K}_d \smallsetminus \mathcal{K}_{2,1}^{-2}$ to be the complement of the tritangent locus; for $d = 4$, one can reinterpret the proof of van Luijk [50] in this context, taking $U = \mathcal{K}_4 \smallsetminus \mathcal{K}_{4,1}^{-2}$ to be the complement of the divisor of quartic K3 surfaces containing a line; for $d = 6$, we take $U = \mathcal{K}_d \smallsetminus \mathcal{K}_{6,1}^{-2}$ to be the complement of the divisor of sextic K3 surfaces containing a line; and for $d = 8$, we take U to be the complement in $\mathcal{K}_8$ of the union of $\mathcal{K}_{8,2}^{0}$, $\mathcal{K}_{8,1}^{-2}$, $\mathcal{K}_{8,2}^{-2}$ (which defines the open locus $\mathcal{K}_8^\circ$ where the discriminant K3 is defined and smooth, as in Proposition 2.1), as well as the complement of the preimage under $\mathcal{K}_8^\circ \to \mathcal{K}_2^\circ$ of the tritangent locus $\mathcal{K}_{2,1}^{-2}$.

It remains to show that T is Zariski dense in $\mathcal{K}_d$. In all cases, this holds by applying the following general result to the unirational varieties $\mathcal{K}_d$ for $d \leq 8$.

Lemma 4.1. *Let X be an equidimensional $\mathbb{Z}$-scheme with nonempty generic fiber $X_\mathbb{Q}$ of dimension n and fix a $\mathbb{Q}$-point x of X reducing to an $\mathbb{F}_p$-point x_p of the reduction X_p modulo a prime p. Assume that there is a dominant rational map $f : \mathbb{P}^n_\mathbb{Z} \dashrightarrow X$ such that $x_p = f(z_p)$ for some $\mathbb{F}_p$-point z_p of $\mathbb{P}^n_{\mathbb{F}_p}$. Then the set T of $\mathbb{Q}$-points of $X_\mathbb{Q}$ that reduce to x_p modulo p is Zariski dense in $X_\mathbb{Q}$.*

We believe that this should be a standard weak approximation type result for unirational varieties but could not find an explicit reference, so we include a proof below.

Proof. First we prove the result for $X = \mathbb{P}^n_\mathbb{Z}$. Suppose to the contrary that there is a closed proper subvariety $V \subset \mathbb{P}^n_\mathbb{Z}$ whose $\mathbb{Q}$-points contains T. We can assume, without loss of generality, that V is codimension 1, i.e., $V = V(g)$ for some homogenous polynomial g of degree d. For any prime q, the Schwartz–Zippel lemma implies that the number of $\mathbb{F}_q$-points of V_q is at most dq^{n-1}, an amount that becomes strictly less than $q^n + q^{n-1} + \cdots + 1$ as q increases. Thus, for all but finitely many primes q, there exists at least one $\mathbb{F}_q$-point of $\mathbb{P}^n_{\mathbb{F}_q}$ that is not in V_q.

Fix one such prime $q \neq p$, one such $\mathbb{F}_q$-point y_q of $\mathbb{P}^n_{\mathbb{F}_q}$, and let y be a lift of y_q to a $\mathbb{Q}$-point of $\mathbb{P}^n_\mathbb{Q}$. Now consider the $\mathbb{Q}$-point $qx + py$ of $\mathbb{P}^n_\mathbb{Q}$, where we perform addition in some affine patch $\mathbb{A}^n_\mathbb{Q}$ containing x and y. Since $qx + py$ reduces to x_p modulo p we have that $qx + py \in T$ hence is a $\mathbb{Q}$-point of V. However, this contradicts the fact that $qx + py$ reduces to y_q modulo q, which by construction is not contained in V_q. Hence T must be Zariski dense in $\mathbb{P}^n_\mathbb{Q}$.

Now, assume we have a dominant rational map $f : \mathbb{P}^n_\mathbb{Z} \dashrightarrow X$ such that x_p lifts to an $\mathbb{F}_p$-point z_p. By the above argument, the set S of $\mathbb{Q}$-points of $\mathbb{P}^n_\mathbb{Q}$ that reduce to z_p modulo p is Zariski dense in $\mathbb{P}^n_\mathbb{Q}$. But since f is dominant, the image of a dense subset is dense, so we obtain that $f(S)$ is Zariski dense in $X_\mathbb{Q}$. Since $f(z_p) = x_p$ we have that $f(S) \subseteq T$, and hence that T is Zariski dense in $X_\mathbb{Q}$ as well. ☐

Finally, we can give a proof of our main result.

Proof of Theorem 1. For each $d \leq 8$, we construct a prime p and a smooth K3 surface X_p of degree d over $\mathbb{F}_p$ with geometric Picard rank 2. For $d = 2$, this was first done by Elsenhans and Jahnel [16, Example 28(ii)], [17, Example 6.1(ii)] in several different ways; for $d = 4$, this was the original case done by van Luijk [50, § 3]; for $d = 6$, we use the K3 surface in Corollary 3.10; and for $d = 8$, this was first done by Elsenhans and Jahnel [17, § 8], there is another example in [36, §5.4], and we provide yet another example in Corollary 2.4. In the cases $d \leq 6$, the geometric Picard group is generated by the polarization and a line, whereas in the case $d = 8$, the isogenous discriminant K3 has geometric Picard group generated by the polarization and a line. For $d \leq 6$, taking a lift to $\mathbb{Q}$ that is contained in the open $U = \mathcal{K}_d \smallsetminus \mathcal{K}_{d,1}^{-2}$, i.e., that does not contain a line, forces the lift to have geometric Picard rank 1 by Theorem 3.9. For $d = 8$, we take a lift to $\mathbb{Q}$ whose isogenous discriminant K3 is contained in the open $U = \mathcal{K}_2 \smallsetminus \mathcal{K}_{2,1}^{-2}$, i.e., not containing a tritangent line, forcing geometric Picard rank 1 by Proposition 2.3.

Lemma 4.1 ensures that the set of lifts to $\mathbb{Q}$ is Zariski dense in $\mathcal{K}_d$, since for $d \leq 8$, where the general polarized K3 surface is a complete intersection, we have that $\mathcal{K}_d$ admits a model that is unirational over $\mathbb{Z}$, given as a quotient of an affine space by a linear algebraic group defined over $\mathbb{Z}$. The intersection of this dense subset of $\mathcal{K}_D(\mathbb{Q})$ with the open subvariety U is still dense in $\mathcal{K}_d$ and consists of polarized K3 surfaces that are Noether–Lefschetz general. □

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