Clifford sequences in the theory of line bundle-valued quadratic forms over schemes

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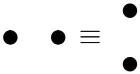
Boca Raton, Florida
Friday 30 October 2009, 2:30 - 2:50pm
Special Session on Arithmetic Geometry, I

Accidental isomorphisms

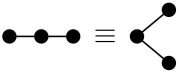
$A_1 \equiv B_1 \equiv C_1$

$A_1^2 \equiv D_2$



$B_2 \equiv C_2$

$A_3 \equiv D_3$



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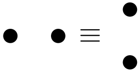

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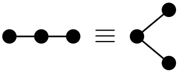

$$\left\{ \begin{array}{c} \text{quaternion algebras} \\ \text{with structure} \end{array} \right\} \updownarrow \left\{ \begin{array}{c} \text{quadratic forms} \\ \text{of rank 3} \end{array} \right\}$$

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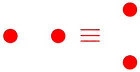

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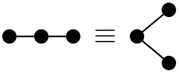

$$\left\{ \begin{array}{c} \text{pair of quaternion algebras} \\ \text{with structure} \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{oriented quadratic forms} \\ \text{of rank 4} \end{array} \right\}$$

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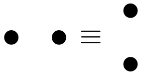
$A_1^2 \equiv D_2$



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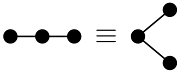

$$\left\{ \begin{array}{c} \text{quadratic forms} \\ \text{of rank 5} \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{symplectic Azumaya algebras} \\ \text{of rank 16} \end{array} \right\}$$

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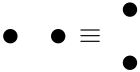

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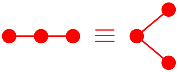

$$\left\{ \begin{array}{c} \text{2-torsion Azumaya algebras} \\ \text{of rank 16, with structure} \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{oriented quadratic forms} \\ \text{of rank 6} \end{array} \right\}$$

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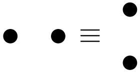

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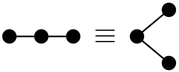
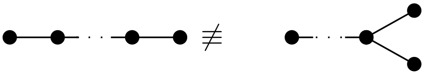


$B_n \not\equiv C_n \qquad n > 2$

$A_n \not\equiv D_n \qquad n > 3$



Quadratic forms

over rings and schemes

ring R

E projective R-module

$q : E \to R$

$q(a\,v) = a^2\,q(v)$

$E \times E \to R$ sym. bilinear

$(v, w) \mapsto q(v+w) - q(v) - q(w)$

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scheme X

$\mathscr{E}$ locally free $\mathscr{O}_X$-module

$q : \mathscr{E} \to \mathscr{O}_X$

$$q \in \Gamma(X, S^2\mathscr{E}^\vee)) \cong \Gamma(X, (S_2\mathscr{E})^\vee)$$

Line bundle-valued quadratic forms

line bundle = invertible module

ring R

E projective R-module

$q : E \to L$

$q(a\,v) = a^2\,q(v)$

$E \times E \to L$ sym. bilinear

$(v, w) \mapsto q(v+w) - q(v) - q(w)$

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$\mathscr{E}$ locally free $\mathscr{O}_X$-module

$q : \mathscr{E} \to \mathscr{L}$

$$q \in \Gamma(X, (S^2\mathscr{E}^\vee) \otimes \mathscr{L}))$$

$$\wr\|$$

$$\Gamma(X, \mathscr{H}om(S_2\mathscr{E}, \mathscr{L}))$$

Goal:

Extend the accidental classification theorems to line bundle-valued (and degenerate) quadratic forms.

Regularity

nice forms

$$q : \mathscr{E} \to \mathscr{O}_X \quad \rightsquigarrow \quad b : \mathscr{E} \times \mathscr{E} \to \mathscr{O}_X$$

$$\rightsquigarrow \quad \varphi : \mathscr{E} \to \mathscr{E}^\vee$$

$$v \mapsto w \mapsto b(v, w)$$

Regularity

nice forms

$$
\begin{aligned}
q : \mathscr{E} \to \mathscr{O}_X \quad & \rightsquigarrow & b : \mathscr{E} \times \mathscr{E} &\to \mathscr{O}_X \\
& \rightsquigarrow & \varphi : \mathscr{E} &\to \mathscr{E}^\vee \\
& & v &\mapsto w \mapsto b(v, w) \\
& \rightsquigarrow & \det \varphi : \det \mathscr{E} &\to \det \mathscr{E}^\vee
\end{aligned}
$$

Regularity

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& & v &\mapsto w \mapsto b(v, w) \\
& \rightsquigarrow & d\varphi : (\det \mathscr{E})^{\otimes 2} &\to \mathscr{O}_X
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$(\mathscr{E}, q)$ *regular* $\Leftrightarrow \varphi$ isomorphism $\Leftrightarrow d\varphi$ isomorphism

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$(\mathscr{E}, q)$ *regular* $\Leftrightarrow \varphi$ isomorphism $\Leftrightarrow d\varphi$ isomorphism

Odd rank

$$q(x) = a\,x^2 \quad \rightsquigarrow \quad b(x, y) = 2a\,xy \quad \rightsquigarrow \quad \varphi : x \mapsto 2a\,x^\vee$$

Regularity

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Discriminants

$(\det \mathscr{E}, (-1)^{n(n-1)/2} d\varphi)$ *signed discriminant* form

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Discriminants

$(\det \mathscr{E}, (-1)^{n(n-1)/2} d\varphi)$ *signed discriminant* form

$D(\mathscr{E}, q)$ rank 2 *discriminant algebra* refining $\mathscr{O}_X \oplus \det \mathscr{E}$

Algebraic groups

are great

$\mathbf{GO}(q) : U \mapsto \{\varphi \in \mathbf{GL}(\mathscr{E}|_U) : q|_U \circ \varphi = \lambda_\varphi q|_U,\ \lambda_\varphi \in \mathbb{G}_m(U)\}$
orthogonal similitude group scheme (fppf)

$\mathbf{O}(q) : U \mapsto \{\varphi \in \mathbf{GL}(\mathscr{E}|_U) : q|_U \circ \varphi = q|_U\}$
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$$1 \to \mathbf{O}(q) \to \mathbf{GO}(q) \xrightarrow{\lambda} \mathbb{G}_m \to 1$$

multiplier sequence (fppf)

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$\mathbf{SO}(q) = \ker(\mathbf{O}(q) \to \mathrm{Aut}_{\mathscr{O}_X} D(\mathscr{E}, q))$

special orthogonal group scheme

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special orthogonal group scheme

$(\mathscr{E}, q)$ (semi)regular $\Rightarrow$ $\mathbf{GO}(q), \mathbf{O}(q), \mathbf{SO}(q)$ smooth
linear algebraic groups

Torsors
too many

$(\mathscr{E}, q)$ (semi)regular quadratic form of rank n

alg groups	category of torsors
$\mathbf{O}(q)$	
$\mathbf{GO}(q)$	
$\mathbf{SO}(q)$	
$\mathbf{GSO}(q)$	
$\mathbf{GL}_n$	
$\mathbf{PGL}_n$	
$\mathbf{GL}_n/\mu_2$	

Torsors

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$(\mathscr{E}, q)$ (semi)regular quadratic form of rank n

alg groups	category of torsors
$\mathbf{O}(q)$	
$\mathbf{GO}(q)$	$\left\{ \begin{array}{c} \text{Ob: (semi)regular quadratic forms} \\ (\mathscr{E}', q') \text{ of rank } n \\ \text{Mor: isometries} \end{array} \right\}$
$\mathbf{SO}(q)$	
$\mathbf{GSO}(q)$	
$\mathbf{GL}_n$	$\left\{ \begin{array}{c} \text{Ob: semi(regular) line bundle-valued} \\ \text{forms} \\ (\mathscr{E}', q', \mathscr{L}) \text{ of rank } n \\ \text{Mor: similarities} \end{array} \right\}$
$\mathbf{PGL}_n$	
$\mathbf{GL}_n/\mu_2$	

Torsors

too many

$(\mathscr{E}, q)$ (semi)regular quadratic form of rank n

alg groups	category of torsors
$\mathbf{O}(q)$	
$\mathbf{GO}(q)$	
$\mathbf{SO}(q)$	$\left\{ \begin{array}{c} \text{Ob: oriented (semi)regular quadratic} \\ \text{forms } (\mathscr{E}', q', \zeta) \text{ of rank } n, \\ \zeta : D(\mathscr{E}', q') \rightsquigarrow D(\mathscr{E}, q) \\ \text{Mor: orientated isometries} \end{array} \right\}$
$\mathbf{GSO}(q)$	$\left\{ \begin{array}{c} \text{Ob: oriented (semi)regular} \\ \text{line bundle-valued quadratic forms} \\ (\mathscr{E}', q', \mathscr{L}, \zeta) \text{ of rank } n \\ \zeta : D(\mathscr{E}', q', \mathscr{L}) \rightsquigarrow D(\mathscr{E}, q) \\ \text{Mor: orientated isometries} \end{array} \right\}$
$\mathbf{GL}_n$	
$\mathbf{PGL}_n$	
$\mathbf{GL}_n/\mu_2$	

Torsors

too many

$(\mathscr{E}, q)$ (semi)regular quadratic form of rank n

alg groups	category of torsors
$\mathbf{O}(q)$	
$\mathbf{GO}(q)$	$\left\{ \begin{array}{c} \text{Ob: locally free } \mathscr{O}_X\text{-modules} \\ \mathscr{E}' \text{ of rank } n \\ \text{Mor: } \mathscr{O}_X\text{-module isomorphisms} \end{array} \right\}$
$\mathbf{SO}(q)$	
$\mathbf{GSO}(q)$	
$\mathbf{GL}_n$	
$\mathbf{PGL}_n$	$\left\{ \begin{array}{c} \text{Ob: Azumaya } \mathscr{O}_X\text{-algebras} \\ \mathscr{A} \text{ of rank } n^2 \\ \text{Mor: } \mathscr{O}_X\text{-algebra isomorphisms} \end{array} \right\}$
$\mathbf{GL}_n/\mu_2$	

Torsors

too many

$(\mathscr{E}, q)$ (semi)regular quadratic form of rank n

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$\mathbf{GO}(q)$	
$\mathbf{SO}(q)$	
$\mathbf{GSO}(q)$	
$\mathbf{GL}_n$	
$\mathbf{PGL}_n$	
$\mathbf{GL}_n/\mu_2$	$\left\{ \begin{array}{c} \text{Ob: 2-torsion datum } (\mathscr{A}, \mathscr{V}, \psi) \\ \mathscr{A} \text{ Azumaya rank } n^2 \\ \mathscr{V} \text{ locally free rank } n^2 \\ \psi : \mathscr{A} \otimes \mathscr{A} \xrightarrow{\sim} \mathscr{E}nd(\mathscr{V}) \\ \mathscr{O}_X\text{-algebra isomorphism} \\ \text{Mor: compatible isomorphisms} \end{array} \right\}$

$A_1 \equiv B_1 (\equiv C_1)$

$$\left\{ \begin{array}{c} \text{Azumaya algebras} \\ \mathscr{A} \text{ of rank 4} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{oriented semiregular} \\ \text{quadratic forms} \\ (\mathscr{E}, q, \zeta) \text{ of rank 3} \end{array} \right\}$$

$$C_0(\mathscr{E}, q) \longleftarrow\!\!\shortmid \quad (\mathscr{E}, q.\zeta)$$

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$$C_0(\mathscr{E}, q) \longleftarrow (\mathscr{E}, q.\zeta)$$

$$(ab, bc) \longleftarrow a\,x^2 + b\,y^2 + c\,z^2$$

$A_1 \equiv B_1 (\equiv C_1)$

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$$C_0(\mathscr{E}, q) \longleftarrow (\mathscr{E}, q.\zeta)$$

$$\det : S^2(\mathscr{O}_X^2) \to \mathscr{O}_X$$

$$\begin{pmatrix} x & y \\ y & z \end{pmatrix} \mapsto xz - y^2$$

semiregular, $\frac{1}{2}\, d \det = 1$

$\mathbf{SO}_{1,2} = \mathbf{SO}(\det)$

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$$\left\{\begin{matrix}\text{Azumaya algebras} \\ \mathscr{A} \text{ of rank } 4\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{oriented semiregular} \\ \text{quadratic forms} \\ (\mathscr{E}, q, \zeta) \text{ of rank } 3\end{matrix}\right\}$$

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$$1 \to \mu_2 \to \mathbf{SL}_2 \to \mathbf{SO}_{1,2} \to 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad + bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix}$$

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$$\mathbf{SO}_{1,2} = \mathbf{SO}(\det)$$

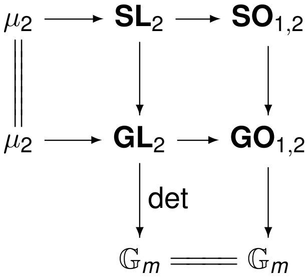
$$1 \to \mu_2 \to \mathbf{SL}_2 \to \mathbf{SO}_{1,2} \to 1$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^2 & 2ac & c^2 \\ ab & ad + bc & cd \\ b^2 & 2bd & d^2 \end{pmatrix}$$

$$\mathbf{PGL}_2 \xrightarrow{\sim} \mathbf{SO}_{1,2}$$

$A_1 \equiv B_1 (\equiv C_1)$

line bundle-valued point of view



$A_1 \equiv B_1 (\equiv C_1)$

line bundle-valued point of view

$$\begin{array}{ccccc}
\mu_2 & \longrightarrow & \mathbf{SL}_2 & \longrightarrow & \mathbf{SO}_{1,2} \\
\| & & \downarrow & & \downarrow \\
\mu_2 & \longrightarrow & \mathbf{GL}_2 & \longrightarrow & \mathbf{GO}_{1,2} \\
 & & \downarrow{\scriptstyle \det} & & \downarrow \\
 & & \mathbb{G}_m & = & \mathbb{G}_m
\end{array}$$

$$\mathbf{GL}_2/\mu_2 \rightsquigarrow \mathbf{GO}_{1,2}$$

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$$\mathbf{GL}_2/\mu_2 \xrightarrow{\sim} \mathbf{GO}_{1,2}$$

$$\left\{ \begin{array}{c} (\mathscr{A}, \mathscr{V}, \psi) \\ \mathscr{A} \text{ Azumaya rank 4} \\ \psi : \mathscr{A} \otimes \mathscr{A} \xrightarrow{\sim} \mathscr{E}nd(\mathscr{V}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{semiregular line bundle-} \\ \text{valued quadratic forms} \\ (\mathscr{E}, q, \mathscr{L}) \text{ of rank 3} \end{array} \right\}$$

$$(C_0(\mathscr{E}, q, \mathscr{L}), C_1(\mathscr{E}, q, \mathscr{L}), \mathrm{can}) \longleftarrow (\mathscr{E}, q, \mathscr{L})$$

$A_1 \equiv B_1 (\equiv C_1)$

line bundle-valued point of view

$$
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$$

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Extension to degenerate forms by Venkata Balaji and Voight

Clifford diagrams

$(\mathscr{E}, q)$ (semi)regular
rank $2, 3$ mod 4

$$\begin{array}{ccccc}
\mu_2 & \longrightarrow & \mathbf{Spin}(q) & \longrightarrow & \mathbf{SO}(q) \\
\| & & \downarrow & & \downarrow \\
\mu_2 & \longrightarrow & \Gamma_+(q) & \longrightarrow & \mathbf{GO}(q) \\
 & & \downarrow & & \downarrow \\
 & & \mathbb{G}_m & = & \mathbb{G}_m
\end{array}$$

$(\mathscr{E}, q)$ (semi)regular
rank $0, 1$ mod 4

$$\begin{array}{ccccc}
\mu_2 & \longrightarrow & \mathbf{Spin}(q) & \longrightarrow & \mathbf{SO}(q) \\
\downarrow & & \downarrow & & \downarrow \\
\mu_4 & \longrightarrow & \mathbf{S}\Gamma(q) & \longrightarrow & \mathbf{GSO}(q) \\
\downarrow & & \downarrow & & \downarrow \lambda \\
\mu_2 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\;2\;} & \mathbb{G}_m
\end{array}$$

Clifford diagrams

$(\mathscr{E}, q)$ (semi)regular
rank $2, 3$ mod 4

$$\begin{CD}
\mu_2 @>>> \mathbf{Spin}(q) @>>> \mathbf{SO}(q) \\
@| @VVV @VVV \\
\mu_2 @>>> \Gamma_+(q) @>>> \mathbf{GO}(q) \\
@. @VVV @VVV \\
@. \mathbb{G}_m @= \mathbb{G}_m
\end{CD}$$

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rank $0, 1$ mod 4

$$\begin{CD}
\mu_2 @>>> \mathbf{Spin}(q) @>>> \mathbf{SO}(q) \\
@VVV @VVV @VVV \\
\mu_4 @>>> \mathbf{S}\Gamma(q) @>>> \mathbf{GSO}(q) \\
@VVV @VVV @VV{\lambda}V \\
\mu_2 @>>> \mathbb{G}_m @>{2}>> \mathbb{G}_m
\end{CD}$$

These provide cohomological invariants in $H^2_{\text{ét}}(X, \mu_2)$ or $H^2_{\text{ét}}(X, \mu_4)$ for (semi)regular line bundle-valued quadratic forms!

Application to Milnor Conjectures

over schemes

X complete curve/local field k (char k $\neq 2$), $X(k) \neq \emptyset$
$I^2(X) \subset W(X)$ quadratic forms of trivial signed discriminant

Theorem (Parimala-Sridharah '92)

$I^2(X) \xrightarrow{e_2} {}_2\mathrm{Br}(X)$ *surjective* $\Leftrightarrow$ *theta-characteristic is rational*

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Moral: when working with quadratic forms over schemes, you must consider the line bundle-valued forms!