Self-dual Galois representations

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$K/\mathbb{Q}$	finite Galois extension with group $\mathrm{Gal}(K/\mathbb{Q})$
$\rho : G \hookrightarrow \mathrm{GL}_n(\mathbb{C})$	faithful finite dimensional complex representation of $G = \mathrm{Gal}(K/\mathbb{Q})$
$\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$	continuous finite dimensional complex Galois representation of the absolute Galois group $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
$L(s, \rho)$	Artin L-function of ρ
$W(\rho) = \epsilon(\frac{1}{2}, \rho)$	root number, sign of the functional equation of the completed Artin L-function of ρ
$\rho^\vee$	dual representation, $\rho^\vee(g) = \rho(g^{-1})^t$

1. Root numbers of *self-dual* Galois representations

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(a) Symplectic representations
- Quaternion representations
- Galois module structure of algebraic integers

(b) Orthogonal representations
- Theorem of Fröhlich and Queyrut
- Deligne's interpretation in terms of Stiefel-Whitney classes

2. Consider of “essentially self-dual” Galois representations

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for a fixed 1-dimensional Galois representation λ.

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(a) Symplectic essentially self-dual
- 2-dimensional representations

(b) Orthogonal essentially self-dual
- A new generalization of the 2nd Stiefel-Whitney class

Lemma. *If ρ is irreducible and self-dual then either*

- *ρ is symplectic, i.e. ρ is isomorphic to $\rho' : G_{\mathbb{Q}} \to \mathrm{Sp}_{2n}(\mathbb{C})$, equivalently, to $\rho'' : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{H})$, or*

- *ρ is orthogonal, i.e. ρ is isomorphic to $\rho' : G_{\mathbb{Q}} \to \mathrm{O}_n(\mathbb{C})$, equivalently, to $\rho'' : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{R})$.*

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Proof. Fixing an isomorphism $\varphi : \rho \xrightarrow{\sim} \rho^\vee$,

$$\mathrm{Hom}_{\mathbb{C}}(\rho, \rho) \cong \rho^\vee \otimes \rho \overset{\varphi}{\xrightarrow{\sim}} \rho^\vee \otimes \rho^\vee \cong \mathrm{Sym}^2(\rho^\vee) \oplus \bigwedge^2(\rho^\vee),$$

which, by Schur's lemma, has 1-dimensional $G_{\mathbb{Q}}$-invariants.

Example. Generalized quaternion groups (or binary dihedral groups):

$$H_{4n} =< x, y \mid x^4 = 1, y^n = x^2, xyx^{-1} = y^{-1} >$$

of order $4n$, for $n \geq 2$. Then H_{4n} has an irreducible symplectic 2-dimensional complex representation

$$\begin{aligned} \rho : H_{4n} &\rightarrow \mathrm{Sp}_2(\mathbb{C}) \subset \mathrm{GL}_2(\mathbb{C}) \\ x &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ y &\mapsto \begin{pmatrix} \zeta_{2n} & 0 \\ 0 & \zeta_{2n}^{-1} \end{pmatrix} \end{aligned}$$

Example. Dihedral groups:

$$D_{2n} =< x, y \mid x^2 = y^n = 1, xyx^{-1} = y^{-1} >$$

of order $2n$, for $n \geq 3$. Then D_{2n} has an irreducible orthogonal 2-dimensional complex representation

$$\begin{aligned} \rho : D_{2n} &\longrightarrow \mathrm{GL}_2(\mathbb{C}) \\ x &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ y &\longmapsto \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \end{aligned}$$

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$$\begin{array}{rcl} \rho : D_{2n} & \to & \mathrm{GL}_2(\mathbb{C}) \\ x & \mapsto & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ y & \mapsto & \begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix} \end{array} \qquad \begin{array}{rcl} \rho' : D_{2n} & \to & \mathrm{O}_2(\mathbb{R}) \\ x & \mapsto & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ y & \mapsto & \begin{pmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) \\ -\sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix} \end{array}$$

Functional equation for the completed Artin L-function

$$\Lambda(1-s,\rho) = W(\rho)\,\Lambda(s,\overline{\rho}),$$

where $W(\rho) \in \mathbb{C}$ is the root number:

- $|W(\rho)| = 1$
- $W(\rho_1 \oplus \rho_2) = W(\rho_1)W(\rho_2)$
- $W(\rho) = \prod_v W_v(\rho_v)$

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- $W(\rho) = \prod_v W_v(\rho_v)$
- $W(\rho) = \pm 1$ if ρ is self-dual

Question. How to determine the sign of $W(\rho)$ for self-dual representations?

Symplectic representations. Work of Serre, Armitage, Fröhlich, Cassou-Noguès, Taylor.

Let $K/\mathbb{Q}$ be a finite extension with $G = \mathrm{Gal}(K/\mathbb{Q})$. Assume $K/\mathbb{Q}$ is tamely ramified, i.e. $p \not| e_p$ for all primes p.

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Theorem. [Noether, 1932] *The ring of integers $\mathcal{O}_K$ is a locally free $\mathbb{Z}[G]$-module iff $K/\mathbb{Q}$ is at most tamely ramified.*

Assume $K/\mathbb{Q}$ is at most tamely ramified, $G = \mathrm{Gal}(K/\mathbb{Q})$.

Classical results:

- (Hilbert) If G is abelian, then $\mathcal{O}_K \cong \mathbb{Z}[G]$.

- (Martinet, 1969) If $G \cong D_{2p}$, with p an odd prime, then $\mathcal{O}_K \cong \mathbb{Z}[G]$.

- (Martinet, 1971) If $G \cong Q = H_8$, then there exists $K/\mathbb{Q}$ with $\mathcal{O}_K \ncong \mathbb{Z}[G]$.

(Martinet, 1971) Then there is exactly one non-trivial isomorphism class M of locally free $\mathbb{Z}[Q]$-modules of rank 1. If $K/\mathbb{Q}$ is a tame Q-extension with discriminant $D_{K/\mathbb{Q}}$, then

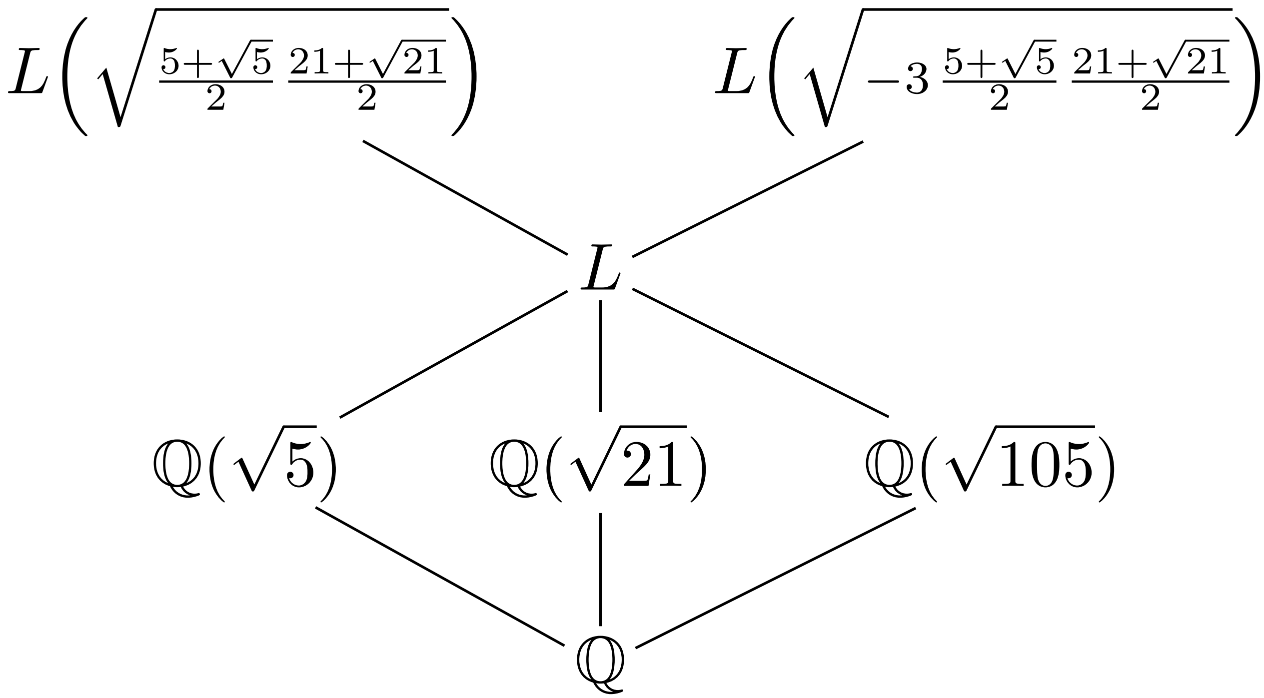
$$\delta \prod_{p \mid D_{K/\mathbb{Q}}} p \equiv u \, \frac{1 + d_1 + d_2 + d_3}{4} \mod 4,$$

where $d_i = D_{L_i/\mathbb{Q}}$, and

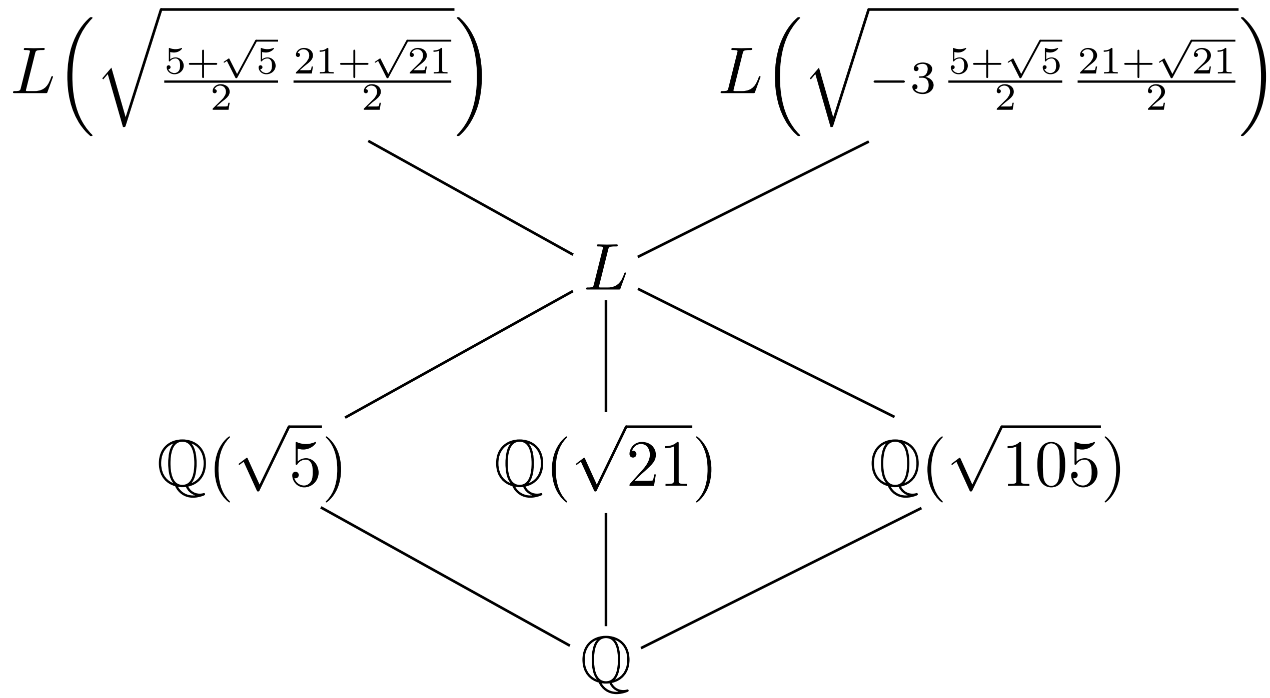
$$\delta = \begin{cases} 1 & K \text{ is totally real} \\ -1 & K \text{ is totally complex} \end{cases}$$

$$u = \begin{cases} 1 & \mathcal{O}_K \cong \mathbb{Z}[Q] \\ -1 & \mathcal{O}_K \cong M \end{cases}$$

Example.


$L\left(\sqrt{\frac{5+\sqrt{5}}{2}\,\frac{21+\sqrt{21}}{2}}\right)$
$L\left(\sqrt{-3\,\frac{5+\sqrt{5}}{2}\,\frac{21+\sqrt{21}}{2}}\right)$
L
$\mathbb{Q}(\sqrt{5})$
$\mathbb{Q}(\sqrt{21})$
$\mathbb{Q}(\sqrt{105})$
$\mathbb{Q}$

Example.



$$\delta \cdot 3 \cdot 5 \cdot 7 \equiv u \cdot \frac{1+5+21+105}{4} \mod 4$$

- (Fröhlich, 1972) For $K/\mathbb{Q}$ a tame Q-extension, then

$$W(K/\mathbb{Q}) = u(K/\mathbb{Q}) = \begin{cases} 1 & \mathcal{O}_K \cong \mathbb{Z}[Q] \\ -1 & \mathcal{O}_K \cong M \end{cases},$$

  where $W(K/\mathbb{Q}) = W(\rho)$, where $\rho : Q \to \mathrm{Sp}_2(\mathbb{C})$ is the unique irreducible symplectic representation of Q.

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- (Taylor, 1981) Let $K/\mathbb{Q}$ be a tame G-extension, then the $\mathbb{Z}[G]$-module structure of $\mathcal{O}_K$ is "completely determined" by the root numbers of symplectic representations of G. In particular, if $W(\rho) = 1$ for all symplectic representations ρ of G, then $\mathcal{O}_K \cong \mathbb{Z}[G]$.

Orthogonal representations. From the global point of view, there isn't much to say here:

Theorem. [Fröhlich and Queyrut, 1973] *Let* ρ *be an orthogonal Galois representation, then*

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Use induction from the cases:

- quadratic characters
  $\chi_a : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\sqrt{a})/\mathbb{Q}) \to \{\pm 1\}$,
- $\psi \oplus \overline{\psi}$ for cyclic characters $\psi : G_{\mathbb{Q}} \twoheadrightarrow C_n \hookrightarrow \mathbb{C}^\times$, and
- dihedral representations $\rho : G_{\mathbb{Q}} \twoheadrightarrow D_{2n} \hookrightarrow \mathrm{O}_2(\mathbb{C})$.

Deligne's approach. There's a decomposition into local root numbers,

$$W(\rho) = \prod_v W_v(\rho_v),$$

where v ranges over the places of $\mathbb{Q}$ and

$$\rho_v : G_{\mathbb{Q}_v} \to \mathrm{GL}_n(\mathbb{C})$$

is the local v-adic component of ρ, i.e. restriction to the a decomposition group of v. Here $W_v(\rho_v)$ is defined using Deligne's theory of local ϵ-constants.

Lemma. *For ρ_v a local self-dual Galois representation,*

$$W_v(\rho_v) \in \{\pm 1, \pm i\}.$$

Proof. For local Galois representations,

$$W_v(\rho_v) W_v(\rho_v^\vee) = \det(\rho)(-1),$$

where -1 is considered in $G_{\mathbb{Q}_v}$ via the norm-residue symbol of local class field theory.

Example. (Quadratic characters) Let χ_a be the quadratic character associated to $a \in \mathbb{Q}_v^\times/\mathbb{Q}_v^{\times 2}$,

$$\chi_a(g) = \frac{g\sqrt{a}}{\sqrt{a}}, \quad g \in G_{\mathbb{Q}_v}.$$

When v is real, we have:

a	1	-1
$W_v(\chi_a)$	1	$-i$

When $v = p$ is a finite place, local root numbers are evaluated by certain local Gauss sums. These calculation were first performed by Gauss.

	a	1	-1	2	-2	5	-5	10	-10
$p = 2$	$W_2(\chi_a)$	1	$-i$	i	i	1	i	-1	$-i$

	a	1	u	p	up
$p \equiv 1(4)$	$W_p(\chi_a)$	1	1	1	-1
$p \equiv 3(4)$	$W_p(\chi_a)$	1	1	$-i$	i

With some work, one also proves, for ρ_v a local orthogonal Galois representation,

$$W(\rho_v)/W(\det(\rho_v)) \in \{\pm 1\}.$$

Deligne identifies this sign with a Stiefel-Whitney class $sw_2(\rho_v) \in H^2(G_{\mathbb{Q}_v}, \{\pm 1\}) \cong \{\pm 1\}$.

This class is the obstruction to lifting an orthogonal representation to a representation of the "pin" group:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Pin}_n(\mathbb{C}) & \longrightarrow & \mathrm{O}_n(\mathbb{C}) & \longrightarrow & 1 \\ & & & & & \nwarrow & \uparrow \rho_v & & \\ & & & & & & G_{\mathbb{Q}_v} & & \end{array}$$

Via the "long exact sequence" of pointed non-abelian Galois cohomology sets of trivial Galois modules,

$$\begin{array}{rcl} H^1(G_{\mathbb{Q}_v}, \mathrm{Pin}_n(\mathbb{C})) \to H^1(G_{\mathbb{Q}_v}, \mathrm{O}_n(\mathbb{C})) & \xrightarrow{d^1} & H^2(G_{\mathbb{Q}_v}, \{\pm 1\}) \\ [\rho_v] & \mapsto & sw_2(\rho_v) \end{array}$$

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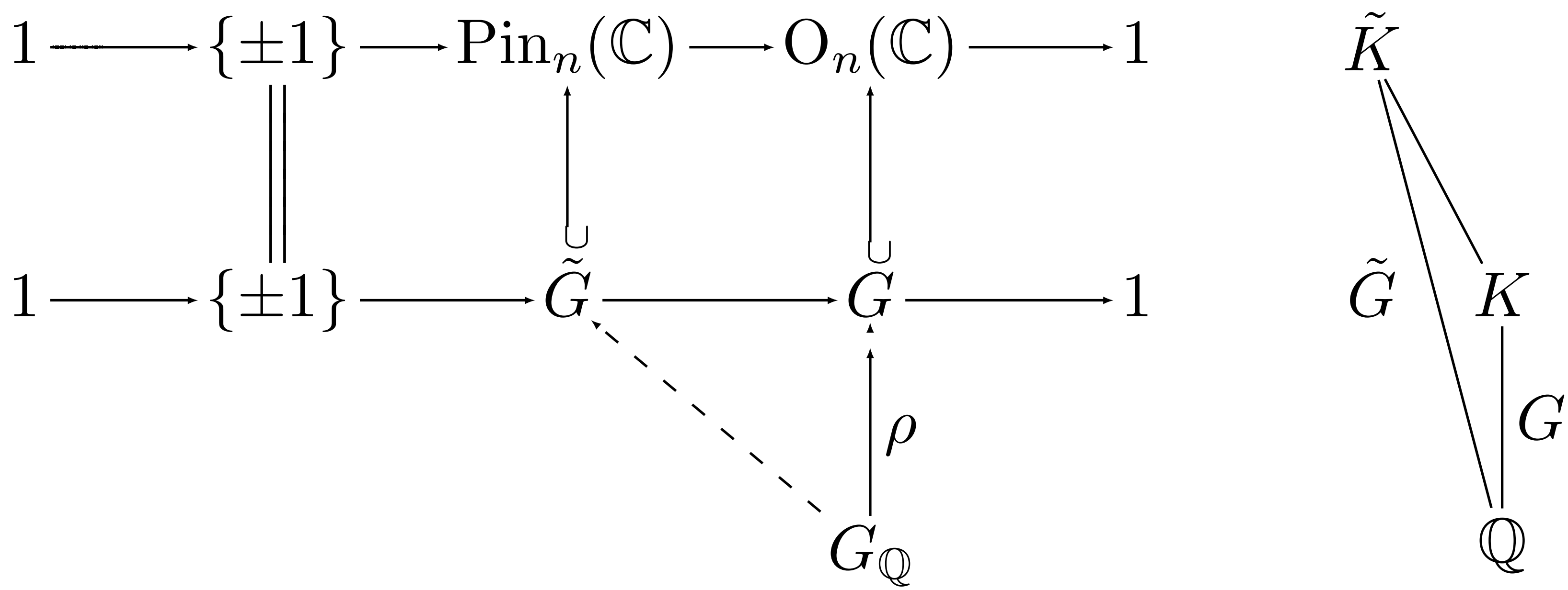
Theorem. [Deligne, 1976] *Let* ρ_v *be a local orthogonal Galois representation, then*

$$sw_2(\rho_v) = W_v(\rho_v)/W_v(\det(\rho_v)),$$

under the identification $H^2(G_{\mathbb{Q}_v}, \{\pm 1\}) \cong \{\pm 1\}$*.*

Embedding problems in inverse Galois theory.

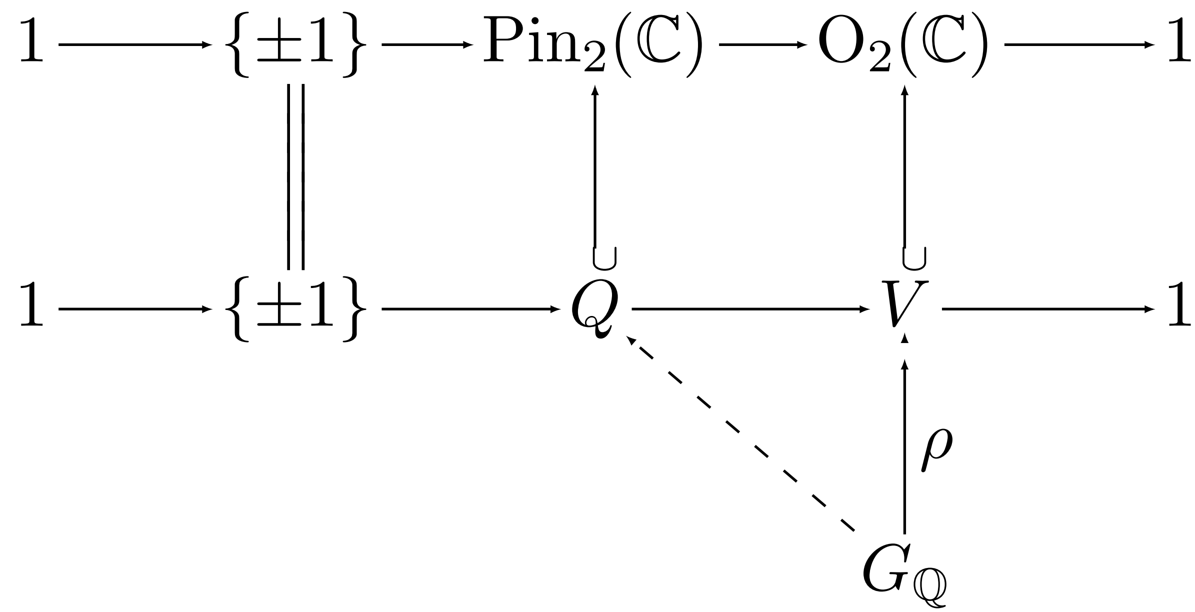
ρ is the inflation from a finite Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$



Then $sw_2(\rho) \in H^2(G_{\mathbb{Q}}, \{\pm 1\})$ is precisely the obstruction to the embedding problem $(\tilde{G}, K/\mathbb{Q})$.

Klein four inside quaternion extensions.

V the Klein four group, $V \hookrightarrow \mathrm{O}_2(\mathbb{C})$ diagonal embedding



$K = \mathbb{Q}(\sqrt{a}, \sqrt{b})/\mathbb{Q}$ can be embedded into a quaternion extension iff

$$W_v(\chi_a)\, W_v(\chi_b)/W_v(\chi_{ab}) = 1 \quad \text{for every place } v$$

Proof. (of the Fröhlich-Queyrut theorem) In the same way, the global orthogonal representation,

$$\rho : G_{\mathbb{Q}} \to \mathrm{O}_n(\mathbb{C}),$$

has a Stiefel-Whitney class $sw_2(\rho) \in H^2(G_{\mathbb{Q}}, \{\pm 1\})$. Now,

$$\begin{aligned} W(\rho) &= W(\rho)/W(\det(\rho)) = \prod_v W_v(\rho_v)/W_v(\det(\rho_v)) \\ &= \prod_v sw_2(\rho_v) = \prod_v sw_2(\rho)_v = 1 \end{aligned}$$

using finally the Hasse-Brauer-Noether exact sequence,

$$1 \to H^2(G_{\mathbb{Q}}, \{\pm 1\}) \to \bigoplus_v H^2(G_{\mathbb{Q}_v}, \{\pm 1\}) \xrightarrow{\prod} \{\pm 1\} \to 1.$$

Essentially self-dual. Fix a Galois character $\lambda : G_{\mathbb{Q}} \longrightarrow \mathbb{C}^{\times}$. Then a λ-self-dual Galois representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_n(\mathbb{C})$ satisfies

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Example. For any character λ,

$$\lambda = \lambda^{\vee} \otimes \lambda^2,$$

so λ is a λ^2-self-dual representation.

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If ρ_1, ρ_2 are λ-self-dual, then so is $\rho_1 \oplus \rho_2$.

If ρ_i is λ_i-self-dual then $\rho_1 \otimes \rho_2$ is $\lambda_1 \otimes \lambda_2$-self-dual.

Lemma. *Let ρ be an irreducible λ-self-dual Galois representation, then either*

- *ρ is symplectic similitude, i.e. ρ is isomorphic to a representation $\rho' : G_{\mathbb{Q}} \to \mathrm{GSp}_n(\mathbb{C})$, or*
- *ρ is orthogonal similitude, i.e. ρ is isomorphic to a representation $\rho' : G_{\mathbb{Q}} \to \mathrm{GO}_n(\mathbb{C})$.*

$$\begin{aligned} \mathrm{GO}_n(\mathbb{C}) &= \{M \in \mathrm{GL}_n(\mathbb{C}) : M^t M = \mu(M)\, I_n\} \\ \mathrm{GSp}_n(\mathbb{C}) &= \{M \in \mathrm{GL}_{2n}(\mathbb{C}) : M^t J_n M = \mu(M)\, J_n\} \end{aligned}$$

Symplectic similitude representations. (The case $n = 1$)

The matrix equation,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & ad-bc \\ -(ad-bc) & 0 \end{pmatrix},$$

proves that $\mathrm{GSp}_1(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C})$. In terms of representations,

$$\rho \cong \rho^\vee \otimes \det(\rho),$$

for any $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. Any 2-dimensional representation is "determinant-self-dual" of symplectic type!

There's not much hope studying root numbers of arbitrary 2-dimensional representations (though this may be feasible since the finite subgroups of $\mathrm{GL}_2(C)$ are classified).

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Question. Does there exist a Deligne style approach to studying root numbers of orthogonal similitude representations using characteristic classes?

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A new construction. A characteristic class gw_2 for λ-self-dual representations interpolating between the Stiefel-Whitney class and the 1st Chern class.

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \mathrm{Pin}_n & \longrightarrow & \mathrm{O}_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\kappa}_n & \longrightarrow & \Gamma_n & \longrightarrow & \mathrm{GO}_n & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow N & & \downarrow \mu & & \\
1 & \longrightarrow & \mu_2 & \longrightarrow & \mathbb{G}_m & \xrightarrow{\;2\;} & \mathbb{G}_m & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

$$\begin{array}{ccccc}
H^1_{\text{ét}}(X, \mathrm{Pin}_n) & \longrightarrow & H^1_{\text{ét}}(X, \mathrm{O}_n) & \xrightarrow{sw_2} & H^2_{\text{ét}}(X, \mu_2) \\
\downarrow & & \downarrow & & \downarrow \\
H^1_{\text{ét}}(X, \Gamma_n) & \longrightarrow & H^1_{\text{ét}}(X, \mathrm{GO}_n) & \xrightarrow{gw_2} & H^2_{\text{ét}}(X, \boldsymbol{\kappa}) \\
\downarrow N & & \downarrow \mu & & \downarrow \\
H^1_{\text{ét}}(X, \mathbb{G}_m) & \xrightarrow{2} & H^1_{\text{ét}}(X, \mathbb{G}_m) & \xrightarrow{c_1} & H^2_{\text{ét}}(X, \mu_2)
\end{array}$$

Applications.

- Embedding problems of the form

$$1 \to \boldsymbol{\kappa} \to G' \to G \to 1$$

where $\boldsymbol{\kappa} \cong \mathbb{Z}/4\mathbb{Z}$ or $\boldsymbol{\kappa} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and $G \hookrightarrow \mathrm{GO}_n$.

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- Normalized root number

$$W(\rho)/W(\lambda)$$

where $\rho : G_{\mathbb{Q}} \to \mathrm{GO}_n$ and $\mu : \mathrm{GO}_n \to \mathbb{G}_m$ is the multiplier coefficient homomorphism.