STABLE RATIONALITY OF QUADRIC BUNDLES

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1. Universal CH_0-triviality

This notion will likely come up a few times this week.

Definition 1.1. *A smooth proper variety X over a field k is* universally CH_0-trivial *if the degree map* $\deg : \mathrm{CH}_0(X_F) \to \mathbb{Z}$ *is an isomorphism for every field extension F/k.*

Variants of this notion were introduced by Bloch in his proof of Mumford's theorem for algebraic surfaces, and further developed in the work of Bloch and Srinivas. As stated, the notion was first considered in a paper by Merkurjev, and then developed in a paper by Auel, Colliot-Thélène, and Parimala and promoted for its use in rationality problems.

Remark 1.2. When X is rationally connected, then any two points defined over an algebraically closed field $\overline{F}$ are connected by a rational curve defined over $\overline{F}$. As a consequence, $\deg : \mathrm{CH}_0(X_{\overline{F}}) \to \mathbb{Z}$ is an isomorphism for every algebraically closed field $\overline{F}/k$. This is the condition studied in the work of Bloch and Srinivas. It has two consequences:

- (Bloch–Srinivas) There exists $N \geq 1$ such that $N\Delta_X = P \times X + Z \in \mathrm{CH}^n(X \times X)$, where P is a 0-cycle and Z is a cycle supported in $X \times V$ for a closed subvariety $V \subsetneq X$ (here $n = \dim(X)$). This is called a rational decomposition of the diagonal.
- (Colliot-Thélène) There exists $N \geq 1$ such that the kernel of the degree map $\deg : \mathrm{CH}_0(X_F) \to \mathbb{Z}$ is N-torsion for every field extension F/k. In this case, we say that $\mathrm{CH}_0(X)$ is universally N-torsion.

We now know that these consequences are actually equivalent. The following result, whose proof is easy in hindsight, is one of the key lemmas proved in our paper.

Lemma 1.3. *Let X be a smooth proper variety of dimension n over a field k. Then the following are equivalent:*

1. *X is universally CH_0-trivial.*
2. *X has a 0-cycle of degree 1 and $\deg : \mathrm{CH}_0\big(X_{k(X)}\big) \to \mathbb{Z}$ is an isomorphism.*
3. *X admits an* integral decomposition of the diagonal *$\Delta_X = P \times X + Z \in \mathrm{CH}^n(X \times X)$ where P is a 0-cycle of degree 1 and Z is a cycle supported in $X \times V$ for a closed subset $V \subsetneq X$.*

Proof. (1) implies (2) by definition. Letting P be a 0-cycle of degree 1 on X and $\eta \in X \times_k k(X)$ the "diagonal generic point" (which is a 0-cycle of degree 1 on $X_{k(X)}$), then (2) implies that $\eta = P$ in $\mathrm{CH}_0(X \times_k k(X))$, which yields (3) upon taking Zariski closure. Thinking of the integral decomposition of the diagonal as a correspondence $\mathrm{CH}_0(X) \to \mathrm{CH}_0(X)$ shows that every 0-cycle is equivalent to a multiple of P. Since the integral decomposition of the diagonal persists after extending scalars to F, we arrive at (1). $\square$

There is also a version of this lemma for universal N-torsion.

In our paper, we asked whether there exists a Fano variety that fails to be CH_0-universally trivial. After reading our paper, Voisin provided the first example, and in the process developed her degeneration method for obtructing universal CH_0-triviality.

Universal CH_0-triviality is a stable birational invariant of smooth proper k-varieties (Colliot-Thélène–Coray, Fulton) and has good specialization properties (Voisin, Colliot-Thélène–Pirutka).

Examples 1.4. *The following are examples of universally* CH_0*-trivial varieties:*

- *X stably rational over k (since $\mathbb{P}^n$ is universally CH_0-trivial)*
- *X a smooth projective surface over $\mathbb{C}$ with $\mathrm{CH}_0(X) = \mathbb{Z}$ (i.e., $p_g(X) = q(X) = 0$ and X satisfies Bloch's conjecture) and $\mathrm{NS}(X)$ torsionfree, e.g., X a Barlow surface.*

Because of the first example, universal CH_0-triviality is an obstruction to stable rationality, but it is very hard to compute! Mostly, we obstruct one of the following consequences, which are all arrived at by considering the integral decomposition of the diagonal as a correspondence acting on various cohomology theories:

- If X is a smooth projective surface over $\mathbb{C}$ then transcendental cohomology $T(X) = \mathrm{NS}(S)^{\perp} = 0$. Bloch used this to give a new proof of Mumford's theorem.
- If X is a smooth projective variety over k, then $H^0(X, \Omega^i_X) = 0$ for all $i > 0$. In characteristic zero, this was proved by Bloch; in characteristc p, a proof was given by Totaro using Gros' theory of cycle class map in logarithmic de Rham cohomology.
- If X is a smooth projective variety over a field of characteristic p, the Frobenius slope $[0, 1)$ part of rigid cohomology $H^i_{[0,1)}(X/k) = 0$ for all $i > 0$. This was proved by Esnault, and used, together with an argument involving the Lefschetz trace formula, to prove that $|X(\mathbb{F}_q)| \equiv 1 \pmod q$ if X is defined over $\mathbb{F}_q$.
- If X is any variety over k, then the unramified cohomology groups $H^i_{\mathrm{ur}}(X/k)$ are trivial for $i \geq 0$, meaning that the natural map $H^i(k) \to H^i_{\mathrm{ur}}(X/k)$ is an isomorphism. This was proved by Merkurjev. We always use coefficients $\mathbb{Q}/\mathbb{Z}(i-1)$ in degree i.

  In low degree the unramified cohomology groups are well-known. Purity implies that $H^1_{\mathrm{ur}}(X/k) = H^1_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z})$ (for X locally factorial) and $H^2_{\mathrm{ur}}(X/k) = \mathrm{Br}(X)$ (for X regular). A result of Colliot-Thélène and Voisin says that

  $$H^3_{\mathrm{ur}}(X/\mathbb{C}) = H^{2,2}(X) \cap H^4(X, \mathbb{Z}) / \operatorname{im}(\mathrm{CH}^2(X) \to H^4(X, \mathbb{Z}))$$

  (for X is smooth projective and rationally connected over $\mathbb{C}$), i.e., $H^3_{\mathrm{ur}}(X/\mathbb{C})$ is the obstruction to the integral Hodge conjecture for codimension 2 cycles on X.

  Merkurjev proves more generally that X is universally CH_0-triviality if and only if all unramified cohomology groups arising from all cycle modules (in the sense of Rost) are universally trivial.

We remark that the first three consequences above (except for triviality of differential forms in characteristic p) only require CH_0 to be universally N-torsion (e.g., rationally connected), while unramified cohomology is more sensitive to torsion phenomena.

2. Voisin's degeneration method d'après Colliot-Thélène–Pirutka

Let X be a smooth proper geometrically integral variety over k. Voisin's degeneration method, as developed by Colliot-Thélène and Piritka, proceeds as follows:

(1) Fit X into a flat proper family $\mathcal{X} \to B$ over a scheme of finite type with central fiber X_0 possibly singular.

(2) Find a universally CH_0-trivial resolution $f : \widetilde{X}_0 \to X_0$, i.e., the pushforward map $f_* : \mathrm{CH}_0(\widetilde{X}_{0,F}) \to \mathrm{CH}_0(X_{0,F})$ is an isomorphism for all F/k. A sufficient condition is that every scheme-theoretic fiber of the resolution is universally CH_0-trivial over the residue field.

(3) Show that $\widetilde{X}_0$ is not universally CH_0-trivial, e.g., by obstructing one of the above consequences (differential forms, unramified cohomology, etc.)

The outcome is that the very general fiber of the family (though perhaps not X itself) will not be universally CH_0-trivial.

Remark 2.1. The notion of a universally CH_0-trivial resolution defines a new class of singularities that should be classified in the spirit of the minimal model program. Some are known: ordinary double points and, more generally, singular loci with rational exceptional divisors, but the exceptional locus could also be a Barlow surface.

When X is equipped with a quadric bundle structure $X \to S$ over a rational variety S, I'll discuss part (3). A program to address part (1) has been layed out by Hassett, Kresch, and Tschinkel using deformation theory. Part (2) can be complicated to check in special cases.

3. Quadric bundles

Let S be a smooth projective geometrically integral variety. A quadric bundle will be a proper flat morphism $X \to S$ whose fibers are quadric hypersurfaces of positive dimension in a projective space bundle over S. The discriminant divisor $D \subset S$ parameterizes the singular fibers. We assume that the generic fiber of $X \to S$ is a smooth quadric over $k(S)$ but we do not necessarily assume that X is smooth.

Examples 3.1. *The following are motivating examples:*

- *Conic bundles over rational surfaces, e.g., those considered by Artin and Mumford.*
- *Given a smooth cubic hypersurface $Y \subset \mathbb{P}^{n+1}$, with $n \geq 3$, and a line $\ell \subset Y$, the blow-up Y_ℓ of Y along the line resolves the projection from the line, and defines a conic bundle $Y_\ell \to \mathbb{P}^{n-1}$ whose discriminant is a quintic hypersurface. Interesting examples are cubic threefolds and fourfolds, where we get conic bundles over $Y_\ell \to \mathbb{P}^2$ and $Y_\ell \to \mathbb{P}^3$, respectively.*

- *Given a smooth cubic hypersurface $Y \subset \mathbb{P}^{2m+1}$ containing a linear space $P = \mathbb{P}^m$. The blow-up $Y_P \to \mathbb{P}^m$ resolves the projection from P, and defines a quadric bundle with discriminant divisor of degree $m+4$. When Y is a cubic fourfold containing a plane, where we get a quadric surface bundle $Y_\ell \to \mathbb{P}^2$ with sextic discriminant.*

Now I will describe a method to compute the unramified cohomology groups $H^i_{\mathrm{ur}}(X/k)$ of the total space of a quadric bundle, developed by Colliot-Thélène and Ojanguren.

By definition, $H^i_{\mathrm{ur}}(X/k)$ is the cohomology in degree 0 of the Gersten complex in Galois cohomology

$$H^i(k(X)) \xrightarrow{\oplus\partial^i_v} \bigoplus_v H^{i-1}(k(v)) \xrightarrow{\oplus\partial^{i-1}_v} \bigoplus_w H^{i-2}(k(w)) \xrightarrow{\oplus\partial^{i-1}_v} \cdots$$

where the first sum is taken over all discrete (rank 1) valuations v on the function field $k(X)$, whose valuation ring $\mathscr{O}_v$ contains k and has residue field $k(v)$, and where ∂^i_v are residue maps defined as Gysin coboundary maps in étale cohomology for the closed embedding $\operatorname{Spec} k(v) \to \operatorname{Spec} \mathscr{O}_v$.

When X is smooth and proper, one only needs to compute residues at divisorial valuations centered at codimension 1 points $x \in X^{(1)}$. Also, by Bloch–Ogus theory, $H^i_{\mathrm{ur}}(X/k) = H^0(X, \mathcal{H}^i)$ is the set of global sections of the Zariski sheaf associated to the presheaf $U \mapsto H^i_{\text{ét}}(U, \mathbb{Q}/\mathbb{Z}(i-1))$. Here, the torsion should stay away from any residue characteristic.

Now we assume that $S = \mathbb{P}^2$ and that k is algebraically closed. Consider the following diagram arising from separating out those valuations on $k(X)$ that are trivial on $k(S)$ and those that dominate higher codimensional points of S. In the diagram, we will only consider residues at divisorial valuations, which in the case when X is singular, singles out a possibly larger set than the full unramified cohomology of X:

$$\begin{array}{ccccccc} H^i_{\mathrm{ur}}(X/k) \hookrightarrow & H^i_{\mathrm{ur}}(X_{k(S)}/k(S)) & \xrightarrow{\oplus\partial^i_s} & \bigoplus_{s\in S^{(1)}} H^{i-1}(k(X_s)) & \xrightarrow{\oplus\partial^i_t} & \bigoplus_{t\in S^{(2)}} H^{i-2}(k(X_t)) \\ & \uparrow{\scriptstyle r_X} & & \uparrow{\scriptstyle \oplus r_{X_s}} & & \uparrow \\ 0 \longrightarrow & H^i(k(S)) & \xrightarrow{\oplus\partial^i_s} & \bigoplus_{s\in S^{(1)}} H^{i-1}(k(s)) & \xrightarrow{\oplus\partial^{i-1}_t} & \bigoplus_{t\in S^{(2)}} H^{i-2}(k(t)) \end{array} \tag{3.1}$$

We will only be interested in the sequence for $i = 2, 3$. Of particular concern are the following facts concerning the kernel and cokernel of the restriction maps r_X and r_{X_s}.

Theorem 3.2 (Arason, Pfister, Kahn–Rost–Sujatha)**.** *Let Q be a smooth quadric over a field K of characteristic $\neq 2$. Then for $i \leq 2$ or $i = 3$ and Q not an anisotropic Albert quadric, the restriction map $H^i(K) \to H^i_{\mathrm{ur}}(Q/K)$ is surjective.*

Corollary 3.3 (Colliot-Thélène)**.** *If S is a smooth proper surface over $\mathbb{C}$ and $X \to S$ is a quadric bundle, then $H^3_{\mathrm{ur}}(X/k) = 0$.*

Proof. The function field $\mathbb{C}(X)$ is a C_2 field, so has cohomological dimension 2 and any quadric of dimension at least 3 has a $\mathbb{C}(X)$ rational point. Thus $H^3(\mathbb{C}(S)) \to H^3_{\mathrm{ur}}(X_{k(S)}/k(S))$ is surjective (in relative dimensions 1 and 2 by the result above, and in relative dimension at least 3 since $X_{k(S)}$ is $k(S)$-rational). But $H^3(\mathbb{C}(S)) = 0$, so we are done. □

We now assume, in diagram (3.1), that $H^3_{\text{ét}}(S, \mu_2^{\otimes 2}) = 0$ and that the cycle class map $\mathrm{CH}^2(S)/2 \to H^4_{\text{ét}}(S, \mu_2^{\otimes 2})$ is an isomorphism. For example, we could take $S = \mathbb{P}^n$. From the Bloch–Ogus spectral sequence, these conditions ensure that $H^1(S, \mathcal{H}^i) = 0$ for $i = 2, 3$, ensuring that the bottom row in the diagram is exact.

Then by a diagram chase in (3.1), we see that

$$H^i_{\mathrm{ur}}(X/k) \subset \frac{\ker(\oplus r_{X_s}) \cap \ker(\oplus \partial_t^{i-1})}{\ker(r_X)}.$$

Hence we are particularly interested in the kernel of the restriction map $H^j(K) \to H^j(K(Q))$ for a (possibly singular) quadric Q over a field K, for $j = 1, 2, 3$.

- Case $j = 1$. The restriction map is injective whenever Q is geometrically integral over K; there is a kernel in the case when Q is geometrically a union of two disjoint lines, planes, etc., and in this case, the kernel is generated by $\delta \in H^1(K, \mathbb{Z}/2\mathbb{Z})$, the class of the (quadratic) field of definition of the components of Q. When Q is a nonreduced double line or plane, then the correct analogue of the restriction map (which is twice the restriction map the the reduced subscheme of Q) has kernel all of $H^1(K, \mathbb{Z}/2\mathbb{Z})$.
- Case $j = 2$. For Q a smooth conic, the kernel is generated by the quaternion class $c(Q)$ associated to Q. If Q is geometrically a union of disjoint lines, then the kernel is $\delta \cup H^1(K)$. For Q a smooth quadric surface, the restriction is injective if Q has nontrivial discriminant (i.e., $\mathrm{Pic}(Q) = \mathbb{Z}$), and generated by the Clifford invariant $c(Q)$ when the discriminant is trivial (i.e., $\mathrm{Pic}(Q) = \mathbb{Z}^2$). For Q a cone over a smooth conic Q', the kernel of the restriction is generated $c(Q')$. For Q geometrically a union of two disjoint planes, the kernel of the restriction is $\delta \cup H^1(K)$.
- Case $j = 3$. For Q a smooth conic, the kernel is $c(Q) \cup H^1(K)$. For Q a smooth quadric surface, the kernel is nontrivial and was computed by Arason.

3.1. **Examples (conic bundles over surfaces).** Let S be a smooth projective rational surface over $\mathbb{C}$ and $X \to S$ a conic bundle with discriminant divisor $D \subset S$ and associated Brauer class $\alpha \in \mathrm{Br}(\mathbb{C}(S))$. Then $\partial_s^2 \alpha \neq 0$ if and only if s is the generic point of a component of D. By the above, $H^i_{\mathrm{ur}}(X/\mathbb{C}) = 0$ for $i = 3$, so we focus on $i = 2$.

3.1.1. Assume D is smooth irreducible. This implies that X is smooth. In this case $\ker(r_{X_s})$ is trivial except when s is the generic point of D, when the group is generated by $\partial_s^2 \alpha$. Hence $\ker(\oplus r_{X_s})$ is generated by the image of α. But also $\ker(r_X)$ is generated by α. Hence $H^2_{\mathrm{ur}}(X/\mathbb{C}) = 0$. In fact, this argument works exactly the same as long as D is only assumed to be irreducible (not necessarily smooth).

3.1.2. *(Artin–Mumford).* Assume $D = D_1 \sqcup D_2$ disjoint union of smooth irreducible curves. This implies that X is smooth. Then $\ker(\oplus r_{X_s}) = \partial_{s_1}^2 \alpha \, \mathbb{Z}/2\mathbb{Z} \oplus \partial_{s_2}^2 \alpha \, \mathbb{Z}/2\mathbb{Z}$ and this is contained in $\ker(\oplus \partial_t^1)$. However, $\ker(r_X) = \alpha \, \mathbb{Z}/2\mathbb{Z}$. Hence $H^2_{\mathrm{ur}}(X/\mathbb{C})$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ modulo the diagonal $\mathbb{Z}/2\mathbb{Z}$, so is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This is an unramified cohomological description of the Brauer class discovered by Artin and Mumford.

3.1.3. *(Colliot-Thélène).* Assume $D = \cup D_i$ is a strict normal crossings divisor such that X is smooth (in this case the conic bundle $X \to S$ is standard). Each irreducible component C_i contributes $\partial^2_{s_i}\alpha\,\mathbb{Z}/2\mathbb{Z}$ to $H^2_{\mathrm{ur}}(X/\mathbb{C})$, though if $t \in C_i \cap C_i$ and the residues ∂^1_t are nonzero, then the choice of class at C_i and C_j must agree in order to lie in $\ker(\partial^1_t)$. Colliot-Thélène used this method to give a precise formula for $H^2_{\mathrm{ur}}(X/\mathbb{C})$.

3.2. **Examples (quadric surface bundle over surfaces).** Let S be a smooth projective rational surface over $\mathbb{C}$ and $X \to S$ a quadric surface bundle with discriminant divisor $D \subset S$. As before, $H^i_{\mathrm{ur}}(X/\mathbb{C}) = 0$ for $i = 3$ so we focus on $i = 2$.

3.2.1. Assume D is smooth irreducible. In particular, X is smooth and all fibers X_s are geometrically integral, being smooth conics for s away from D and being geometrically the union of two disjoint lines for s the generic point of D. Hence $\ker(\oplus r_{X_s}) = 0$ and thus $H^2_{\mathrm{ur}}(X/\mathbb{C}) = 0$. In fact, this argument works as long as D is only assumed to be reduced.

3.2.2. Assume $D = 2D_1 \sqcup 2D_2$ is the disjoint union of two nonreduced divisors whose reduced subschemes are smooth. In this case, the generic fiber $X_{\mathbb{C}(S)}$ is a quadric surface with trivial discriminant, so there is a Clifford invariant $\alpha = c(X_{\mathbb{C}(S)}) \in \mathrm{Br}(\mathbb{C}(S))$ generating $\ker(r_X)$. Then $\ker(\oplus r_{X_s}) = \partial^2_{s_1}\alpha\,\mathbb{Z}/2\mathbb{Z} \oplus \partial^2_{s_2}\alpha\,\mathbb{Z}/2\mathbb{Z}$ and this is contained in $\ker(\oplus\partial^1_t)$. Hence $H^2_{\mathrm{ur}}(X/\mathbb{C})$ is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ modulo the diagonal, so is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

This seems reminicient of the Artin–Mumford example! In fact, given $Y \to S$ a conic bundle whose discriminant is a disjoint union of two smooth curves $D = D_1 \sqcup D_2$, then $X = Y \times_S Y \to S$ is a quadric surface bundle with discriminant $D = 2D_1 \sqcup 2D_2$ such that Y is birational to $X \times \mathbb{P}^1$. In particular, X and Y have isomorphic unramified cohomology groups. Thus such examples are really explained by the Artin–Mumford construction. Algebraically, if $\alpha = (a, b) \in \mathbb{C}(S)$ is the Brauer class associated to the Artin–Mumford example $Y \to S$, then the generic fiber of $X \to S$ is the quadric associated to the Pfister form $\langle 1, a, b, ab\rangle$. However, in such examples, X has a curve (of positive genus) of singularities above D, and so will likely not have a universally CH_0-trivial resolution.

3.2.3. *(Pirutka).* Gave the first explicit examples of quadric surface bundles $X \to S$ with $H^2_{\mathrm{ur}}(X/k) \neq 2$ and with nontrivial generic discriminant. She also gave a formula for $H^2_{\mathrm{ur}}(X/k)$ using these methods, in the spirit of Colliot-Thélène's formula for conic bundles.

3.3. **What about H^3_{ur}?** As we have seen before, if $X \to S$ is a quadric bundle with $H^3_{\mathrm{ur}}(X/\mathbb{C})$ nontrivial we must have $\dim(S) \geq 3$ or be working over a nonalgebraically closed base field. In the former case, examples of quadric threefold bundles $X \to \mathbb{P}^3$ with $H^3_{\mathrm{ur}}(X/\mathbb{C}) \neq 0$ where discovered by Colliot-Thélène–Ojanguren. In the later case, this method was used by Auel–Colliot-Thélène–Parimala to prove $H^3_{\mathrm{ur}}(X/F) = H^3(F)$ for a smooth cubic fourfold X over $\mathbb{C}$ containing a plane and $F/\mathbb{C}$ any field extension.

One interesting case is the conic bundle $X_\ell \to \mathbb{P}^3$ arising from a smooth cubic fourfold. In this case the discriminant D is irreducible but with isolated ordinary double points. Though Voisin has proved the integral Hodge structure in this case, one might wonder if the irreducibility of D might generally imply the triviality of $H^3_{\mathrm{ur}}(X/\mathbb{C})$ using this method.