

Azumaya algebras without involution

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Galois Cohomology and the Brauer Group

Albert's Theorem

k field

$\text{Br}(k)$ Brauer group

A central simple k -algebra, $[A] \in \text{Br}(k)$

Definition. An *involution* (of the first kind) on A is a k -algebra isomorphism $\sigma : A \rightarrow A^{\text{op}}$ such that $\sigma^{\text{op}} \circ \sigma = \text{id}_A$.

Example. Assume $\text{char}(k) \neq 2$.

$A = \langle i, j : i^2 = a, j^2 = b, ij = -ji \rangle$ quaternion algebra

$a + bi + cj + dij \mapsto a - bi - cj - dij$ standard involution

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Proposition. A has an involution $\implies \text{per}(A) = 2$

$$A \otimes A \longrightarrow \text{End}(A)$$

$$x \otimes y \longmapsto (z \mapsto x z \sigma(y))$$

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Theorem (Albert). $\text{per}(A) = 2 \implies A$ has an involution

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Theorem (Knus–Parimala–Srinivas). $\text{per}(A) = 2 \implies M_2(A)$ has an involution, i.e., there exists locally free A -module P of rank 2 such that $\text{End}_A(P)$ has an involution

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Question. If $\text{per}(A) = 2$ then can A fail to have an involution?

Split Algebras

P locally free R -module of finite rank

L invertible R -module

$b : P \times P \rightarrow L$ nondegenerate (skew-)symmetric bilinear form

$\psi_b : P \rightarrow \text{Hom}(P, L)$ isomorphism

Definition. The *adjoint involution* associated to b is

$$\begin{aligned}\sigma_b : \text{End}(P) &\longrightarrow \text{End}(P)^{\text{op}} \\ f &\longmapsto \psi^{-1} \circ f^{\vee L} \circ \psi\end{aligned}$$

where $f^{\vee L} : \text{Hom}(P, L) \rightarrow \text{Hom}(P, L)$ is the L -dual of f .

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Theorem (Saltman). Any involution on $\text{End}(P)$ is adjoint to some $b : P \times P \rightarrow L$.

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Corollary. If $P \not\cong P^\vee \otimes L$ for any invertible R -module L then $\text{End}(P)$ has no involution.

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Examples.

- 1 (A) $X = \mathbb{P}^1$, $\mathcal{P} = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$
 $\text{End}(\mathcal{P})$ has no involution
- 2 (U. First) R ring of integers in a number field K/\mathbb{Q} with $\text{Cl}(R) \cong \mathbb{Z}/4\mathbb{Z}$

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Question'. If $[A] \in {}_2\text{Br}(R)$ then can every Brauer equivalent algebra of the same degree fail to have an involution?

Preliminary Reductions

Question'. If $[A] \in {}_2\text{Br}(R)$ then can every Brauer equivalent algebra of the same degree fail to have an involution?

Degree considerations:

- 1 $\deg(A) = 2 \implies A$ has (standard) involution
More generally, can deal with $\text{ind}(A) = 2$.
- 2 $\deg(A)$ odd $\implies A$ is split
We know how to handle this case.
- 3 A is always Brauer equivalent to $A_1 \otimes A_2$ where $\text{ind}(A_1)$ is odd and $\text{ind}(A_2) = 2^n$
(Antieau–Williams) A not always isomorphic to $A_1 \otimes A_2$

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So we can reduce to considering $\text{ind}(A) = 2^n$ with $n \geq 2$.
First open case is $\deg(A) = \text{ind}(A) = 4$.

Results

Theorem (A–First). For every $n \geq 2$, there exists a ring R and an Azumaya R -algebra A of degree 2^n and period 2 such that no Azumaya R -algebra B of degree 2^n Brauer equivalent to A has an involution.

Results

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Remark. For $n = 2$, i.e., $\deg(A) = 4$, we can take R to be a finitely generated \mathbb{C} -algebra of dimension 3.

Examples with R of smaller dimension over more complicated fields are also possible.

Sketch of Proof

$$G = \mathbf{GL}_4/\mu_2$$

\mathbf{BG} classifying “space” (topological, simplicial, stack)

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Facts.

- 1 G -tors $\longleftrightarrow (A, V, \varphi)$ *2-torsion data*
A Azumaya algebra of degree 4
 V is locally free of rank 16
 $\varphi : A \otimes A \rightarrow \text{End}(V)$ isomorphism
- 2 (A, V, φ) universal 2-torsion data on \mathbf{BG}
 $(A, V, \varphi) = f^*(A, V, \varphi)$ on X for some $f : X \rightarrow \mathbf{BG}$
- 3 $\text{Br}(\mathbf{BG}) \cong \mathbb{Z}/2\mathbb{Z}$ generated by A

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Sketch of Proof

Proposition. The Azumaya algebra \mathbf{A} on \mathbf{BG} has no involution.

Proof.

- $A = \text{End}(P)$ split Azumaya algebra of degree 4 with no involution on X , e.g., $X = \mathbb{P}^1$ and $P = \mathcal{O}^{\oplus 3} \oplus \mathcal{O}(1)$
- A extends to 2-torsion datum $(\text{End}(P), P \otimes P, \varphi_P)$
 $\varphi_P : \text{End}(P)^{\otimes 2} \rightarrow \text{End}(P^{\otimes 2})$ canonical isomorphism
- via the classifying map $f : X \rightarrow \mathbf{BG}$
 $(\text{End}(P), P \otimes P, \varphi_P) = f^*(\mathbf{A}, \mathbf{V}, \varphi)$
- if \mathbf{A} had an involution then $\text{End}(P) = f^* \mathbf{A}$ would □

Sketch of Proof

Details left out:

Proving that no Azumaya algebra B of degree 4 on BG , Brauer equivalent to A , has an involution.

The passage from “space” BG to an affine scheme using quasi-projective approximation of BG à la Totaro.
This step is not new (e.g., Antieau–Williams).

Questions

Find examples with R of minimal dimension (say over \mathbb{C})?
For Azumaya algebras of degree 4, we can take “generic”
algebra constructions?

Find examples over projective varieties?