Azumaya algebras without involution

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Galois Cohomology and the Brauer Group

Albert's Theorem

k field
$\mathrm{Br}(k)$ Brauer group
A central simple k-algebra, $[A] \in \mathrm{Br}(k)$

Definition. An *involution* (of the first kind) on A is a k-algebra isomorphism $\sigma : A \to A^{\mathrm{op}}$ such that $\sigma^{\mathrm{op}} \circ \sigma = \mathrm{id}_A$.

Example. Assume $\mathrm{char}(k) \neq 2$.
$A = < i, j : i^2 = a, j^2 = b, ij = -ji >$ quaternion algebra
$a + bi + cj + dij \mapsto a - bi - cj - dij$ standard involution

Proposition. A has an involution $\implies$ $\mathrm{per}(A) = 2$

$$\begin{aligned} A \otimes A &\longrightarrow \mathrm{End}(A) \\ x \otimes y &\longmapsto \big(z \mapsto x\, z\, \sigma(y)\big) \end{aligned}$$

Theorem (Albert). $\mathrm{per}(A) = 2 \implies A$ has an involution

Saltman's Theorem

R ring
A Azumaya R-algebra
$\sigma : A \to A^{\mathrm{op}}$ involution

Proposition. A has an involution $\implies$ $\mathrm{per}(A) = 2$

Theorem (Saltman). $\mathrm{per}(A) = 2 \implies A$ is Brauer equivalent to an Azumaya algebra with involution

Theorem (Knus–Parimala–Srinivas). $\mathrm{per}(A) = 2 \implies M_2(A)$ has an involution, i.e., there exists locally free A-module P of rank 2 such that $\mathrm{End}_A(P)$ has an involution

Question. If $\mathrm{per}(A) = 2$ then can A fail to have an involution?

Split Algebras

P locally free R-module of finite rank
L invertible R-module
$b : P \times P \to L$ nondegenerate (skew-)symmetric bilinear form
$\psi_b : P \to \mathrm{Hom}(P, L)$ isomorphism

Definition. The *adjont involution* associated to b is

$$\begin{aligned} \sigma_b : \mathrm{End}(P) &\longrightarrow \mathrm{End}(P)^{\mathrm{op}} \\ f &\longmapsto \psi^{-1} \circ f^{\vee L} \circ \psi \end{aligned}$$

where $f^{\vee L} : \mathrm{Hom}(P, L) \to \mathrm{Hom}(P, L)$ is the L-dual of f.

Theorem (Saltman). Any involution on $\mathrm{End}(P)$ is adjoint to some $b : P \times P \to L$.

Trivial Answer

Theorem (Saltman). Any involution on End(P) is adjoint to some $b : P \times P \to L$.

Corollary. If $P \ncong P^\vee \otimes L$ for any invertible R-module L then End(P) has no involution.

Examples.

1. (A) $X = \mathbb{P}^1$, $\mathscr{P} = \mathscr{O} \oplus \mathscr{O} \oplus \mathscr{O} \oplus \mathscr{O}(1)$
   $\mathscr{E}nd(\mathscr{P})$ has no involution
2. (U. First) R ring of integers in a number field $K/\mathbb{Q}$ with $\mathrm{Cl}(R) \cong \mathbb{Z}/4\mathbb{Z}$

Question'. If $[A] \in {}_2\mathrm{Br}(R)$ then can every Brauer equivalent algebra of the same degree fail to have an involution?

Preliminary Reductions

Question'. If $[A] \in {}_2\mathrm{Br}(R)$ then can every Brauer equivalent algebra of the same degree fail to have an involution?

Degree considerations:

1. $\deg(A) = 2 \implies A$ has (standard) involution
   More generally, can deal with $\mathrm{ind}(A) = 2$.
2. $\deg(A)$ odd $\implies A$ is split
   We know how to handle this case.
3. A is always Brauer equivalent to $A_1 \otimes A_2$ where $\mathrm{ind}(A_1)$ is odd and $\mathrm{ind}(A_2) = 2^n$
   (Antieau–Williams) A not always isomorphic to $A_1 \otimes A_2$

So we can reduce to considering $\mathrm{ind}(A) = 2^n$ with $n \geq 2$.
First open case is $\deg(A) = \mathrm{ind}(A) = 4$.

Results

Theorem (A–First). For every $n \geq 2$, there exists a ring R and an Azymaya R-algebra A of degree 2^n and period 2 such that no Azumaya R-algebra B of degree 2^n Brauer equivalent to A has an involution.

Remark. For $n = 2$, i.e., $\deg(A) = 4$, we can take R to be a finitely generated $\mathbb{C}$-algebra of dimension 3.

Examples with R of smaller dimension over more complicated fields are also possible.

Sketch of Proof

$G = \mathbf{GL}_4/\mu_2$
$\boldsymbol{B}G$ classifying "space" (topological, simplicial, stack)

Facts.

1. G-tors $\longleftrightarrow$ (A, V, φ) *2-torsion data*
   A Azumaya algebra of degree 4
   V is locally free of rank 16
   $\varphi : A \otimes A \to \mathrm{End}(V)$ isomorphism
2. $(\boldsymbol{A}, \boldsymbol{V}, \boldsymbol{\varphi})$ universal 2-torsion data on $\boldsymbol{B}G$
   $(A, V, \varphi) = f^*(\boldsymbol{A}, \boldsymbol{V}, \boldsymbol{\varphi})$ on X for some $f : X \to \boldsymbol{B}G$
3. $\mathrm{Br}(\boldsymbol{B}G) \cong \mathbb{Z}/2\mathbb{Z}$ generated by $\boldsymbol{A}$

Proposition. The Azumaya algebra $\boldsymbol{A}$ on $\boldsymbol{B}G$ has no involution.

Sketch of Proof

Proposition. The Azumaya algebra $\boldsymbol{A}$ on $\boldsymbol{B}G$ has no involution.

Proof.

- $A = \mathrm{End}(P)$ split Azumaya algebra of degree 4 with no involution on X, e.g., $X = \mathbb{P}^1$ and $P = \mathscr{O}^{\oplus 3} \oplus \mathscr{O}(1)$
- A extends to 2-torsion datum $(\mathrm{End}(P), P \otimes P, \varphi_P)$
  $\varphi_P : \mathrm{End}(P)^{\otimes 2} \to \mathrm{End}(P^{\otimes 2})$ canonical isomorphism
- via the classifying map $f : X \to \boldsymbol{B}G$
  $(\mathrm{End}(P), P \otimes P, \varphi_P) = f^*(\boldsymbol{A}, \boldsymbol{V}, \varphi)$
- if $\boldsymbol{A}$ had an involution then $\mathrm{End}(P) = f^*\boldsymbol{A}$ would $\square$

Sketch of Proof

Details left out:

Proving that no Azumaya algebra B of degree 4 on $\boldsymbol{B}G$, Brauer equivalent to $\boldsymbol{A}$, has an involution.

The passage from "space" $\boldsymbol{B}G$ to an affine scheme using quasi-projective approximation of $\boldsymbol{B}G$ à la Totaro. This step is not new (e.g., Antieau–Williams).

Questions

Find examples with R of minimal dimension (say over $\mathbb{C}$)?
For Azumaya algebras of degree 4, we can take “generic” algebra constructions?

Find examples over projective varieties?