FLORIAN POP'S MATH 702 FALL 2005
Topics in Algebra: Seminar in Model Theory and Forms

1. AN INTRODUCTION TO MODEL THEORY

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1.1. **First Order Predicate Calculus.**

Definition 1. Given index sets I and J, families of natural numbers $\mu = (\mu_i)_{i\in I}$ and $\nu = (\nu_j)_{j\in J}$, and a set K, a *first order language* or *language* $\mathcal{L} = (\mu, \nu, K)$ is the free semigroup generated by the following elements

- *variable symbols* $X_1, X_2, \ldots$
- *constant symbols*, i.e. the elements of K
- for each $i \in I$, a μ_i-ary *relation* symbol R_i
- the binary relation symbol $=$ called *equals*
- for each $j \in J$, a ν_j-ary *function symbol* F_j
- the logical symbols $\forall, \exists, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$, called the *universal quantifier*, *existential quantifier*, *negation*, *and* (or *disjunction*), *or* (or *conjunction*), *implication*, and *double implication* symbols, respectively
- the parentheses $(,)$ and bracket $[,]$ symbols.

Words of $\mathcal{L}$ are called *strings*. The natural numbers μ_i and ν_j are called the *arity* of R_i and F_j, respectively.

Remark 2. One need not include all the above logical symbols as independent symbols of the language $\mathcal{L}$, since for the purposes of first order logic, some of them can be defined in terms of the others. For example, the convention in [3] is to include only $\neg$, $\vee$ and $\exists$, since for any strings $\varphi, \psi \in \mathcal{L}$ we may define,

- $\varphi \wedge \psi$ as $\neg[\neg\varphi \vee \neg\psi]$
- $\varphi \rightarrow \psi$ as $\neg\varphi \vee \psi$
- $\varphi \leftrightarrow \psi$ as $[\varphi \rightarrow \psi] \wedge [\psi \rightarrow \varphi]$
- $(\forall X_l)[\phi]$ as $\neg(\exists X_l)[\neg\phi]$.

Being even more clever, one needs in fact only define the logical symbol $|$ called *alternative denial* and then define

- $\neg\varphi$ as $\varphi|\varphi$
- $\varphi \vee \psi$ as $\neg\varphi|\neg\psi$.

Intuitively, $\varphi|\psi$ means "$\neg[\varphi \wedge \psi]$." Similarly, one need only take the "xor" symbol. Thus it's sufficient to take $|$ and $\exists$ as the only logical symbols for our language, though we'll stick with the convention of [3].

We now define several distinguished subsets of $\mathcal{L}$:

terms: The subset $\texttt{terms} \subset \mathcal{L}$ of *terms* is the smallest subset containing the variable and constant symbols and closed under function symbols, i.e. $\texttt{terms}$ contains $\{X_n\}_{n\in\mathbb{N}} \cup K$ and satisfies

$$F_j(t_1, \ldots, t_{\nu_j}) \in \texttt{terms}, \quad \text{for all } t_1, \ldots, t_{\nu_j} \in \texttt{terms}, j \in J.$$

atomic formulae: The subset $\mathtt{atoms} \subset \mathcal{L}$ of *atomic formulae* is the set of all relations among terms, i.e. the set

$$\{t = t', R_i(t_1, \ldots, t_{\mu_i}) : t, t', t_1, \ldots, t_{\mu_i} \in \mathtt{terms}, i \in I\}.$$

formulae: Finally, the subset $\mathtt{formulas} \subset \mathcal{L}$ of *formula* is the smallest subset containing $\mathtt{atoms}$ and satisfying

f1: $\neg\phi \in \mathtt{formulas}$ for every $\phi \in \mathtt{formulas}$

f2: $\varphi \vee \varphi' \in \mathtt{formulas}$ for every $\varphi, \varphi' \in \mathtt{formulas}$

f3: $(\exists X_l)[\phi] \in \mathtt{formulas}$ for every $\phi \in \mathtt{formulas}$

Example 3. The first order language of groups $\mathcal{L}_{\mathrm{grp}}$ is generated by a single binary function symbol $\cdot$, where we usually write $\cdot(t, t')$ as $(t \cdot t')$ for terms t, t', and constant symbol e. We also define the language of rings $\mathcal{L}_{\mathrm{ring}}$, which is generated by two binary operations usually denoted by $+$ and $\cdot$, and two constant symbols 0 and 1. We might as well also define the language of ordered rings $\mathcal{L}_{\mathrm{ordring}}$ by adding a single additional binary relation $\leq$ to $\mathcal{L}_{\mathrm{rign}}$. The following are example of strings, terms, atomic formula, and formula in $\mathcal{L}_{\mathrm{ordring}}$:

- $)\neg((1 \vee X_1 0 \leq [0 \cdot 111]\exists 0 X_3 \rightarrow 111($
- $(1 \cdot (1 + 1)) \cdot (0 \cdot (1 + (0 + 1)))$
- $1 \leq (0 \cdot (1 + 0))$
- $(\exists X_1)[X_1 \cdot (1 + 1) = X_2] \rightarrow (\exists X_3)[X_3 \cdot (1 + 1 + 1) = X_2]$.

A languages is just a set of strings of symbols.

In fact, we can write any formula $\varphi(X_1, \ldots, X_m)$ of a language $\mathcal{L}$ in *prenex disjunctive normal form*

$$(Q_1 Y_1) \ldots (Q_n Y_n) [\vee_i \wedge_j A_{ij}]$$

where each Q_i is a quantifier ($\exists$ or $\forall$), and each A_{ij} is an atomic formula, and where "write" we mean the two formulae are "logically equivalent," where we'll find out later on what that means.

Remark 4. By the recursive construction of the set of formulas, we may prove properties of formulas by an *induction on structure* or *induction of formulas*. Given a property P of strings, we first prove P holds for all atomic formula and then supposing it holds for formula φ and φ' we show that P holds for $\neg\varphi$, $\varphi \vee \varphi'$ and $(\exists X_l)[\varphi]$ for all $l \in \mathbb{N}$.

Definition 5. Let $\mathcal{L}$ be a language and X a variable symbol. Then we'll define the notion of a *free occurrence* of a variable X in a formula by induction on structure. We'll say that

- any occurrence of X in any atomic formula is free
- if an occurrence of X in a formula φ is free then this occurrence is
  - also free in $\neg\varphi$
  - also free in $\varphi \vee \varphi'$ for any formula φ'
  - also free in $(\exists X')[\varphi]$ for any variable symbol X' distinct from X and not free in $(\exists X)[\varphi]$.

Any occurrence of X in φ that is not free is called *bound*. If X has a free occurrence in a formula φ, then X is called a *free variable* of φ, and we'll write φ as $\varphi(X_1, \ldots, X_m)$ to indicate that $X_1, \ldots, X_m$ include all free variables of φ. We'll call a formula without free variables a *sentence* of $\mathcal{L}$, and denote by $\mathtt{sents}$ the set of sentences of $\mathcal{L}$.

1.2. **Structures.**

Though a language is a completely formal object, we usually "interpret" a language through a structure, and usually don't even realize there is an ambient language. Unless otherwise stated, let $\mathcal{L} = (\mu, \nu, K)$ be a fixed language.

Definition 6. A *structure* $\mathcal{A}$ for $\mathcal{L}$ (or an $\mathcal{L}$-*structure*) is a 4-tuple $\mathcal{A} = (A, \phi, (R_i^{\mathcal{A}})_{i \in I}, (F_j^{\mathcal{A}})_{j \in J})$, where

- $\phi : K \to A$ is a map of sets
- $R_i^{\mathcal{A}} \subset A^{\mu_i}$ is a μ_i-ary relation on A
- $F_j^{\mathcal{A}} : A^{\nu_i} \to A$ is a μ_i-ary operation on A

and where the elements of $\phi(K) \subset A$ are called *constants* and the set A is called the *domain* of $\mathcal{A}$..

We "see" or "interpret" languages through their structures via substitutions.

Definition 7. Let $\mathcal{A}$ be an $\mathcal{L}$-structure, then a *substitution* from $\mathcal{L}$ into $\mathcal{A}$ is a function f : $\texttt{terms} \to A$ defined by images

- $f : X_i \mapsto x_i \in A$ for all variables X_i, $i \in \mathbb{N}$
- $f : k \mapsto \phi(k) \in A$ for all constant symbols $k \in K \subset \texttt{terms}$

such that for each $j \in J$ the following diagram commute

$$\begin{array}{ccc} \texttt{terms}^{\nu_j} & \xrightarrow{F_j} & \texttt{terms} \\ f \downarrow & & \downarrow f \\ A^{\nu_j} & \xrightarrow{F_j^{\mathcal{A}}} & A \end{array}$$

i.e. $f(F_j(t_1, \ldots, t_{\nu_j})) = F_j^{\mathcal{A}}(f(t_1), \ldots, f(t_{\nu_j}))$ for all terms $t_1, \ldots, t_{\nu_j}$ of $\mathcal{L}$.

Definition 8. Let $\mathcal{A}$ be an $\mathcal{L}$-structure. Then we'll define the *truth value* of a formula φ under a substitution f or the f-*truth* of φ by induction on structure:

- $t = t'$ is true if $f(t) = f(t') \in A$
- $R_i(t_1, \ldots, t_{\mu_i})$ is true if $(f(t_1), \ldots, f(t_{\mu_i})) \in R_i^{\mathcal{A}}$
- and supposing that formulas φ, φ' have a true value under every substitution then
  - $\neg\varphi$ is true if φ is false
  - $\varphi \vee \varphi'$ is true if φ is true and/or if φ' is true
  - $(\exists X)[\varphi]$ is true if there exists $x \in A$ such that φ is true under the modified substitution $f_{X \mapsto x}$ defined by $X \mapsto x$ and $X' \mapsto f(X')$ for all variables X' distinct from X.

It's clear that the f-truth of a formula $\varphi = \varphi(X_1, \ldots, X_m)$ only depends on the images of $f : X_i \mapsto x_i$, and so if φ is f-true then we'll write

$$\mathcal{A} \models_f \varphi \quad \text{or} \quad \mathcal{A} \models \varphi(x_1, \ldots, x_m).$$

In particular, if φ is a sentence, then the truth value of φ is independent of the substitution f and so we'll write simply $\mathcal{A} \models \varphi$ if φ is f-true for any substitution f of $\mathcal{L}$ into $\mathcal{A}$.

Remark 9. The standard truth tables hold for the composite logical symbols $\wedge, \to, \leftrightarrow, \forall$, as expected.

Example 10. How could you tell the structures $\mathbb{N}, \mathbb{Z}$, and $\mathbb{Z}/3$ apart as structures of the language $\mathcal{L}_{\text{grp}}$?

1.3. **Models.**

Models are structures for a language satisfying axioms. They are what we're used to calling groups, rings, fields, etc.

Definition 11.

- A *theory* of $\mathcal{L}$ or an *$\mathcal{L}$-theory* is a subset $T \subset \texttt{sents}$ of sentences of $\mathcal{L}$.
- An $\mathcal{L}$-structure $\mathcal{A}$ is called a *model* of a theory T if
$$\mathcal{A} \models \varphi \quad \text{for every } \varphi \in T,$$
and in this case we'll write $\mathcal{A} \models T$.
- If T' is another $\mathcal{L}$-theory, then we'll write $T' \models T$ if
$$\mathcal{A} \models T' \quad \Rightarrow \quad \mathcal{A} \models T.$$
- If Π is an $\mathcal{L}$-theory and
$$T = \{\varphi \in \texttt{sents} : \Pi \models \varphi\},$$
then we'll say that Π is a set of *axioms* for T.
- Denote by $\mathrm{Mod}(T)$ the class of all models of a theory T. If $\mathcal{A}$ is an $\mathcal{L}$-structure, denote by $\mathrm{Th}(\mathcal{A}, \mathcal{L})$ the set of all sentences of $\mathcal{L}$ that are true in $\mathcal{A}$, called the *theory of $\mathcal{A}$*.

For an $\mathcal{L}$-structure $\mathcal{A}$, we'll denote by $\mathcal{L}(A)$ the language $\mathcal{L}(\mu, \nu, K \sqcup A)$ gotten by adding a constant symbol for each element of A.

Example 12. We write the standard axioms for the algebraic objects we know and love.

- Axioms for a group: in $\mathcal{L}_{\mathrm{grp}}$ we have the following set Π of axioms for the theory T of groups,
$$\begin{aligned} &(\forall X)(\forall Y)(\forall Z)[(X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)] \\ &(\forall X)[e \cdot X = X \cdot e = X] \\ &(\forall X)(\exists Y)[X \cdot Y = Y \cdot X = e] \end{aligned}$$
Then any model $G \models T$ is a group. Now for a given group H, in the language $\mathcal{L}_{\mathrm{grp}}(H)$ we can extend Π by the *positive diagram* of H, i.e. all sentences $a \cdot b = c$ for $a, b, c \in H$ that are true in H. Denote this extension by $\Pi(H)$. Then a model of $\Pi(H)$ will be a group G such that there's a homomorphism $H \to G$.
- Axioms for ordered groups: in $\mathcal{L}_{\mathrm{ordgrp}}$ we extend the axioms for a group by the following order axioms
$$\begin{aligned} &(\forall X)(\forall Y)[(e \leq X) \wedge (e \leq Y) \to (e \leq X \cdot Y)] \\ &(\forall X)[((e \leq X) \wedge (X \leq e)) \to (X = e)]. \end{aligned}$$
- Axioms for fields: in $\mathcal{L}_{\mathrm{ring}}$ we can write the standard axioms for a field.
- Axioms for fields of a given characteristic: we extend the axioms for a field by the sentence $p = 0$, where we're thinking of $p = 1 + \cdots + 1$, if we want a characteristic p field, and by the set $\{\neg(p = 0) : p \text{ prime}\}$ if we want a characteristic zero field.
- Axioms for algebraically closed fields: we extend the axioms for a field by the sentences
$$(\forall A_0 \cdots \forall A_{n-1})(\exists X)[X^n + A_{n-1}X^{n-1} + \cdots + A_0]$$
for each $n \geq 1$.

1.4. **Morphisms of structures.**

Definition 13. Fix a language $\mathcal{L} = (\mu, \nu, K)$ and let $\mathcal{A} = (A, (R_i^{\mathcal{A}}), (F_i^{\mathcal{A}}), \phi)$ and $\mathcal{B} = (B, (R_i^{\mathcal{B}}), (F_j^{\mathcal{B}}), \phi')$ be $\mathcal{L}$-structures, then a *morphism* of structures $f : \mathcal{A} \to \mathcal{B}$ is a set map $f : A \to B$ such that

- $f(\phi(k)) = f(\phi'(k))$ for all $k \in K$
- $(a_1, \dots, a_{\mu_i}) \in R_i^{\mathcal{A}} \Leftrightarrow (f(a_i), \dots, f(a_{\mu_i})) \in R_i^{\mathcal{B}}$ for each $i \in I$
- $f(F_j^{\mathcal{A}}(a_1, \dots, a_{\nu_i})) = F_j^{\mathcal{B}}(f(a_i), \dots, f(a_{\nu_i}))$, for all $j \in J$.

Thus the class of all structures of a language $\mathcal{L}$ form the objects of a category $\mathrm{Str}(\mathcal{L})$. Call a morphism $f : \mathcal{A} \to \mathcal{B}$ an *embedding* of structures if $f : A \to B$ is injective. We'll note that $f : \mathcal{A} \to \mathcal{B}$ is a categorical isomorphism iff $f : A \to B$ is bijective. Call $\mathcal{A}$ a *substructure* of $\mathcal{B}$ (and write $\mathcal{A} \subset \mathcal{B}$) if $f : \mathcal{A} \to \mathcal{B}$ is an embedding and $f^{\times \mu_i}(R_i^{\mathcal{A}}) = f(A)^{\times \mu_i} \cap R_i^{\mathcal{B}}$ (as opposed to just $\subset$ for an embedding).

1.5. **Elementary equivalence.**

Definition 14. Call two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$ *elementarily equivalent* (we'll write $\mathcal{A} \equiv_{\mathcal{L}} \mathcal{B}$ or just $\mathcal{A} \equiv \mathcal{B}$) if $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$. We'll say that $\mathcal{A}$ is an *elementary substructure* of $\mathcal{B}$ or that $\mathcal{B}$ is an *elementary extension* of $\mathcal{A}$ (we'll write $\mathcal{A} \prec \mathcal{B}$) if $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{A} \equiv_{\mathcal{L}(A)} \mathcal{B}$ as structures of the language $\mathcal{L}(A)$, i.e. considering $A \subset B$ than for each formula $\varphi(X_1, \dots, X_m)$ of $\mathcal{L}$ and for every $a_1, \dots, a_m \in A$,

$$\mathcal{A} \models \varphi(a_1, \dots, a_m) \Leftrightarrow \mathcal{B} \models \varphi(a_1, \dots, a_m).$$

Concerning the freedom of choosing cardinalities of structures and elementary substructures we have the following pair of theorems.

Theorem 15 (Downward Skolem-Löwenhein)**.** *Let $\mathcal{L}$ be a countable language, $\mathcal{B}$ be an $\mathcal{L}$-structure, and $A_0 \subset B$ be countable. Then $\mathcal{B}$ has a countable elementary substructure $\mathcal{A}$ such that $A_0 \subset A$.*

Proof. See [3] Prop. 7.4.2. □

Theorem 16 (Upward Skolem-Löwenhein)**.** *Let $\mathcal{L}$ be a language, $\mathcal{A}$ be an infinite $\mathcal{L}$-structure, and κ be a cardinal such that $\kappa \geq |A|, |\mathcal{L}|$. Then $\mathcal{A}$ has an elementary extension $\mathcal{B}$ with $\kappa = |B$.*

Proof. The proof involves taking high ultrapowers, which we'll introduce shortly. □

2. Ultrafilters and Ultraproducts

An ultrafilter on a set S is a collection of subsets of S that we would like to consider as being "big."

Definition 17. Let S be a set, then a *filter* $\mathcal{D}$ on S is a nonempty collection of subset of S satisfying the following properties:

- $\emptyset \notin \mathcal{D}$ (and $S \in \mathcal{D}$ ensures $\mathcal{D}$ is nonempty)
- $A, B \in \mathcal{D} \Rightarrow A \cap B \in \mathcal{D}$
- if $A \subset B \subset S$ then $A \in \mathcal{D} \Rightarrow B \in \mathcal{D}$

in addition, $\mathcal{D}$ is called an *ultrafilter* on S if

- $A \subset S \Rightarrow A \in \mathcal{D}$ or $S \setminus A \in \mathcal{D}$.

If $\mathcal{D}$ is an ultrafilter we see that for every $A \subset S$, both $A \in \mathcal{D}$ and $S \setminus A \in \mathcal{D}$ is impossible, and that $\mathcal{D}$ also satisfies

- $A \cup B \in \mathcal{D} \Rightarrow A \in \mathcal{D}$ or $B \in \mathcal{D}$.

Note that the notion of a filter can be made on any poset (partially ordered set), here we're always using the powerset poset 2^S.

Example 18.

- The family of all cofinite sets of S is a filter on S.
- Let $a \in S$, then the family of all subsets of S containing a is an ultrafilter on S, called a *principal* ultrafilter. Note that from the final property of ultrafilters, an ultrafilter is principal iff it contains a finite set.
- If we did not assume $\emptyset \notin \mathcal{D}$, then we could have the *improper* ultrafilter $\mathcal{D} = 2^S$.
- Let $\mathcal{D}_0$ be a family of nonempty subsets of S closed under finite intersections , then the filter $\mathcal{D}_1$ *generated* by $\mathcal{D}_0$ is the family of all subsets $B \subset S$ which contain sets of $\mathcal{D}_0$, indeed $\mathcal{D}_1$ is a filter on S.

Lemma 19. *A filter $\mathcal{D}$ on a set S is an ultrafilter iff it's maximal.*

Proof. To "$\Rightarrow$." By the final condition of an ultrafilter, it's clear that it is maximal.

To "$\Leftarrow$." Let $\mathcal{D}$ be a maximal filter on S and let $A \subset S$. If $S \setminus A \in \mathcal{D}$ then we're done. So assume $S \setminus A \notin \mathcal{D}$. Now $\mathcal{D}' = \mathcal{D} \cup \{A \cap D : D \in \mathcal{D}\} \cup \{A\}$ is closed under finite intersections and consists of all nonempty sets. Indeed, for any $D', D'' \in \mathcal{D}$, we have $D = D' \cap D' \in \mathcal{D}$ and so $(D' \cap A) \cap (D'' \cap A) = (D' \cap D'') \cap A = D \cap A \in \mathcal{D}'$. Also, if $D \cap A = \emptyset$, then $D \subset S \setminus A$, so that $S \setminus A \in \mathcal{D}$, a contradiction. Thus by the above remark, there exists a filter $\mathcal{D}''$ containing $\mathcal{D}'$ hence containing $\mathcal{D}$, but by maximality of $\mathcal{D}$, $\mathcal{D} = \mathcal{D}''$, so in particular $A \in \mathcal{D}$. Thus $\mathcal{D}$ is an ultrafilter. ☐

Corollary 20.

- *Every family of subsets of S closed under finite intersections is contained in an ultrafilter.*
- *Let $\mathcal{D}'$ be a family of sets that satisfies the property that every finite intersections is infinite, then $\mathcal{D}'$ is contained in an ultrafilter.*

Proof. For the first statement, by Zorn's lemma we can pick a maximal filter containing such a family.

For the second statement, note that the stated family along with all cofinite sets of S has the finite intersection property. ☐

2.1. **Regular ultrafilters.**

Since we're thinking of ultrafilters as the "big" sets of our set S, we say that an ultrafilter is regular if it also doesn't contain any "small" sets. The small sets have an axiomatization dual to that of a filter.

Definition 21. Let S be a set, then a family of *small sets* of S is a family $\mathcal{F}$ of subsets of S satisfying the following properties:

- $S \notin \mathcal{F}$ (and $\emptyset \in \mathcal{F}$ ensures $\mathcal{F}$ is nonempty)
- $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$
- if $A \subset B \subset S$ then $B \in \mathcal{F} \Rightarrow A \in \mathcal{F}$

Note that in the general theory of posets, a family of small sets is also called an *ideal*.

Example 22. The family of all finite sets of S is a family of small sets.

Fix a family of small set $\mathcal{F}$ of S, and for $A, B \subset S$ we'll say that A is *almost contained* in B if $A \setminus B \in \mathcal{F}$. If A and B are almost contained in each other, i.e. $(A \setminus B) \cup (B \setminus A) \in \mathcal{F}$, then we'll say they are *almost equal* (we'll write $A \approx B$.

We'll say that an ultrafilter $\mathcal{D}$ on S is $\mathcal{F}$*-regular* (or just *regular*) if $\mathcal{D} \cap \mathcal{F} = \emptyset$. In particular, if $A \approx B$ then $A \in \mathcal{D} \Rightarrow B \in \mathcal{D}$. Note that nonprincipal ultrafilters are on S are regular with respect to the family of finite sets of S.

Lemma 23. *Let S be a set and $\mathcal{F}$ a family of small sets of S. Suppose that a family $\mathcal{D}_0$ of subsets of S satisfies $A, A' \in \mathcal{D}_0 \Rightarrow A \cap A' \notin \mathcal{F}$. Then there exists a regular ultrafilter $\mathcal{D}$ on S containing $\mathcal{D}_0$.*

2.2. **Ultraproducts.**

Definition 24. Let $\mathcal{L} = (\mu, \nu, K)$ be a language and S a set together with an ultrafilter $\mathcal{D}$. For each $s \in S$, let $\mathcal{A}_s = (A_s, \phi_s, (R_{is})_{i\in I}, (F_{js})_{j\in J})$ be an $\mathcal{L}$-structure. We construct a new $\mathcal{L}$-structure called the *ultraproduct* $\mathcal{A}^* = (A^*, \phi^*, (R_i^*)_{i\in I}, (F_j^*)_{j\in J})$ of the $\mathcal{A}_s$ as follows. First, define an equivalence relation $\sim$ on the direct product $\prod_{s\in S} A_s$ by

$$a \sim b \Longleftrightarrow \{s \in S : a_s = b_s\} \in \mathcal{D}.$$

Then define

- the domain of $\mathcal{A}^*$ to be the set $A^* = \prod_{s\in S} A_s / \sim$,
- the relations $R_i^* \subset (A^*)^{\mu_i}$, for $i \in I$, by

$$(a_1, \ldots, a_{\mu_i}) \in R_i^* \Longleftrightarrow \{s \in S : (a_{1s}, \ldots, a_{\mu_i s}) \in R_{is}\} \in \mathcal{D},$$

  for any choice of representative $(a_{ks})_{s\in S}$ of $a_k \in A^*$, for $1 \le k \le \mu_i$,
- the set map $\phi^* : K \to A^*$ via the quotient $\phi : K \to \prod A_s \to \prod A_s / \sim$,
- the functions $F_j^* : A^{\nu_j} \to A$, for $j \in J$, by

$$F_j^*(a_1, \ldots, a_{\nu_j}) = (F_{js}(a_{1s}, \ldots, a_{\nu_j s}))_{s\in S} \mod \mathcal{D},$$

  for any choice of representative $(a_{ks})_{s\in S}$ of $a_k \in A$, for $1 \le k \le \nu_j$.

Then one checks that these are all well defined. We'll sometimes denote $\mathcal{A}^*$ by $\prod \mathcal{A}_s/\mathcal{D}$. Note that in the above construction, we only require that $\mathcal{D}$ be a filter (not necessarily an ultrafilter). The resulting structure $\prod \mathcal{A}_s/\mathcal{D}$ is usually called the *reduced product* of the structures $\mathcal{A}_s$.

Example 25. Let $a \in S$ and $\mathcal{D}_a$ be the principal ultrafilter at a, then $\prod \mathcal{A}_s/\mathcal{D}_a \cong \mathcal{A}_a$.

The fundamental property of ultraproducts is that a sentence of $\mathcal{L}$ is true in the ultraproduct $\mathcal{A}$ iff it is true for almost all $\mathcal{A}_s$.

Theorem 26 (Łoš)**.** *Let $\mathcal{A}^* = \prod \mathcal{A}_s/\mathcal{D}$ be an ultraproduct of $\mathcal{L}$-structures, indexed over a set S. Let f be a substitution of $\mathcal{L}$ into $\mathcal{A}$. Then for every formula φ of $\mathcal{L}$*

$$\mathcal{A} \models_f \varphi \Longleftrightarrow \{s \in S : \mathcal{A}_s \models_{f_s} \varphi\} \in \mathcal{D},$$

where if f is defined by $f : X_n \mapsto x_n$, then choosing representatives $(x_{ns})_{s\in S}$ for each x_n modulo $\mathcal{D}$, define f_s by $X_n \mapsto x_{ns}$ as a substitution of $\mathcal{L}$ into $\mathcal{A}_s$. In particular, if φ is a sentence of $\mathcal{L}$ then

$$\mathcal{A} \models \varphi \Longleftrightarrow \{s \in S : \mathcal{A}_s \models \varphi\} \in \mathcal{D}.$$

Proof. We proceed by induction on structure. First we show that the statement of the theorem follows for atomic formula. Before that, we note that for terms we have the following fundamental

Lemma 27. *Let f and f_s be as in the statement of the theorem. Then for any term t of $\mathcal{L}$ we have*

$$f(t) = (f_s(t))_{s\in S} \mod \mathcal{D}.$$

Proof of lemma. Note that the lemma holds for constants and for terms of the form $F(X_1,\dots,X_n)$, for any function symbol F with arity n, by construction of the ultraproduct. Now suppose that t has the form $t = F(t_1,\dots,t_n)$ where all terms $t_1,\dots,t_n$ satisfy the lemma, i.e. for all $1 \le k \le n$,

$$\begin{aligned} f(t_k) = (f_s(t_k))_{s\in S} \mod \mathcal{D} &\iff \{s\in S : f(t_k)_s = f_s(t_k)\} \in \mathcal{D} \\ &\iff \{s\in S : f(t_k)_s = f_s(t_k),\ \text{for all } 1\le k\le n\} \in \mathcal{D}, \end{aligned}$$

since $\mathcal{D}$ is closed under finite intersections. But then we have the following equalities modulo $\mathcal{D}$,

$$\begin{aligned} f(t) &= f(F(t_1,\dots,t_n)) = F^*(f(t_1),\dots,f(t_n)) = (F_s(f(t_1)_s,\dots,f(t_n)_s))_{s\in S} \\ &= (F_s(f_s(t_1),\dots,f_s(t_n)))_{s\in S} = (f_s(F(t_1,\dots,t_n)))_{s\in S} \\ &= (f_s(t))_{s\in S} \mod \mathcal{D} \end{aligned}$$

i.e. the lemma holds for t. Thus by the recursive construction of the terms of $\mathcal{L}$, the lemma holds for all terms. □

We proceed with our induction on structure. To show that the theorem holds for all atomic formula, let $t_1,\dots,t_n$ be terms of $\mathcal{L}$, and R be a relation symbol with arity n, then

$$\begin{aligned} \mathcal{A} \models_f R(t_1,\dots,t_n) &\iff (f(t_1),\dots,f(t_n)) \in R^* \\ &\iff \{s\in S : (f(t_1)_s,\dots,f(t_n)_s) \in R_s\} \in \mathcal{D} \\ &\iff \{s\in S : (f_s(t_1),\dots,f_s(t_n)) \in R_s\} \in \mathcal{D} \\ &\iff \{s\in S : \mathcal{A}_s \models_{f_s} R(t_1,\dots,t_n)\} \in \mathcal{D}, \end{aligned}$$

where the third equivalence above uses the lemma. Thus the theorem holds for atomic formula.

Now suppose the theorem holds for a formula φ of $\mathcal{L}$, then we show that it holds for $\neg\varphi$. Indeed,

$$\begin{aligned} \mathcal{A} \models_f \neg\varphi &\iff \mathcal{A} \not\models_f \varphi \\ &\iff \{s\in S : \mathcal{A}_s \models_{f_s} \varphi\} \notin \mathcal{D} \\ &\iff \{s\in S : \mathcal{A}_s \not\models_{f_s} \varphi\} \in \mathcal{D} \\ &\iff \{s\in S : \mathcal{A}_s \models_{f_s} \neg\varphi\} \in \mathcal{D}, \end{aligned}$$

where the second equivalence is the induction hypothesis, and the third equivalence is by the ultraproduct property, i.e. that for each $A \subset S$ exactly one of $A \in \mathcal{D}$ or $S \setminus A \in \mathcal{D}$ is true. Note that this is the only place in the proof where we use the ultraproduct property.

Now suppose the theorem holds for formulas vp and ψ and that without loss of generality we can assume both $\mathcal{A} \models_f \varphi$ and $\mathcal{A} \models_f \psi$ by using the above. Then

$$\begin{aligned}
\mathcal{A} \models_f \varphi \vee \psi &\iff \mathcal{A} \models_f \varphi \quad \text{or} \quad \mathcal{A} \models_f \psi \\
&\iff \mathcal{A} \models_f \varphi \quad \text{and} \quad \mathcal{A} \models_f \psi \\
&\iff \{s \in S : \mathcal{A}_s \models_{f_s} \varphi\} \in \mathcal{D} \quad \text{and} \quad \{s \in S : \mathcal{A}_s \models_{f_s} \varphi\} \in \mathcal{D} \\
&\iff \{s \in S : \mathcal{A}_s \models_{f_s} \varphi\} \cap \{s \in S : \mathcal{A}_s \models_{f_s} \varphi\} \in \mathcal{D} \\
&\iff \{s \in S : \mathcal{A}_s \models_{f_s} \varphi \wedge \psi\} = \{s \in S : \mathcal{A}_s \models_{f_s} \varphi \vee \psi\} \in \mathcal{D}.
\end{aligned}$$

Finally, suppose the theorem holds for a formula φ, and let X be a variable of $\mathcal{L}$, then

$$\begin{aligned}
\mathcal{A} \models_f (\exists X)[\varphi] &\iff \exists\, x \in A^*,\ \mathcal{A} \models_{f_{X \mapsto x}} \varphi \\
&\iff \exists x \in A^*,\ \{s \in S : \mathcal{A}_s \models_{f_{s, X \mapsto x_s}} \varphi\} \in \mathcal{D} \\
&\iff \{s \in S : \mathcal{A}_s \models_{f_s} (\exists X)[\varphi]\} \in \mathcal{D} \quad \text{(WHY?)}
\end{aligned}$$

Thus by induction on structure, the theorem holds for every formula φ of $\mathcal{L}$. ☐

Corollary 28. *The ultraproduct of models for a theory T is also a model of T, in particular, the ultraproduct of groups is a group, the ultraproduct of rings is a ring, the ultraproduct of fields is a field, etc.*

Example 29. Let $S = \{p \in \mathbb{N} : p \text{ is prime}\}$, and choose any non-principal ultrafilter $\mathcal{D}$ containing the comaximal filter on S. For each prime number $p \in S$ let $\overline{\mathbb{F}}_p$ be the algebraic closure of the finite field $\mathbb{F}_p$, thought of as an $\mathcal{L}_{\text{ring}}$-structure. Then the ultra product $\mathbb{F}^* = \prod_p \overline{F}_p/\mathcal{D}$ is a field. We claim it has characteristic zero. Indeed, for any prime number $p \in S$ consider the sentence $\neg(p = 0)$, we have

$$\{q \in S : \mathbb{F}_q \models \neg(p = 0)\} = S \smallsetminus \{p\} \in \mathcal{D},$$

since $\mathcal{D}$ contains the cofinite subsets of S. Thus

$$\mathbb{F}^* \models \neg(p = 0) \quad \text{forallprimes} \quad p \in S,$$

i.e. $\mathbb{F}^*$ has characteristic zero. Now also, $\mathbb{F}^*$ is algebraically closed since it is the ultraproduct of algebraically closed fields. Also, it's not too hard to see that $\mathbb{F}^*$ has the cardinality of the continuum. Thus in fact, $\mathbb{F}^* \cong \mathbb{C}$ as fields.

Example 30. Prove that the ultraproduct of structures of finite bounded cardinality is itself finite.

From the Łoš theorem, we get another main result of model theory.

Theorem 31 (Compactness Theorem)**.** *Let T be a theory in a first order language $\mathcal{L}$. If each finite subset of T has a model, then T has a model.*

Proof. Let F be the collection of all finite subsets of T, then for each $\Phi \in F$ let $D_\Phi = \{\Phi' \in F : \Phi \subset \Phi'\}$. Then $D_\Phi \cap D_{\Phi'} = D_{\Phi \cup \Phi'}$, so that the family $\mathcal{D}_0 = \{D_\Phi : \Phi \in F\}$ has is closed under intersections. Thus there exists an ultrafilter $\mathcal{D}$ on F containing $\mathcal{D}_0$. Choosing a model $\mathcal{M}_\Phi$ for each $\Phi \in F$, then $\mathcal{M} = \prod \mathcal{M}_\Phi/\mathcal{D}$ is a model of T, since for every $\varphi \in T$, $\{\Phi \in F : \mathcal{M}_\Phi \models \varphi\} \supset D_{\{\varphi\}}$, so that $\mathcal{M} \models \varphi$. ☐

Example 32. We can show the existence of algebraically closed fields using the compactness theorem. For each $n \geq 1$ let φ_n be the sentence

$$(\forall A_0 \cdots \forall A_{n-1})(\exists X)[X^n + A_{n-1}X^{n-1} + \cdots + A_0].$$

and let

$$F_n = \varinjlim_k \mathbb{F}_{p^{kn!}}.$$

Then F_n is an algebraic extension of $\mathbb{F}_p$ which has no non-trivial extensions of degree $\leq n$. Let $\Phi = \{\varphi_n : n \geq 1\}$. Then for any finite subset of $\varphi \subset \Phi$ letting $n = \max\{l : \varphi_l \in \varphi\}$, we have that $\mathbb{F}_n \models \varphi$. Since Φ together with the axioms of a field form a set of axioms of the theory of algebraically closed fields, by the compactness theorem, there exists an algebraically closed field. By the proof of the compactness theorem, the ultraproduct $F^* = \prod F_n/\mathcal{D}$ is algebraically closed for any non-principal ultrafilter containing the cofinite sets of $\mathbb{Z}^+$.

Another important result of model theory is the following. We'll call a family of subsets of S a *Boolean algebra of sets* if it's closed under unions, intersections, and complements.

Proposition 33 (Ax)**.** *Let S be a set, $\mathcal{F}$ a family of small subsets of S, $\mathcal{A}$ a Boolean algebra of subsets of S containing $\mathcal{F}$, and $C \subset S$ with $C \notin \mathcal{A}$. The there exist two regular ultrafilters $\mathcal{D}$ and $\mathcal{D}'$ on S such that $\mathcal{D} \cap \mathcal{A} = \mathcal{D}' \cap \mathcal{A}$ but $C \in \mathcal{D}$ and $C \notin \mathcal{D}'$.*

Proof. See [3] Prop. 7.6.2. □

Remark 34. Denoting the set of all ultrafilters on a set S by $\mathrm{Ult}(S)$, we can topologize $\mathrm{Ult}(S)$ in the following way. For each $A \subset S$ define $D_A = \{\mathcal{D} \in \mathrm{Ult}(S) : A \in \mathcal{D}\}$. Then in fact, the above two theorems say that the collection $\{D_A\}_{A \subset S}$ is a base for a compact Hausdorff topology on $\mathrm{Ult}(S)$. This topology is in fact homeomorphic to the Stone-Čech compactification of the discrete topology on S, (i.e. the unique up to homeomorphism topological space $\hat{S}$ such that any continuous map $S \to K$ to a compact Hausdorff topological space factors through $\hat{S}$.)

2.3. **Ultrapowers.**

Definition 35. Let S be a set, $\mathcal{D}$ and ultrafilter on S, and structures $\mathcal{A}_s = \mathcal{A}$ all equal to a fixed $\mathcal{L}$-structure. Then the ultraproduct $\prod \mathcal{A}_s/\mathcal{D}$ is called an *ultrapower* of $\mathcal{A}$ to S modulo $\mathcal{D}$ (we'll write $\mathcal{A}^S/\mathcal{D}$ or just $\mathcal{A}^*$ as before.) Consider the diagonal embedding

$$\begin{array}{ccccc} A & \stackrel{\Delta}{\to} & A^S & \to & A^S/\mathcal{D} \\ a & \mapsto & (a)_{s\in S} & \to & (a)_{s\in S} \mod \mathcal{D} \end{array}$$

and note that it's injective. Thus we have a canonical diagonal embedding of structures $\mathcal{A} \to \mathcal{A}^S/\mathcal{D}$.

Proposition 36. *If $\mathcal{D}$ is an ultrafilter on a set S and $\mathcal{A}$ is an $\mathcal{L}$-structure, then the canonical diagonal embedding $\mathcal{A} \to \mathcal{A}^S/\mathcal{D}$ is an elementary extension.*

Remark 37. There is a powerful theorem of Saharon Shelah stating that for two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, we have that $\mathcal{A} \equiv \mathcal{B}$ iff there exists a set S and an ultrafilter $\mathcal{D}$ on S such that $\mathcal{A}^S/\mathcal{D} \cong \mathcal{B}^S/\mathcal{D}$.

2.4. **Regular ultraproducts.**

Let S be a set together with a family of small subsets $\mathcal{F}$, and for each $s \in S$ let $\mathcal{A}_s$ be an $\mathcal{L}$-structure. Define the *truth set* of a sentence φ of $\mathcal{L}$ by

$$A(\varphi) = \{s \in S : \mathcal{A}_s \models \varphi\}.$$

Note that the map $\varphi \mapsto A(\varphi) : \texttt{sents} \to 2^S$ preserves the Boolean operations:

- $A(\varphi \vee \varphi') = A(\varphi) \cup A(\varphi')$
- $A(\varphi \wedge \varphi') = A(\varphi) \cap A(\varphi')$
- $A(\neg\varphi) = S \smallsetminus A(\varphi)$.

Let T be the theory

$$T = \{\varphi \in \texttt{sents} : \mathcal{A}_s \models \varphi \text{ for almost all } s \in S\},$$

and call $\prod \mathcal{A}_s/\mathcal{D}$ a *regular* ultraproduct if $\mathcal{D}$ is a regular ultraproduct of S.

Proposition 38.

- *Let φ be a sentence in $\mathcal{L}$, then $\varphi \in T$ iff $\mathcal{A} \models \varphi$ for every regular ultrafilter $\mathcal{A}^* = \prod \mathcal{A}_s/\mathcal{D}$.*
- *Every model of T is elementarily equivalent to a regular ultraproduct $\mathcal{A}^* = \prod \mathcal{A}_s/\mathcal{D}$.*

Proof. To the first assertion. To "⇒," note that by Łoš's Theorem, if $vp \in T$ then φ is true for every regular ultraproduct of the $\mathcal{A}_s$.

To "⇐," if $\varphi \notin T$, then $A(\neg\varphi)$ is not small, thus there exists a regular ultraproduct $\mathcal{D}$ on S which contains $A(\neg\varphi)$, so that $\prod \mathcal{A}_s/\mathcal{D} \not\models \varphi$.

To the second assertion. Let $\mathcal{A}$ be a model of T, then by the above remark, the family $\mathcal{D}_0 = \{A(\varphi) : \mathcal{A} \models \varphi\}$ is closed under intersections. Note that no $A(\varphi) \in \mathcal{D}_0$ can be small, since otherwise $\neg\varphi \in T$ so that $\mathcal{A} \models \neg\varphi$. By a previous lemma, there exists a regular ultrafilter $\mathcal{D}$ on S containing $\mathcal{D}_0$, so by Łoš's Theorem, $\mathcal{A} \equiv \prod \mathcal{A}_s/\mathcal{D}$. □

An application of this is a condition for when a theory is *finitely axiomatizable*, i.e. if it has a finite set of axioms.

Proposition 39. *Let C be a class of models of a language $\mathcal{L}$. Then*

- *C is an* elementary class *of models (i.e. $C = \mathrm{Mod}(T)$ for some theory T of $\mathcal{L}$) iff C is closed under ultraproducts and elementary equivalences.*
- *C is a* basic elementary class *of models (i.e. $C = \mathrm{Mod}(\varphi)$ for a sentence φ of $\mathcal{L}$) iff both C and it's complement in the class of all models of $\mathcal{L}$ are closed under ultraproducts and elementary equivalences.*

Proof. See [1] Theorem 4.1.12. □

Corollary 40. *The theories of characteristic zero fields and algebraically closed fields are not finitely axiomatizable.*

Proof. We've already seen examples of an ultraproduct of finite characteristic fields being characteristic zero and the ultraproduct of non algebraically closed fields being algebraically closed. □

REFERENCES

[1] C. C. Chang and H. J. Keisler. *Model Theory*. Studies in Logic and Foundations of Mathematics Vol. 73. North-Holland, Amsterdam, 1973.

[2] Heinz-Dieter Ebbinghaus and Jörg Flum. *Finite Model Theory*. Springer-Verlag, Berlin, 1995.

[3] Michael D. Fried and Moshe Jarden. *Field Arithmetic*. Second edition. Springer-Verlag, Berlin, 2005.