Census projects for K3 surfaces over finite fields

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Abelian varieties over finite fields

Theorem There are finitely many isomorphism classes of abelian varieties of dimension g over $\mathbb{F}_q$.

Challenge Create a database for small g and q.

Honda–Tate Isogeny classes of abelian varieties of dimension g over $\mathbb{F}_q$ are in bijection with specified L-polynomials

$$L(A/\mathbb{F}_q, T) = \det\bigl(1 - TF | H^1_{\mathrm{et}}(A_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\bigr)$$

The LMFDB contains isogeny classes for small g and q... and hopefully will have isomorphism classes soon?

K3 surfaces over finite fields

Tate conjecture (T, O, M, C, MPK, II) + LMS implies

Theorem There are finitely many isomorphism classes of K3 surfaces over $\mathbb{F}_q$.

Challenge Create a database of K3 surfaces for small q.

Like in the case of abelian varieties, we should start with isogeny classes.

Isogenies between K3 surfaces

Def (Shafarevich, Mukai, Buskin, Huybrechts, Z. Yang)
An *isogeny* between K3 surfaces S and S' over $\mathbb{F}_q$ is a $\mathbb{Q}$-correspondence between S and S' inducing an isometry

$$H^2_{\text{et}}(X'_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell) \to H^2_{\text{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)$$

Challenge Create a database of isogeny classes of K3 surfaces for small q.

Honda–Tate for K3 surfaces? Can we characterize the isogeny classes of K3 surfaces in terms of L-polynomials?

Honda–Tate for K3 surfaces?

The zeta function of a K3 surface S over $\mathbb{F}_q$ has the form

$$Z(S/\mathbb{F}_q, T) = \frac{1}{(1-T)L(S/\mathbb{F}_q, T)(1-q^2T)}$$

where the L-polynomial of S has degree 22 and roots of absolute value q consistent with the Weil conjectures

$$L(S/\mathbb{F}_q, T) = \det\big(1 - TF | H^2_{\text{et}}(S_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\big)$$

Question Does the L-polynomial uniquely determine the isogeny class?

Question (Kedlaya–Sutherland) Which L-polynomials are realized by K3 surfaces over $\mathbb{F}_q$?

K3 surfaces over $\mathbb{F}_2$

Theorem (Kedlaya–Sutherland) Complete enumeration of all 1 672 565 potential L-polynomials of K3 surfaces over $\mathbb{F}_2$.

How to determine which are realized?

- **(Taelman)** The transcendental part of each L-polynomial is realized by a K3 surface over $\overline{\mathbb{F}}_q$. Does this help?
- **Brute force** Try to enumerate all K3 surfaces over $\mathbb{F}_q$ admitting a sufficiently small degree polarization. Hope that we find them all?

Enumerating K3 surfaces over $\mathbb{F}_2$

Theorem (Kedlaya–Sutherland) Complete enumeration of all 528 257 orbits of PGL_4 acting on smooth quartic K3s over $\mathbb{F}_2$. These give 52 755 distinct L-polynomials.

What's next?

- **(A–Kulkarni–Petok)** Enumerate all polarized K3 surfaces of degree 2 over $\mathbb{F}_2$ and their L-polynomials.
- **Complete intersections** Enumerate all polarized K3 surfaces of degree 6 and 8.
- **Mukai constructions** Enumerate K3 surfaces of degree $10, 12, 14, 16, 18, 22$.
- **Auxiliary varieties of K3 type** Enumerate Fano fourfolds of K3 type, whose L-polynomials on H^4 give further hints.

K3 surfaces and cubic fourfolds

$X \subset \mathbb{P}^5$ cubic fourfold, Hodge type 0 1 21 1 0 on H^4

$\mathrm{CH}^2(X) = h^2\mathbb{Z}$ for X "very general"

$\mathcal{C}$ moduli space of cubic fourfolds

$\mathcal{C}^{NL} = \bigcup_d \mathcal{C}_d$ Hassett divisors of special cubic fourfolds

Theorem (Hassett) A special X has an associated K3 iff
$d = 14, 26, 38, 42, 62, 74, \ldots$

Theorem (Huybrechts) A special X has an isogenous K3 iff
$d = 8, 14, 18, 24, 26, 32, 38, 42, 50, 54, 56, 62, 72, 74, \ldots$

These statements are true over $\mathbb{C}$, and we believe them over $\mathbb{F}_q$ with appropriate cohomology theories.

Cubic fourfolds over $\mathbb{F}_2$

The zeta function of a cubic fourfold X over $\mathbb{F}_q$ has the form

$$Z(X/\mathbb{F}_q, T) = \frac{1}{(1-T)(1-qT)L(X/\mathbb{F}_q, T)(1-q^3T)(1-q^4T)}$$

where the L-polynomial of S has degree 23 and roots of absolute value q^2 consistent with the Weil conjectures

$$L(X/\mathbb{F}_q, T) = \det\bigl(1 - TF|H^4_{\mathrm{et}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell)\bigr)$$

and $L^{K3}(X/\mathbb{F}_q, T) = L(X/\mathbb{F}_q, qT)/(T-q)$ is the K3-part

Theorem (A–Kulkarni–Petok–Weinbaum) There are 1 069 562 isomorphism classes of smooth cubic fourfolds over $\mathbb{F}_2$ accounting for 86 472 distinct L-polynomials, of which 71 476 have K3-part on Kedlaya–Sutherland's list.

Noncommutative K3

Many of the K3-parts of L-polynomials of cubic fourfolds are not realizable because they would predict K3 surfaces with negative point counts!

Testing Honda–Tate Cubic fourfold with no isogenous K3 surface with K3-part L-polynomial on Kedlaya–Sutherland's list?

Prop (Yang–Yu) A "very general" cubic fourfold containing a cubic scroll and a Veronese surface meeting in 2 points has no isogenous K3.

Census report There are 6 such cubic fourfolds over $\mathbb{F}_2$.