Brill–Noether special cubic fourfolds

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Monday 27 July 2015

Cubic fourfolds

$X \subset \mathbb{P}^5$ smooth cubic hypersurface over $\mathbb{C}$

Torelli Theorem (Voisin). The integral polarized Hodge structure on $H^4(X, \mathbb{Z})$ recovers X up to isomorphism.

Integral Hodge Conjecture (Voisin). The cycle class map is an isomorphism $\mathrm{CH}^2(X) \to H^4(X, \mathbb{Z}) \cap H^{2,2}(X) = A(X)$.

$A(X)$ odd positive definite free $\mathbb{Z}$-lattice
$h^2 \in A(X)$ distinguished element of norm 3

Fact. $A(X) = \mathbb{Z}h^2$ for very general X

Noether–Lefschetz loci

$\mathcal{C}$ moduli space of cubic fourfolds

Noether–Lefschetz locus

$$\{X \in \mathcal{C} \ : \ \operatorname{rk} A(X) > 1\} = \bigcup_d \mathcal{C}_d$$

$X \in \mathcal{C}_d \iff$ exists $T \in A(X)$ such that $\langle h^2, T\rangle \subset A(X)$ is a primitive sublattice of rank 2 and discriminant d

$\iff X$ *special cubic fourfold* of discriminant d

(Hassett) $\mathcal{C}_d \neq \varnothing$ irreducible divisor $\iff d > 6$ and $d \equiv 0, 2 \ (6)$

$\mathcal{C}_d$ called *Hassett divisors*

Interpretation of $\mathcal{C}_d$

The general $X \in \mathcal{C}_d$ contains:

$d = 8$	a plane
$d = 12$	a cubic scroll
$d = 14$	a quartic scroll or a quintic del Pezzo surface
$d = 20$	a Veronese surface
$12 \leq d \leq 38$	certain smooth rational surfaces (Nuer)
$d = 44$	the Fano model of an Enriques surface (Nuer)

Geometry of $\mathcal{C}_d$

(Li/Zhang) Compute the generating function of the degrees of $\mathcal{C}_d$ as a modular form of weight 11 and level 3.
These degrees get large: 3402, 196272, 915678, ...

(Nuer) $\mathcal{C}_d$ is unirational for $d \leq 38$ and has $\mathcal{C}_{44}$ has negative Kodaira dimension.

(Tanimoto/Várilly-Alvarado) $\mathcal{C}_d$ is of general type for $d \gg 0$.
Current state of the art is $d \geq 264$.

Tony's talk on Tuesday, 4:40–5:30 pm in SFEBB 170!

Rationality of cubic fourfolds

Conjecture. The very general cubic fourfold is not rational.

Example. X contains disjoint planes $\implies$ X is rational

(Hassett) $X \in \mathcal{C}_8$ is rational on a countable union of divisors.

$$X \in \mathcal{C}_8 \iff \mathbb{P}^2 \cong P \subset X$$

$$\begin{array}{ccc} \mathrm{Bl}_P X & \hookrightarrow & \mathrm{Bl}_P \mathbb{P}^5 \\ & \overset{\pi}{\searrow} & \downarrow \\ S & \xrightarrow{2:1} & \mathbb{P}^2 \end{array}$$

π quadric surface bundle degenerating along sextic $D \subset \mathbb{P}^2$

S moduli space of rulings, $\beta_X \in \mathrm{Br}(S)$ class of universal ruling

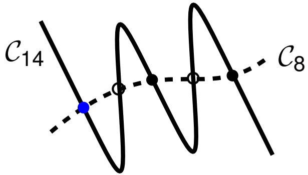
(Hassett) $\beta_X = 0 \implies X$ is rational

(A./Bernardara/Bolognesi/Várilly-Alvarado) There exist
$X \in \mathcal{C}_8$ with X rational but $\beta_X \neq 0$.

Rationality of cubic fourfolds

(Beauville/Donagi, Bolognesi/Russo/Staglianò, A.)
Every $X \in \mathcal{C}_{14}$ is rational.

Challenge. Give new rationality constructions for cubic fourfolds.



(Katzarkov) HMS $\implies$ every $X \in \mathcal{C}_{26}$ is rational

Associated K3 surface

$H^2(S,\mathbb{Z})$	weight 2 signature $(2,20)$		1	20	1	
$H^4(X,\mathbb{Z})$	weight 4 signature $(21,2)$	0	1	21	1	0

Polarized K3 surface (S,H) choice of ample $H \in \mathrm{Pic}(S)$
Marked cubic fourfold (X,K_d) choice of rank 2 $K_d \subset A(X)$

Primitive cohomology $H^2(S,\mathbb{Z})_0 = H^\perp \subset H^2(S,\mathbb{Z})$
Nonspecial cohomology $H^4(X,\mathbb{Z})_0 = K_d^\perp \subset H^4(X,\mathbb{Z})$

(Hassett) Exists a polarized K3 surface (S,H) of degree d with
$H^4(X,\mathbb{Z})_0 \cong H^2(S,\mathbb{Z})_0(-1) \iff 4 \nmid d$, $9 \nmid d$, $p \nmid d$ for $p \equiv 2\ (3)$
$d = 14, 26, 38, 42, 62, 74, \ldots$

S is an *associated K3 surface* to X

(Hassett) $\mathcal{C}_d^{\mathrm{mar}} \hookrightarrow \mathcal{K}_d$ embedding of moduli spaces

Associated K3 category

Semiorthogonal decomposition of the derived category
$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathscr{O}_X, \mathscr{O}_X(1), \mathscr{O}_X(2) \rangle$

$$\mathcal{A}_X = \{E \in \mathrm{D}^{\mathrm{b}}(X) \ : \ \mathrm{Ext}^{\bullet}(\mathscr{O}_X(i), E) = 0, \ i = 0, 1, 2\}$$

$\mathcal{A}_X$ looks like the derived category of a K3 surface

Example. $X \in \mathcal{C}_8 \implies \mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S, \beta_X)$

(Huybrechts) There are finitely many X' such that $\mathcal{A}_X \cong \mathcal{A}_{X'}$.
If X is very general, then $\mathcal{A}_X$ determines X uniquely.

Suspicions and conjectures

Suspicion (Harris, Hassett). X is rational $\rightsquigarrow$ X has an associated K3 surface

Conjecture (Kuznetsov). X is rational $\iff \mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S)$ for a K3 surface S

(Addington/Thomas) $\mathcal{A}_X \cong \mathrm{D}^{\mathrm{b}}(S) \implies X$ has an associated K3 surface S. The converse holds generically on $\mathcal{C}_d$ if $4 \nmid d$, $9 \nmid d$, $p \nmid d$ for $p \equiv 2$ (3).

(Voisin) $4 \nmid d \implies$ every $X \in \mathcal{C}_d$ has universally trivial CH_0

Voisin's plenary talk from week 1!
Alena Pirutka's talk on Tuesday, 2:00–2:50 pm, SFEBB 180!

Brill–Noether general cubic fourfolds

(Mukai) Polarized K3 surface (S, H) is *Brill–Noether general* if

$$h^0(S, N)\, h^0(S, M) < h^0(S, H) = 2 + d/2 = g + 1$$

for any nontrivial decomposition $H = N \otimes M$.

Example. $\mathrm{Pic}(S) = \mathbb{Z}H \implies (S, H)$ is BN general

(Lazarsfeld) $\mathrm{Pic}(S) = \mathbb{Z}H \implies C \in |H|$ is BN general curve

Fact. $C \in |H|$ is BN general curve $\Longrightarrow (S, H)$ is BN general K3

Open question. What about the converse?
Checked for $g \leq 10$ and $g = 12$ by Mukai.

Definition. (X, K_d) marked cubic fourfold is *BN general* if
associated K3 surface (S, H) is BN general

Brill–Noether special cubic fourfolds

Definition. The complement of BN general is *BN special*.

$$\begin{array}{ccc} \mathcal{C}_d^{\mathrm{mar}} & \hookrightarrow & \mathcal{K}_d \\ \uparrow & & \uparrow \\ \mathcal{C}_d^{\mathrm{BN}} & \hookrightarrow & \mathcal{K}_d^{\mathrm{BN}} \end{array}$$

The BN special loci are contained in the union of finitely many Noether–Lefschetz divisors, indexed by *Clifford index*.

(Saint-Donat, Reid, Donagi/Morrison, Green/Lazarsfeld, Mukai, Ciliberto/Pareschi, Knutsen, Johnsen, Lelli-Chiesa)

Classification of BN special K3 surfaces via vector bundles and lattice theory. Completely done for $g \leq 12$.

(Program.) Carry this out for cubic fourfolds.

Brill–Noether special cubic fourfolds

Theorem (A).

- $X \in \mathcal{C}_{14}$ has a BN general marking of discriminant 14 $\iff$ X is pfaffian.
- $X \in \mathcal{C}_{14}$ has a BN special marking of discriminant 14 $\iff$ X contains disjoint planes.

The image of $\mathcal{C}_{14}^{\mathrm{BN}} \hookrightarrow \mathcal{K}_{14}^{\mathrm{BN}}$ is contained in only one of five K3 Noether–Lefschetz divisors, with maximal Clifford index.

Corollary. Every $X \in \mathcal{C}_{14}$ is pfaffian or contains disjoint planes (or both), hence is rational.

Next frontier is $d = 26$. Need good constructions of BN general K3 surfaces of degree $d = 26$.
Input from moduli theory of curves of $g = 14$?