0.1 Topological Groups

Definition 1. A set G is a *topological group* if G is a group, G is a topological space, and the group operations in G are continuous in the topological space G. Equivalently, for all $a, b \in G$, and for every neighborhood W of ab^{-1} there exists neighborhoods U and V of a and b respectively, such that $UV^{-1} \subset W$.

Remark 2. Let G be a topological group, and let $f : G \to \mathbf{R}$. Then f is continuous at $a \in G$ if and only if for every $\epsilon > 0$ there exists a neighborhood U of a such that

$$x \in U \Rightarrow |f(x) - f(a)| < \epsilon.$$

Also, f is continuous at $a \in G$ if and only if for all $\epsilon > 0$ there exists a neighborhood U of the identity such that for all $x \in G$,

$$x\, a^{-1} \in U \Rightarrow |f(x) - f(a)| < \epsilon.$$

Definition 3. Let G be a topological group, let $M \subset G$, and let $f : M \to \mathbf{R}$. Then f is *uniformly continuous* if for every $\epsilon > 0$ there exists a neighborhood U of the identity of G such that for all $x, y \in M$,

$$x\, y^{-1} \in U \Rightarrow |f(x) - f(y)| < \epsilon.$$

Theorem 4. Let G be a topological group, let $M \subset G$ be compact, and let $f : G \to \mathbf{R}$ be continuous. Then f is uniformly continuous. (in both senses...)

Definition 5. Let G be a topological group, let $M \subset G$, and let Δ be a set of function defined on M. Then Δ is *uniformly equicontinuous* if for every $\epsilon > 0$ there exists a neighborhood U of the identity of G such that for all $x, y \in M$ and for all $f \in \Delta$,

$$x\, y^{-1} \in U \Rightarrow |f(x) - f(y)| < \epsilon.$$

Also, Δ is *uniformly bounded* if there exists a $B \in \mathbf{R}$ such that for all $x \in M$ and for all $f \in \Delta$ we have $|f(x)| < B$.

Theorem 6 (Generalization of Arzelà's Theorem). Let G be a topological group, let $M \subset G$ be compact, and let Δ be a uniformly bounded and uniformly equicontinuous set of real function defined on M. Then any sequence of functions in Δ has a uniformly convergent subsequence.

0.2 Invariant Integration

Theorem 7. Let G be a compact topological group. Then there is a unique mapping $f \mapsto \int f(x)dx$ from $C(G)$ to $\mathbf{R}$ satisfying the following conditions:

1. For all $f \in C(G)$ and for all $\alpha \in \mathbf{R}$,
$$\int \alpha f(x)\, dx = \alpha \int f(x)\, dx.$$

2. For all $f, g \in C(G)$,
$$\int (f(x) + g(x))\, dx = \int f(x)\, dx + \int g(x)\, dx.$$

3. For all $f \in C(G)$,
$$f(x) \geq 0 \text{ for all } x \in G \Rightarrow \int f(x)\, dx \geq 0.$$

4. If $f(x) = 1$ for all $x \in G$ then $\int f(x)\, dx = 1$.

5. For all $f \in C(G)$ and for all $a \in G$,
$$\int f(xa)\, dx = \int f(x)\, dx.$$

6. For all $f \in C(G)$ and for all $a \in G$,
$$\int f(ax)\, dx = \int f(x)\, dx.$$

7. For all $f \in C(G)$,
$$\int f(x^{-1})\, dx = \int f(x)\, dx.$$

Furthermore, if there is on G any mapping satisfying conditions 1-5, then the remaining conditions hold. The unique $\int f(x)dx$ will be called the *integral* of f.

The proof of this theorem proceeds in a number of steps. Throughout, let G be a topological group.

Remark 8. Conditions 1-3 are natural for any conception of an integral, and permit the integration of inequalities and make it possible to obtain the usual estimate concerning the integral of absolute values. Namely, let $f, g \in C(G)$, then

$$f(x) \leq g(x) \text{ for all } x \in G \Rightarrow \int f(x)\,dx \leq \int g(x)\,dx,$$

$$\left|\int f(x)\,dx\right| \leq \int |f(x)|\,dx.$$

Proof. Indeed, we see that for all $x \in G$, $f(x) \leq g(x) \Rightarrow g(x) - f(x) \geq 0$ thus by condition 3 we have $\int (g(x) - f(x))\,dx \geq 0$, and by conditions 1 and 2 we have $\int g(x)\,dx - \int f(x)\,dx \geq 0$, i.e.

$$\int f(x)\,dx \leq \int g(x)\,dx.$$

Moreover, for all $x \in G$ we have $-|f(x)| \leq f(x) \leq |f(x)|$, and from what we have just shown we have $-\int |f(x)|\,dx \leq \int f(x)\,dx \leq \int |f(x)|\,dx$, in other words,

$$\left|\int f(x)\,dx\right| \leq \int |f(x)|\,dx.$$

□

Definition 9. Let $f \in C(G)$ and let $A = \{a_1, a_2, \ldots, a_n\} \subset G$. Then define $M(A, f) : G \to \mathbf{R}$ given by

$$M(A, f; x) = \frac{1}{n}\sum_{i=1}^{n} f(xa_i) \qquad \text{for all } x \in G.$$

Remark 10. For all $f \in C(G)$ and $A = \{a_1, a_2, \ldots, a_n\} \subset G$, the function $M(A, f)$ is continuous and the following hold:

1. $\max(M(A, f)) \leq \max(f)$,

2. $\min(M(A, f)) \geq \min(f)$,

3. $\operatorname{osc}(M(A, f)) \geq \operatorname{osc}(f)$, and

4. If $A = \{a_1, a_2, \ldots, a_n\} \subset G$ and $B = \{b_1, b_2, \ldots, b_m\} \subset G$, then

   $$M(A, M(B, f)) = M(AB, f),$$

   where $AB = \{a_i b_j \in G : 0 \leq i \leq n, 0 \leq j \leq m\}$.

Lemma 11. Let $f \in C(G)$ be non-constant. Then there exists an $A \subset G$ such that

$$\operatorname{osc}(M(A, f)) < \operatorname{osc}(f).$$

Proof. Let $f \in C(G)$ be non-constant, let $k = \min(f)$, and let $l = \max(f)$. Then since f is continuous and $k < l$ there exists an open set $U \subset G$ such that for every $x \in G$ we have

$$f(x) \leq h < l \quad \text{for some } h \in \mathbf{R}.$$

Now the collection of open sets of the form Ua^{-1} for some $a \in G$ covers G, so by the compactness of G we can choose an $A = \{a_1, a_2, \dots, a_n\} \subset G$ such that the open sets Ua_i^{-1} for all $a_i \in A$ covers G.

Now for every $x \in G$ and for every $a_i \in A$ we have $f(xa_i^{)} \leq l$, but also for every $x \in G$ there exists an $a_j \in A$ such that $x \in Ua_j^{-1}$, thus $xa_j \in U$ so $f(xa_j) \leq h$. Thus we have that for all $x \in G$,

$$M(A, f; x) = \frac{1}{n} \sum_{i=1}^{n} f(xa_i) \leq \frac{1}{n}((m-1)l + h) \leq h < l.$$

Also we have $k \leq M(A, f; x)$ for all $x \in G$. Thus we have that

$$\begin{aligned} \operatorname{osc}(M(A, f)) &= \max(M(A, f)) - \min(M(A, f)) \\ &\leq \max(M(A, f)) - \min(f) \\ &< \max(f) - \min(f) = \operatorname{osc}(M(A, f)). \end{aligned}$$

□

Definition 12. Let $f \in C(G)$. Then $p \in \mathbf{R}$ is a *right mean* of f if for every $\epsilon > 0$ there exists an $A \subset G$ such that

$$|M(A, f; x) - p| < \epsilon \quad \text{for all } x \in G.$$

Lemma 13. Every $f \in C(G)$ has at least one right mean.

Proof. Fix $f \in C(G)$, and let $\Delta = \{M(A, f) \in C(G) : A = \{a_1, a_2, \dots, a_n\} \subset G\}$. From Remark 10, Δ is uniformly bounded, we will show that Δ is uniformly equicontinuous.

To that end, note that by theorem 4, f is uniformly continuous, thus for any $\epsilon > 0$ there exists a neighborhood U of the identity of G such that for all $x, y \in G$,

$$xy^{-1} \in U \Rightarrow |f(x) - f(y)| < \epsilon.$$

But now, for all $a_i \in A$, we have

$$(xa_i)(ya_i)^{-1} = xy^{-1} \in U \Rightarrow |f(xa_i) - g(xa_i)| < \epsilon.$$

Thus for all $xy^{-1} \in U$, and for all $A = \{a_1, a_2, \ldots, a_n\} \subset G\}$ we have

$$\begin{aligned} |M(A, f; x) - M(A, f; y)| &= \left| \frac{1}{n} \sum_{i=1}^{n} (f(xa_i) - f(ya_i)) \right| \\ &\leq \frac{1}{n} \sum_{i=1}^{n} |f(xa_i) - f(ya_i)| \\ &< \frac{1}{n}(n\epsilon) = \epsilon. \end{aligned}$$

So Δ is uniformly equicontinuous.

Now, let $s = \inf(\{\text{osc}(h) \in \mathbf{R} : h \in \Delta\})$. Then there exists a sequence $\{h_n\} \subset \Delta$ such that $\{\text{osc}(h_n)\} \to s$. But since Δ is uniformly bounded and uniformly equicontinuous we may select from $\{h_n\}$ a uniformly convergent subsequence $\{g_n\}$. Suppose $\{g_n\} \to g$, then $\text{osc}(g) = s$. Now we will show that g is constant, so suppose the contrary. Then by lemma 11, there exists an $A \subset G$, such that

$$\text{osc}(M(A, g)) = s' < s.$$

Let $\epsilon = \frac{s-s'}{3}$. Since $\{g_n\}$ converges uniformly there exists a k such that $|g(x) - g_k(x)| < \epsilon$ for all $x \in G$. Thus $|g(xa_i) - g_k(xa_i)| < \epsilon$ for all $x \in G$ and for all $a_i \in A$. So as before we have for all $x \in G$ and for all $k \in \mathbf{Z}^+$,

$$|M(A, g; x) - M(A, g_k; x)| < \epsilon.$$

Fill this in. Get a contradiction. So g is constant and put $g(x) = p$.

Now since $\{g_n\}$ converges to g uniformly, for every $\epsilon > 0$ there exists an n such that $|g_n(x) - p| < \epsilon$ for every $x \in G$. But now since $g \in \Delta$ we have that

$$|M(A, f; x) - p| < \epsilon$$

for some $A \subset G$. □

By analogy,

Definition 14. Let $f \in C(G)$ and let $B = \{a_1, a_2, \ldots, a_m\} \subset G$. Then define $M'(B, f) : G \to \mathbf{R}$ given by

$$M'(B, f; x) = \frac{1}{m} \sum_{i=1}^{n} f(a_i x) \quad \text{for all } x \in G.$$

Remark 15. Let $f \in C(G)$, let $A = \{a_1, a_2, \ldots, a_n\} \subset G$, and let $B = \{b_1, b_2, \ldots, b_m\} \subset G$. Then

$$M(A, M'(B, f)) = M'(B, M(A, f)).$$

Definition 16. Let $f \in C(G)$. Then $q \in \mathbf{R}$ is a *left mean* of f if for every $\epsilon > 0$ there exists an $B \subset G$ such that

$$|M'(B, f; x) - q| < \epsilon \quad \text{for all } x \in G.$$

Remark 17. Every $f \in C(G)$ has at least one right mean.

Lemma 18. Let $f \in C(G)$. Then f has only one right mean and only one left mean, and furthermore, these two numbers are the same. This unique *mean* for f with be denoted by $M(f)$.

Proof. Let p be any right mean of $f \in C(G)$, and let q be any left mean of f. Then for every $\epsilon > 0$ there exist $A, B \subset G$ such that for all $x \in G$,

$$|M(A, f; x) - p| < \epsilon \text{ and } |M'(B, f; x) - q| < \epsilon.$$

As seen before we then have

$$|M'(B, M(A, f); x) - p| = \left| \frac{1}{n} \sum_{i=1}^{n} (M(A, f; a_i x) - p) \right| < \epsilon$$

And similarly,

$$|M(A, M'(B, f); x) - q| < \epsilon.$$

And by remark 15 we have

$$|p - q| < 2\epsilon.$$

Thus we have that $p = q$. □

Lemma 19. Let $f, g \in C(G)$. Then

$$M(f + g) = M(f) + M(g).$$

Proof. Skip it. □

Lemma 20. Let $f \in C(G)$, and let $a \in G$. Define $f_a(x) = f(xa)$ and $f^a(x) = f(ax)$ for all $x \in M$. Then $M(f_a) = M(f^a) = M(f)$.

Proof. Let $A = \{a_1, a_2, \ldots, a_n\} \subset G$. For all $a, x \in G$ observe that

$$M(A, f_a; x) = \frac{1}{n}\sum_{i=1}^{n} f_a(xa_i) = \frac{1}{n}\sum_{i=1}^{n} f(xa_i a) = M(Aa, f)$$

and that

$$M'(A, f^a; x) = \frac{1}{n}\sum_{i=1}^{n} f^a(a_i x) = \frac{1}{n}\sum_{i=1}^{n} f(aa_i x) = M'(aA, f).$$

Thus the right mean for f_a and f is the same and the left mean for f^a and f is the same. Thus $M(f_a) = M(f^a) = M(f)$. $\square$

We can now proceed to,

Proof of theorem. Let $f \in C(G)$. Then define the mapping from $C(G) \to \mathbf{R}$ by

$$f \mapsto \int f(x)\,dx = M(f).$$

First notice that conditions 3 and 4 are clearly satisfied, while conditions 2,3 and 4 have been proved in the above remarks and lemmas.

To check condition 1, let $\alpha \in \mathbf{R} \setminus \{0\}$ ($\alpha = 0$ holds trivially) and let p be a right mean for f. Thus for any $\epsilon > 0$, there exists an $A \subset G$ such that for all $x \in G$,

$$|M(A, f; x) - p| < \epsilon/|\alpha| \Rightarrow |M(A, \alpha f; x) - \alpha p| < \epsilon$$

so αp is a right mean for αf, i.e. $\int \alpha f(x)\,dx = \alpha \int f(x)\,dx$.

Now, let $\int^* f(x)\,dx$ be any integral defined on G satisfying conditions 1-5. Now let p be a right mean of f, then for all $\epsilon > 0$ there exists an $A \subset G$ such that for all $x \in G$,

$$|M(A, f; x) - p| < \epsilon.$$

As noted earlier, we only need conditions 1-3 in order to integrate this inequality,

$$\left| \int^* M(A, f; x)\,dx - p \right| \leq \epsilon$$

thus by condition 6 we have

$$\left| \int^* f(x)\,dx - p \right| \leq \epsilon.$$

Thus we have

$$\int^* f(x)\,dx = p = M(f) = \int f(x)\,dx$$

Thus any integral satisfying conditions 1-5 is unique.

Now to verify condition 7, define a new integral on G given by

$$\int^* f(x)\,dx = \int f(x^{-1})\,dx.$$

Then we can easily verify conditions 1-5 for this other integral. For instance, verification of 6 is as follows, for any $a \in G$,

$$\begin{aligned}\int^* f(xa)\,dx &= \int f((xa)^{-1})\,dx = \int f(xa)\,dx = \int f(x)\,dx \\ &= \int f(x^{-1})\,dx = \int^* f(x)\,dx.\end{aligned}$$

Thus by the uniqueness just established we have

$$\int f(x^{-1})\,dx = \int f(x)\,dx,$$

which completes the proof. □