Points and conics

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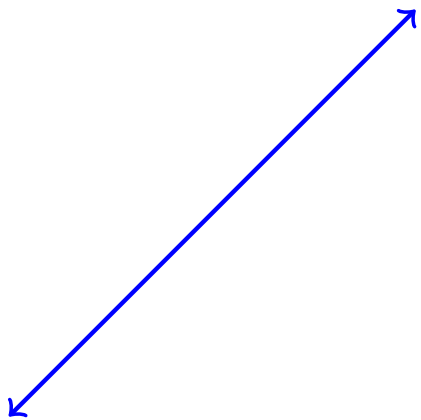
Plane Geometry

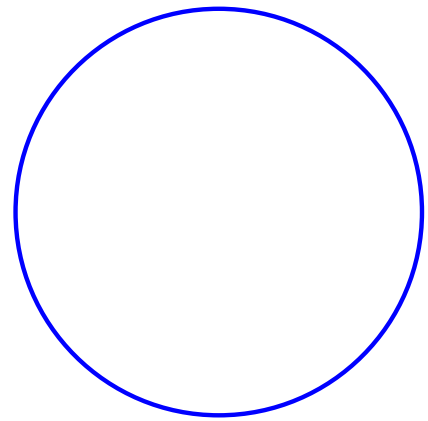
Euclid, -300

Point

Line

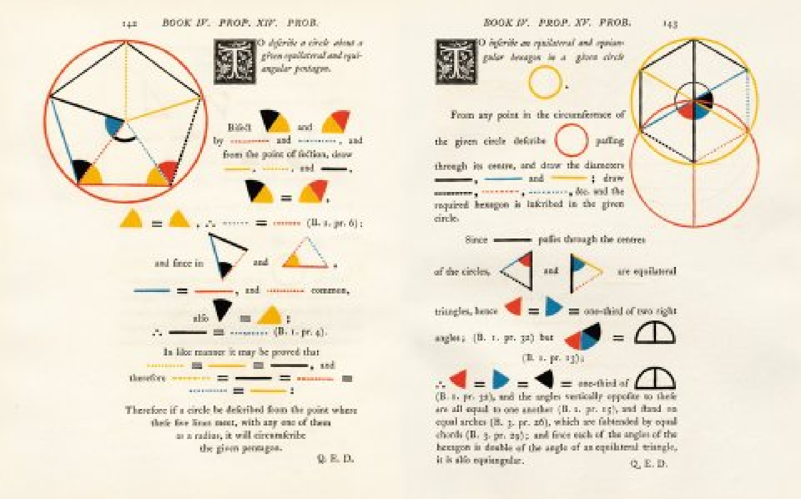


Circle



Plane Geometry

Euclid, -300


142 BOOK IV. PROP. XIV. PROB.

TO describe a circle about a given equilateral and equiangular pentagon.

Bisect and by and, and from the point of section, draw , , and .

= .

= , ∴ = (B. 1. pr. 6);

and since in and ,

= , and common,

also = ;

∴ = (B. 1. pr. 4).

In like manner it may be proved that = = , and therefore = = = = :

Therefore if a circle be described from the point where these five lines meet, with any one of them as a radius, it will circumscribe the given pentagon.

Q. E. D.

BOOK IV. PROP. XV. PROB. 143

TO inscribe an equilateral and equiangular hexagon in a given circle .

From any point in the circumference of the given circle describe passing through its centre, and draw the diameters , and ; draw , , , &c. and the required hexagon is inscribed in the given circle.

Since passes through the centres of the circles, and are equilateral triangles, hence = = one-third of two right angles; (B. 1. pr. 32) but = (B. 1. pr. 13);

∴ = = = one-third of (B. 1. pr. 32), and the angles vertically opposite to these are all equal to one another (B. 1. pr. 15), and stand on equal arches (B. 3. pr. 26), which are subtended by equal chords (B. 3. pr. 29); and since each of the angles of the hexagon is double of the angle of an equilateral triangle, it is also equiangular.

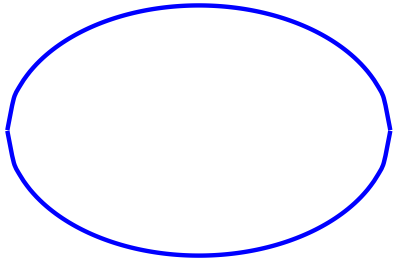
Q. E. D.

Oliver Byrne “The Elements of Euclid” 1847

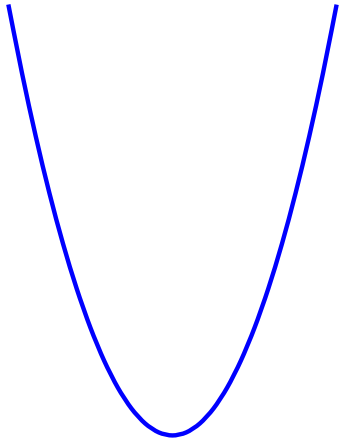
Conic sections

Apollonius, -200

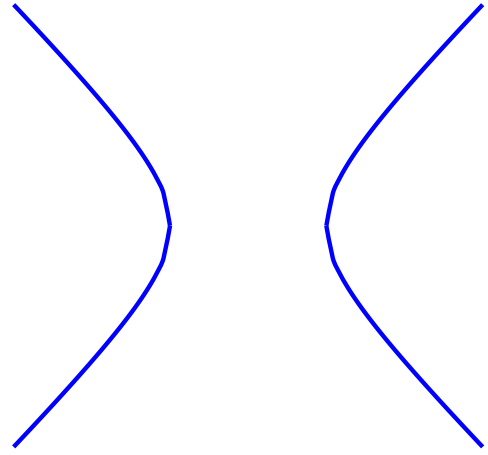
Ellipse



Parabola



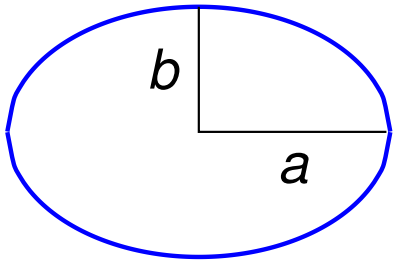
Hyperbola



Conic sections

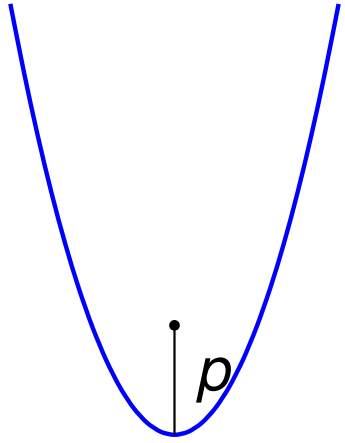
Apollonius, -200

Ellipse



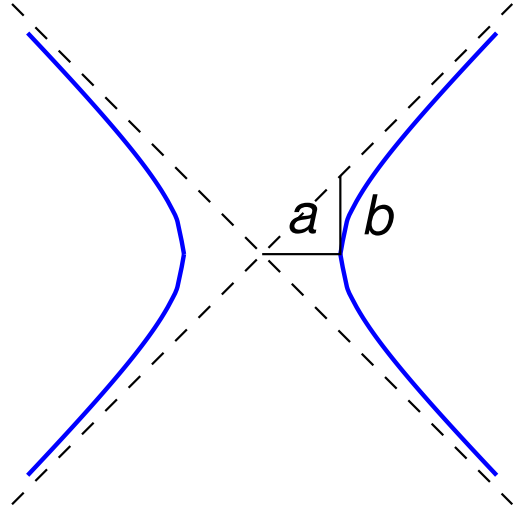
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Parabola



$$y = \frac{x^2}{4p}$$

Hyperbola



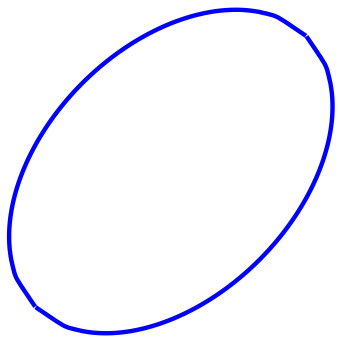
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Canonical forms

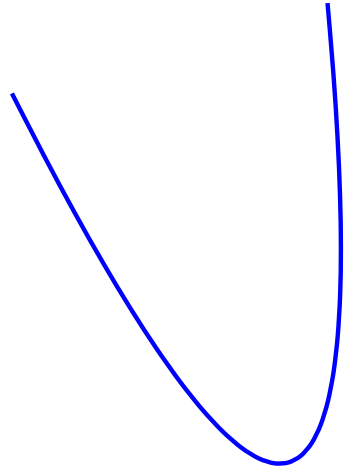
Conic sections

Apollonius, -200

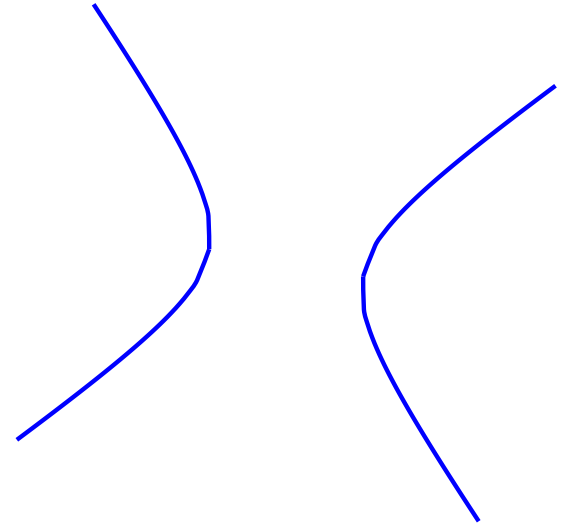
Ellipse



Parabola



Hyperbola

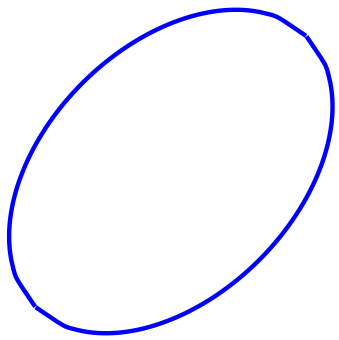


$$A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F = 0$$

Conic sections

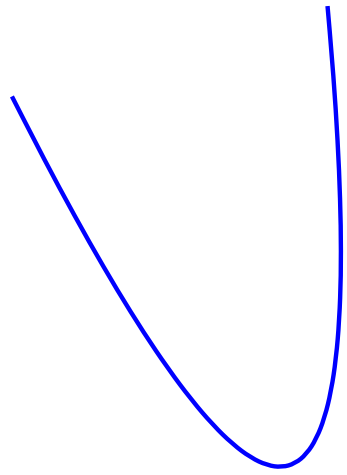
Apollonius, -200

Ellipse



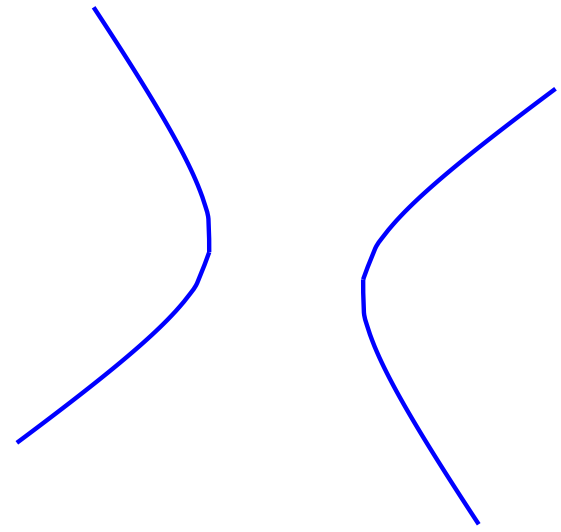
$B^2 - 4AC < 0$

Parabola



$B^2 - 4AC = 0$

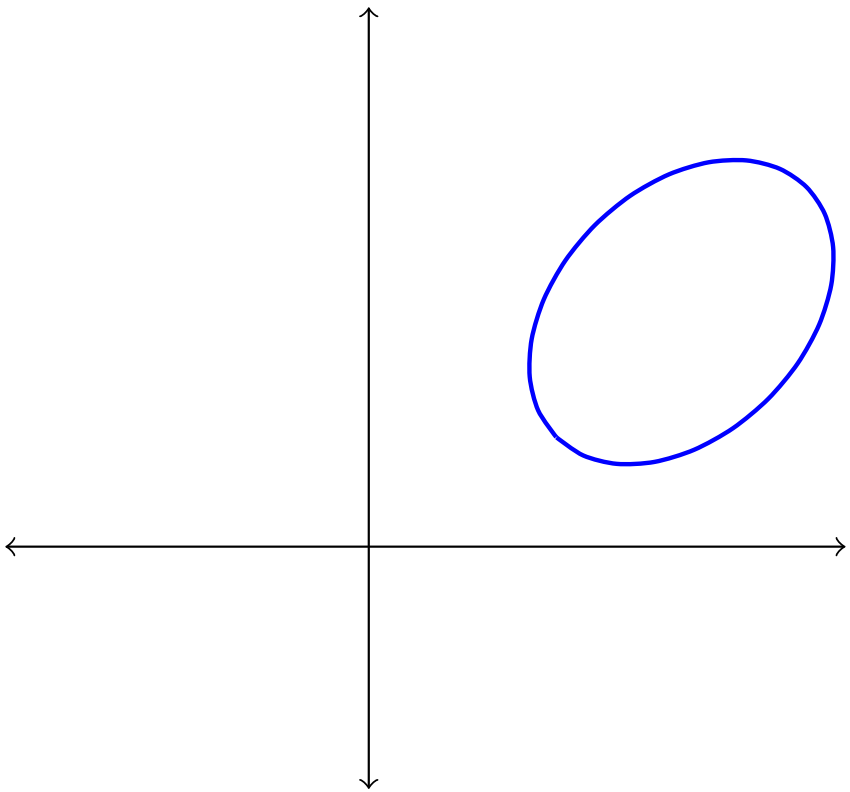
Hyperbola



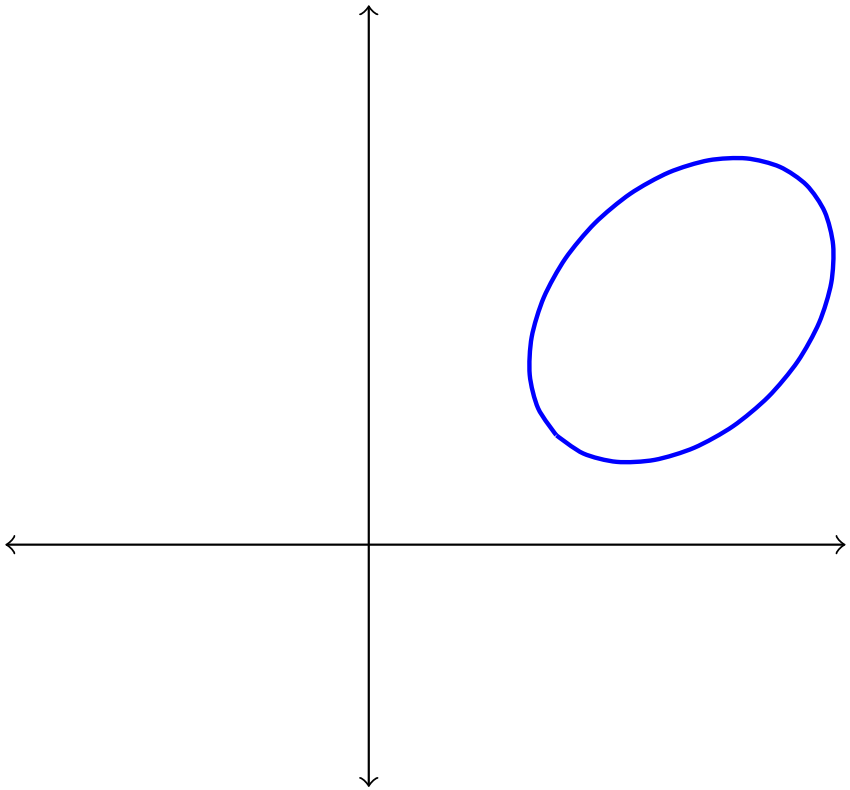
$B^2 - 4AC > 0$

$$A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F = 0$$

$$4x^2 - 2xy + 4y^2 - 14x - 4y + 15 = 0$$



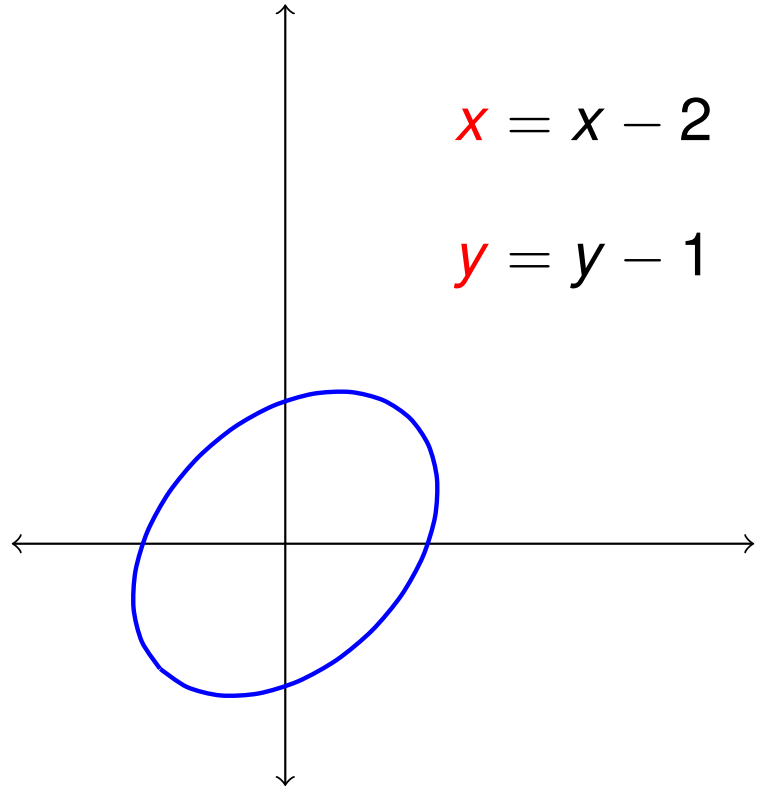
$$4x^2-2xy+4y^2-14x-4y+15=0$$



$$4x^2 - 2xy + 4y^2 = 1$$

$$x = x - 2$$

$$y = y - 1$$



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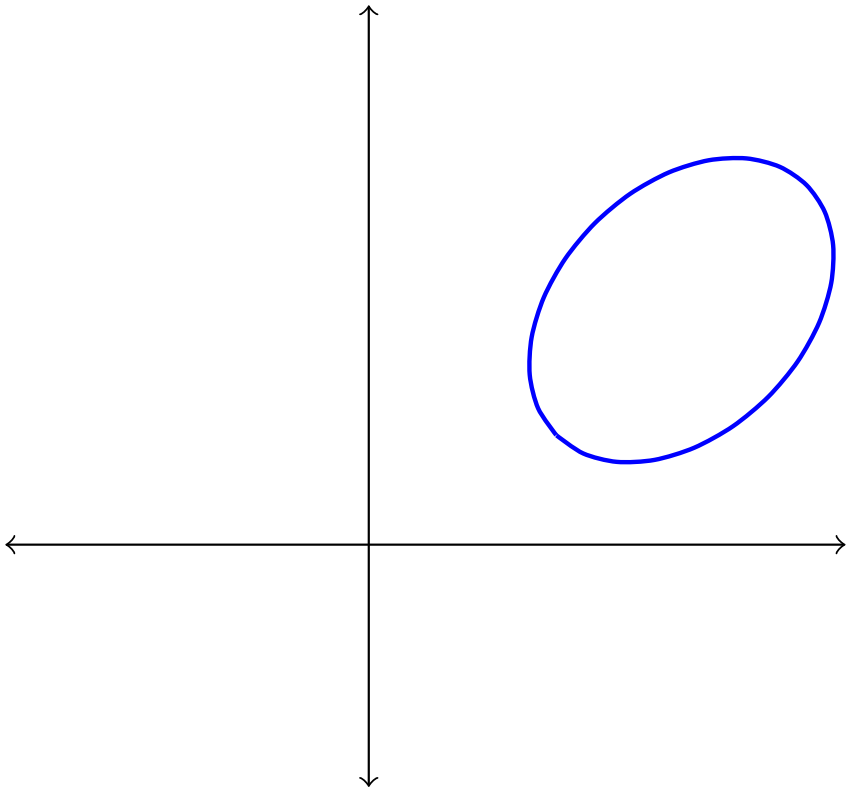
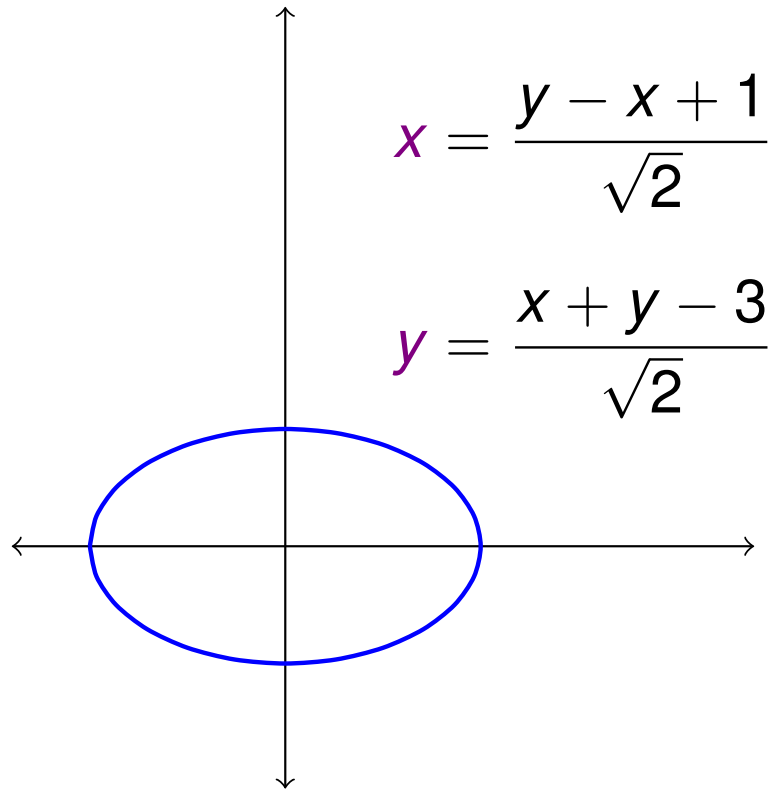


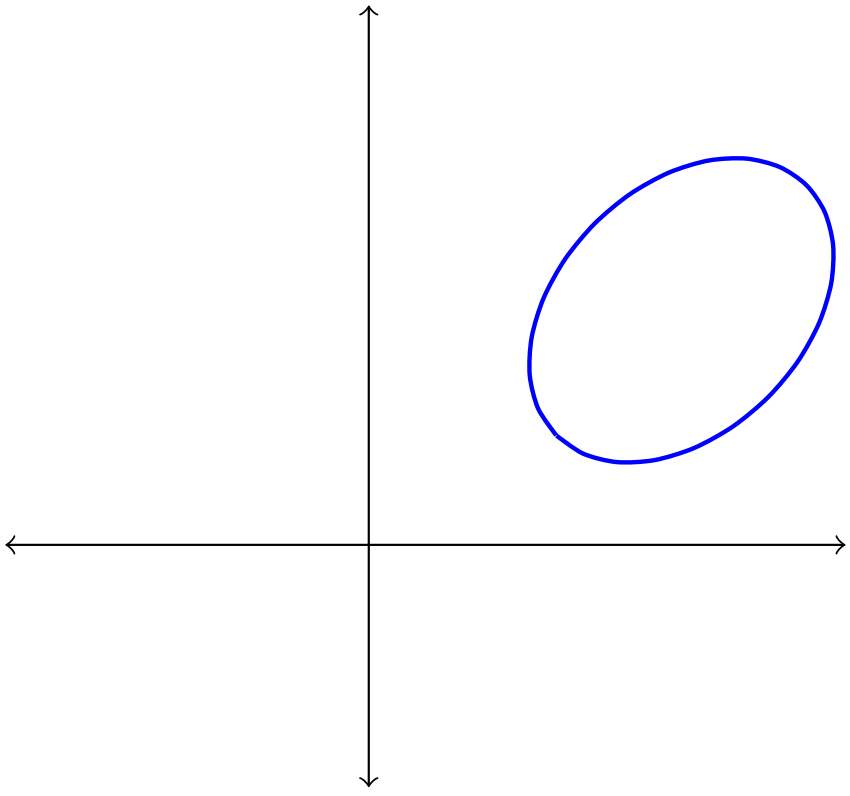
$$3x^2+5y^2=1$$

$$x=\frac{y-x+1}{\sqrt{2}}$$

$$y=\frac{x+y-3}{\sqrt{2}}$$



$$4x^2 - 2xy + 4y^2 - 14x - 4y + 15 = 0$$



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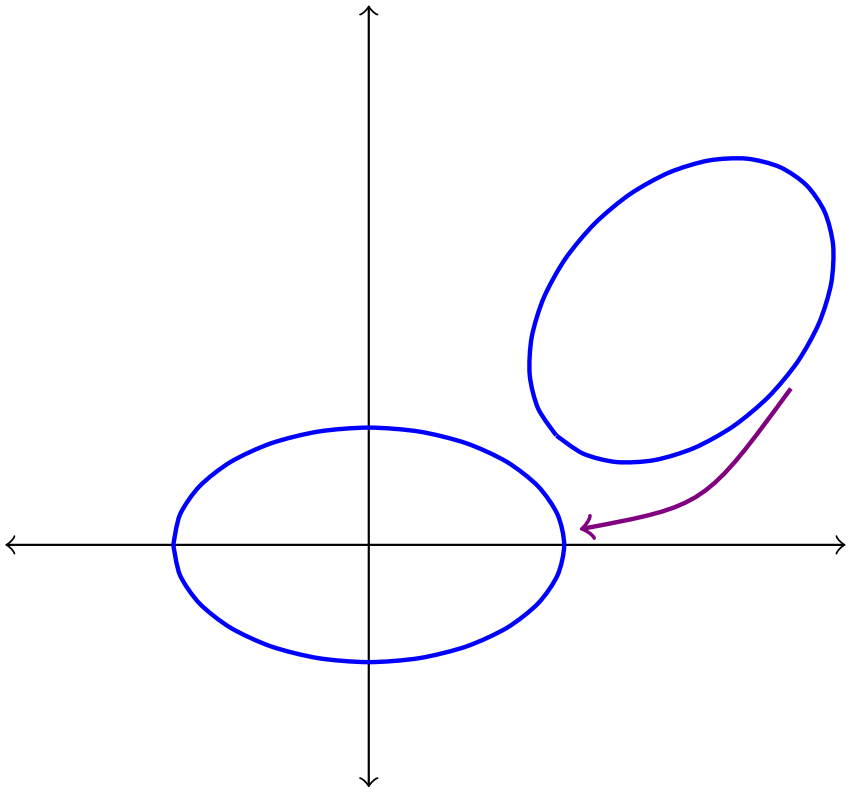
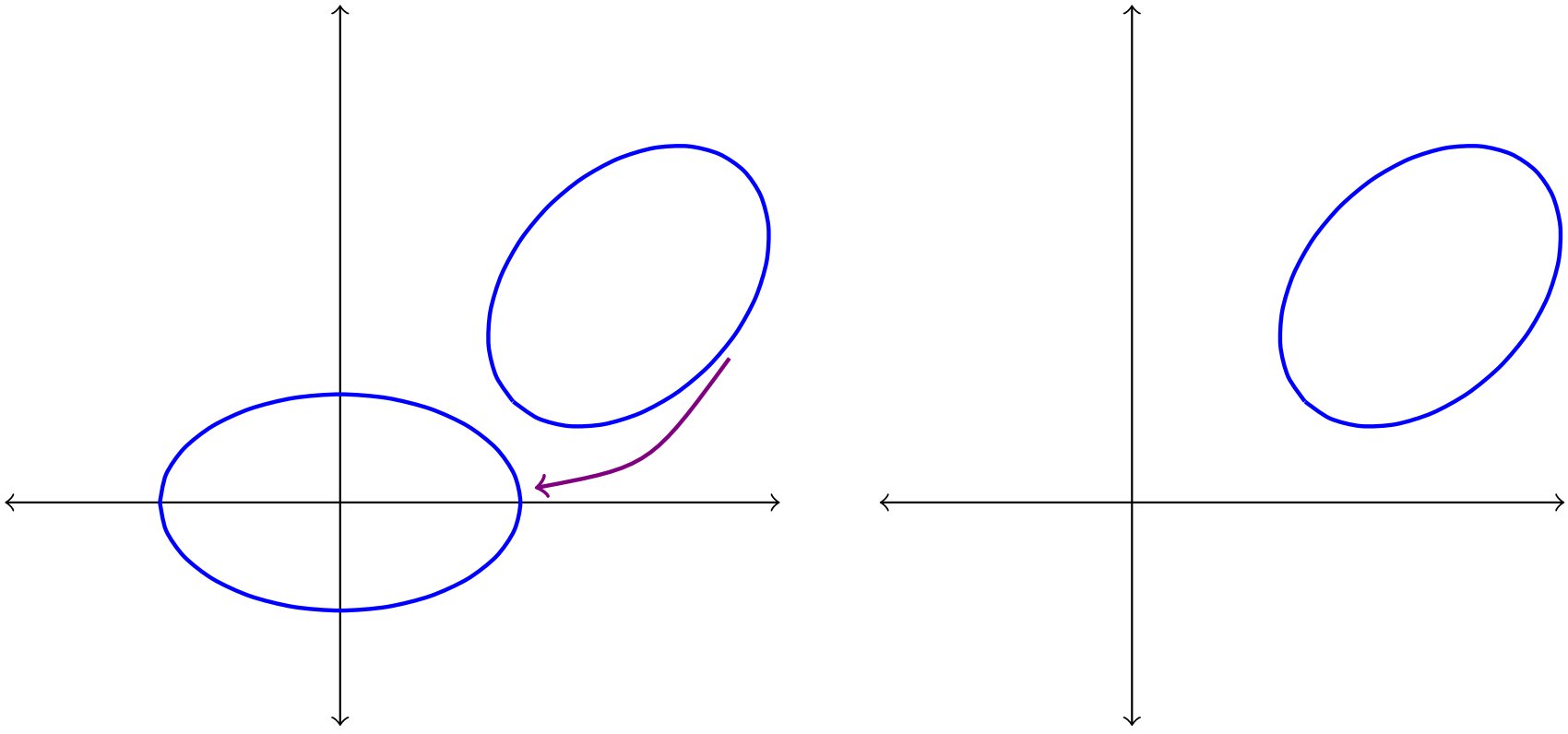


$$3x^2 + 5y^2 = 1$$

euclidean
transformation

$$4x^2 - 2xy + 4y^2 - 14x - 4y + 15 = 0$$



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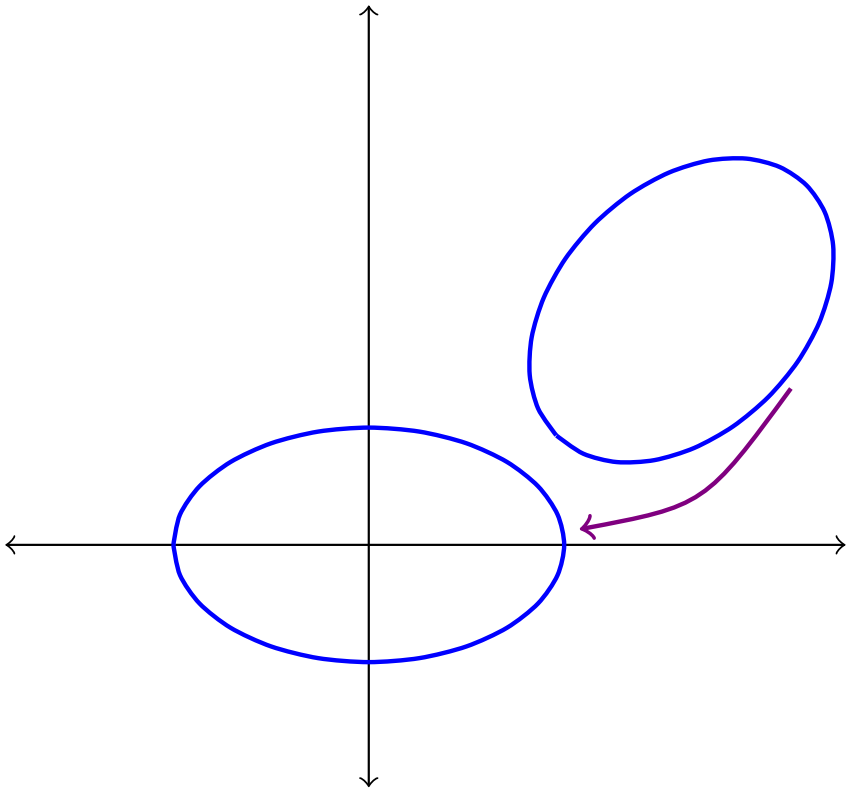

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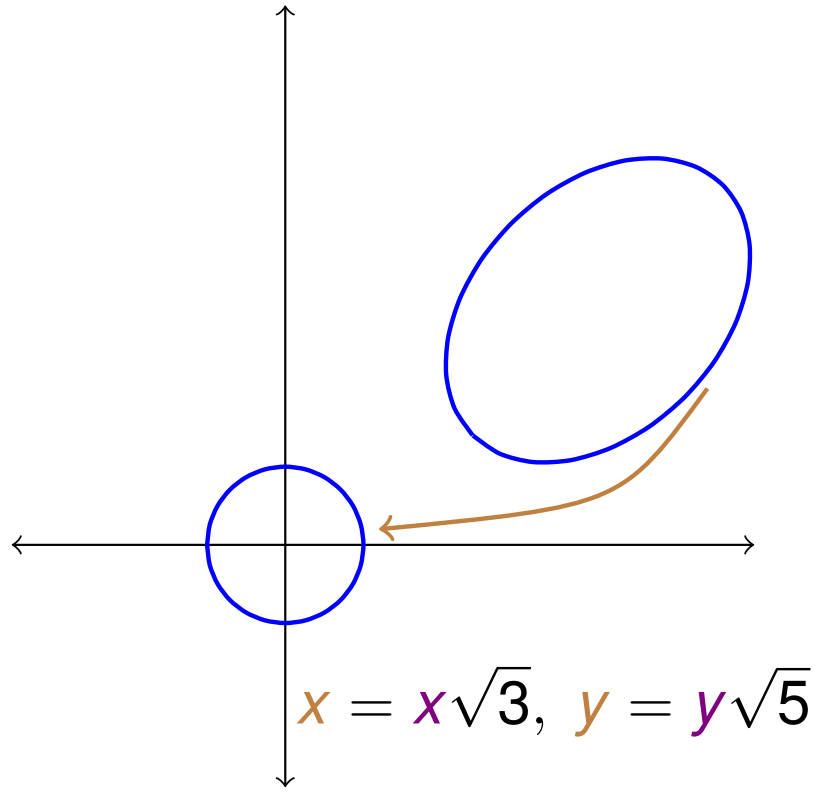

$$x^2 + y^2 = 1$$

general affine
transformation

General Affine Equivalence

$$q(x, y) = A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F$$
$$q'(x, y) = A'\,x^2 + B'\,xy + C'\,y^2 + D'\,x + E'\,y + F'$$

General Affine Equivalence

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$$q(x,y) \approx q'(x,y)$$

$$\Longleftrightarrow$$

$$q(x,y) = c \cdot q'(sx + ty + w, ux + vy + w')$$

General Affine Equivalence

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$$\Longleftrightarrow$$

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$$= c \cdot q'\left(\begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ w' \end{pmatrix}\right)$$

General Affine Equivalence

$$q(x,y) = A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F$$
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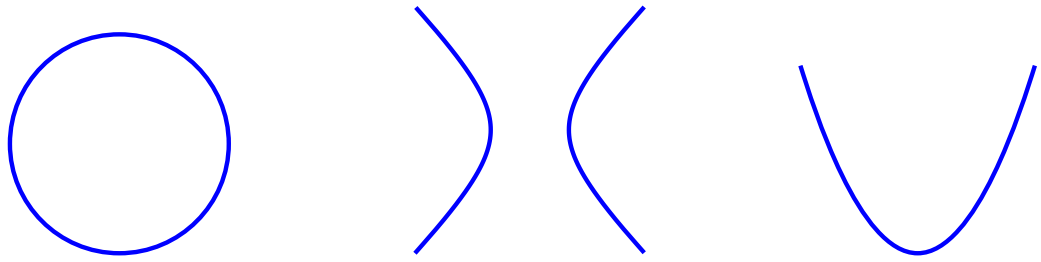
$$= c \cdot q'\left(\begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ w' \end{pmatrix}\right)$$

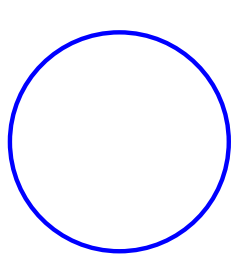
Example: $4x^2 - 2xy + 4y^2 - 14x - 4y + 15 \;\approx\; x^2 + y^2 - 1$

General Affine Classification



$x^2 + y^2 = 1$ $x^2 - y^2 = 1$ $y = x^2$

General Affine Classification



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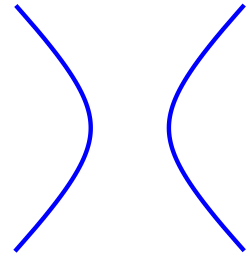
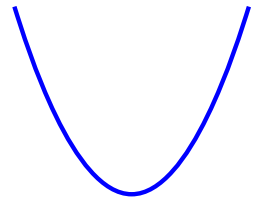


$x^2 - y^2 = 1$

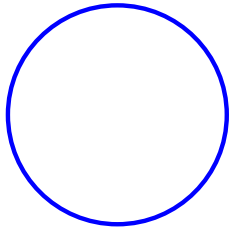


$y = x^2$

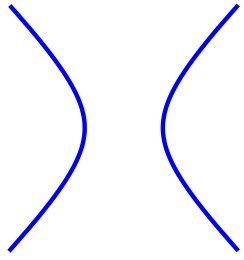
$x^2 + y^2 = -1$

General Affine Classification

smooth/singular



$x^2 + y^2 = 1$



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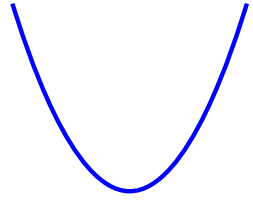
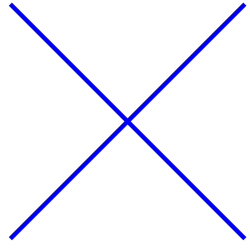


$y = x^2$

$x^2 + y^2 = -1$

$x^2 + y^2 = 0$



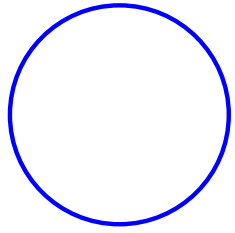
$x^2 - y^2 = 0$

$y^2 = 0$

$y^2 = 1$

General Affine Classification

smooth/singular



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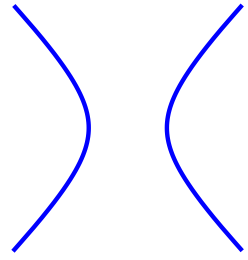
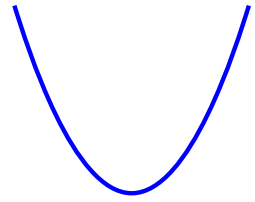


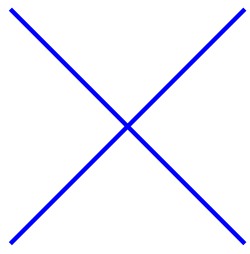
$x^2 - y^2 = 1$



$y = x^2$

$x^2 + y^2 = -1$

$x^2 + y^2 = 0$



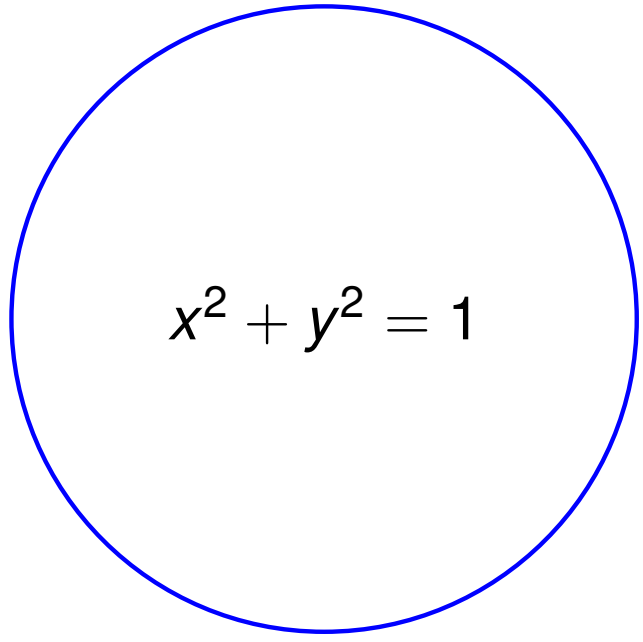
$x^2 - y^2 = 0$

$y^2 = 0$

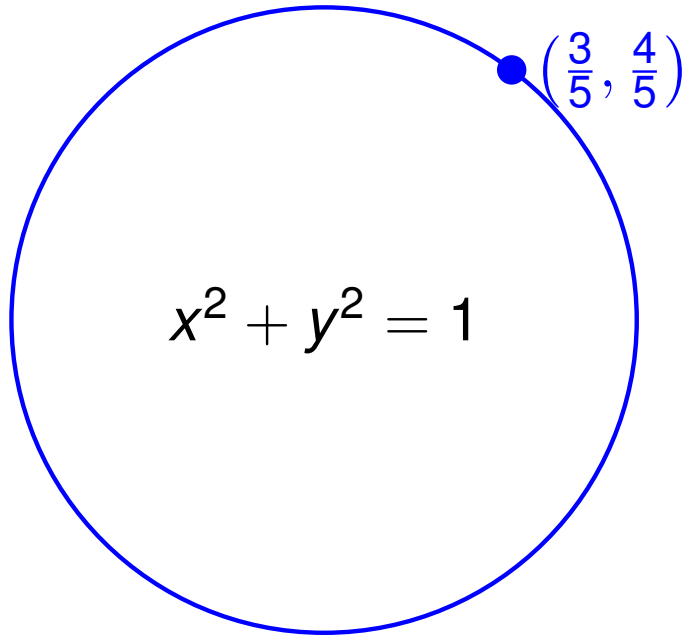
$y^2 = 1$

We'll focus on smooth conics.

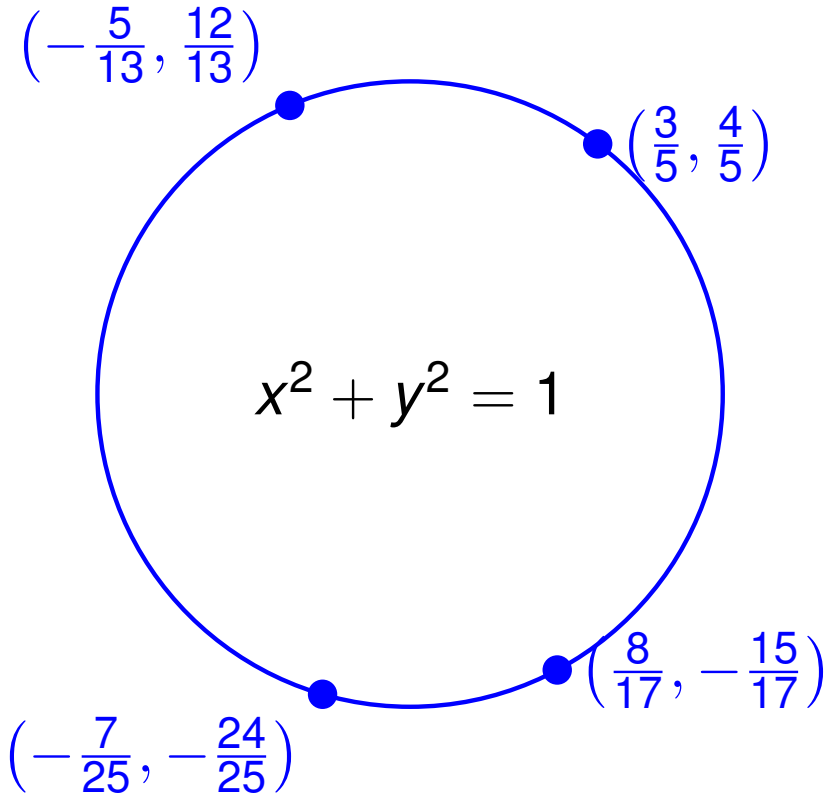
Rational Points



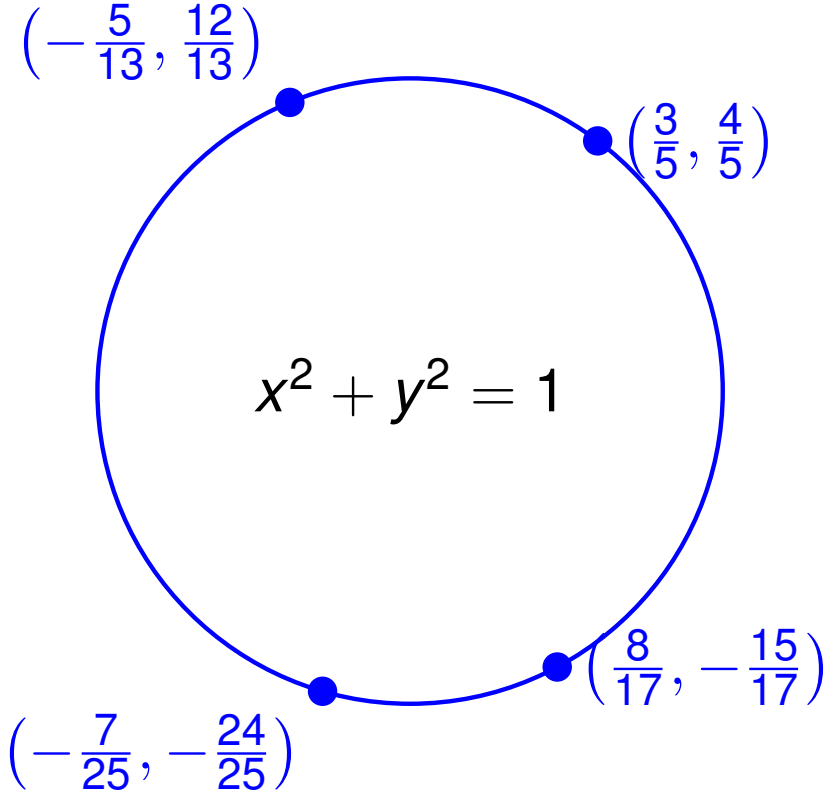
Rational Points



Rational Points



Rational Points



Rational point
$\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 = 1$

$\Longleftrightarrow$

Pythagorean triple
$3^2 + 4^2 = 5^2$

General Affine Equivalence over $\mathbb{Q}$

$\mathbb{Q} = \left\{ \frac{a}{b} \ : \ a,\ b \ \text{integers},\ b \neq 0 \right\}$ set of rational numbers

$$q(x,y) \approx_{\mathbb{Q}} q'(x,y)$$

$$\Longleftrightarrow$$

$$q(x,y) = c \cdot q'(sx + ty + w, ux + vy + w')$$

$$= c \cdot q' \left(\begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} w \\ w' \end{pmatrix} \right)$$

Where we only allow:

$$c \neq 0 \in \mathbb{Q}, \qquad \begin{pmatrix} s & t \\ u & v \end{pmatrix} \in \mathbf{GL}_2(\mathbb{Q}), \qquad \begin{pmatrix} w \\ w' \end{pmatrix} \in \mathbb{Q}^2$$

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Main point: General affine equivalence over $\mathbb{Q}$ gives a bijection on the sets of rational points.

Example. $4x^2 - 2xy + 4y^2 - 14x - 4y + 15 \overset{?}{\approx}_{\mathbb{Q}} x^2 + y^2 - 1$

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Remember $x = x - 2, \quad y = y - 1$

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$$x = \frac{1}{2}(y - x + 1), \quad y = \frac{1}{2}(x + y - 3)$$

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$q(x, y) = A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F$

Discriminant $\Delta(q) = B^2 - 4AC$

Lemma. $q(x, y) \approx_{\mathbb{Q}} q'(x, y) \implies \Delta(q) = d^2\Delta(q')$ for $d \in \mathbb{Q}$.

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compare Δ $\quad -60 \neq -4d^2$

$\sqrt{15}$ irrational

Example. $x^2 + y^2 + 1 \not\approx_{\mathbb{Q}} x^2 + y^2 - 1$ yet both have $\Delta = 4$.

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Conclusion. The discriminant Δ does not necessarily distinguish between general affine equivalence classes over $\mathbb{Q}$.

Diagonalization

Theorem. Every smooth conic over $\mathbb{Q}$ is equivalent to:

$$y = x^2 \qquad \text{or} \qquad a\,x^2 + b\,y^2 = 1, \quad \text{for some } a, b \in \mathbb{Q}$$

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Proof. Case $\Delta \neq 0$ (not a parabola).

$$q(x, y) = A\,x^2 + B\,xy + C\,y^2 + D\,x + E\,y + F$$

Can always clear away the linear terms with a translation by solutions to the system of linear equations:

$$\frac{\partial}{\partial x} q(x, y) = \frac{\partial}{\partial y} q(x, y) = 0 \;\Leftrightarrow\; \begin{cases} 2Ax + By &= -D \\ Bx + 2Cy &= -E \end{cases}$$

$$\begin{aligned} q(x, y) &\approx_{\mathbb{Q}} A\,x^2 + B\,xy + C\,y^2 + F' \\ &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + F \\ &\approx_{\mathbb{Q}} \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} s & t \\ u & v \end{pmatrix}^{\mathrm{t}} \begin{pmatrix} A & B/2 \\ B/2 & C \end{pmatrix} \begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + F \end{aligned}$$

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Proof.

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A "diagonalization" problem. Over $\mathbb{R}$, this can be done by the spectral theorem in linear algebra "every real symmetric matrix can be diagonalized by an orthogonal matrix" (remember Q is orthogonal if $Q^{\mathrm{t}} = Q^{-1}$). Over $\mathbb{Q}$ this is the higher theory of "completing the square."

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$$\begin{aligned} q(x,y) \approx_{\mathbb{Q}} ax^2 + by^2 - 1 \quad &\mapsto \quad [a,b] \\ q(x,y) \approx_{\mathbb{Q}} y - x^2 \quad &\mapsto \quad [1,-1] \end{aligned}$$

(*A priori* depends on the choice of diagonalization.)

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Properties:

- $[a,b] \approx_{\mathbb{Q}} [b,a]$ $\qquad (x,y) \mapsto (y,x)$
- $[a,b] \approx_{\mathbb{Q}} [a,b\,c^2]$ $\qquad (x,y) \mapsto (x,cy)$
- $[a,-a] \approx_{\mathbb{Q}} [1,-1]$ $\qquad \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \mapsto \frac{1}{4}\left(\begin{smallmatrix} a+1 & a-1 \\ a-1 & a+1 \end{smallmatrix}\right)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$
- $[a,1-a] \approx_{\mathbb{Q}} [1,-1]$ $\qquad$ more tricky

Manipulating symbols

Properties:

- $[a, b] \approx_{\mathbb{Q}} [b, a]$
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$$S(\mathbb{Q}) = \{[a, b] \,:\, a, b \in \mathbb{Q} \setminus \{0\}\} / \approx_{\mathbb{Q}} \text{ properties}$$

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Example. For any $a \in \mathbb{Q} \smallsetminus \{0\}$ we have

$$[1, a] \approx_{\mathbb{Q}} [a, 1] \approx_{\mathbb{Q}} [a, 1^2 - a\,0^2] \approx_{\mathbb{Q}} [1, -1]$$

in $S(\mathbb{Q})$. The class of $[1, -1]$ is called the trivial symbol.

Manipulating symbols

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Lemma. The Clifford–Hasse–Witt symbol of $q(x, y)$, taken in $S(\mathbb{Q})$, doesn't depend on the general affine equivalence class.

$$\{\text{conics}\} / \approx_{\mathbb{Q}} \;\longrightarrow\; S(\mathbb{Q})$$

Hasse–Minkowski Theorem

$c, c' \in \mathbb{Q}$ are in the same square class if $c = d^2 c'$ for $d \in \mathbb{Q}$

$\Delta(\mathbb{Q})$ set of rational square classes (including 0)

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Theorem (Hasse–Minkowski). A conic over $\mathbb{Q}$ is uniquely determined, up to general affine equivalence, by its discriminant in $\Delta(\mathbb{Q})$ and its Clifford–Hasse–Witt symbol in $S(\mathbb{Q})$.

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Recall. The trivial symbol is the class of $[1, -1]$ in $S(\mathbb{Q})$.

Theorem. A conic over $\mathbb{Q}$ has a rational point if and only if its Clifford–Hasse–Witt symbol is trivial in $S(\mathbb{Q})$.

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Example. Does

$$q(x, y) = 4x^2 - 2xy + 4y^2 - 14x - 4y + 15 = 0$$

have a rational point?
To use the theorem, we already calculated

$$q(x, y) \quad \approx_{\mathbb{Q}} \quad 6x^2 + 10y^2 - 1 \quad \mapsto \quad [6, 10]$$

Note that $10 = 4^2 - 6 \cdot 1^2$, so $[6, 10] = [1, -1]$ is trivial in $S(\mathbb{Q})$.
So $q(x, y) = 0$ has a rational point?

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$$6\left(\frac{1}{4}\right)^2 + 10\left(\frac{1}{4}\right)^2 = 1$$

$$4 \cdot 2^2 - 2 \cdot 2 \cdot \frac{3}{2} + 4\left(\frac{3}{2}\right)^2 - 14 \cdot 2 - 4\frac{3}{2} + 15 = 0$$

Legendre's Theorem

$ax^2 + by^2 = 1$ has a solution in rationals $x = \frac{X}{Z}$ and $y = \frac{Y}{Z}$

$$\Longleftrightarrow$$

$aX^2 + bY^2 = Z^2$ has a solution in integers X, Y, Z

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Theorem (Legendre's Theorem). Let a and b be positive squarefree integers. Then

$$aX^2 + bY^2 = Z^2$$

has a nontrivial solution if and only if a is a square modulo b and b is a square modulo a and $-\frac{ab}{d^2}$ is a square modulo d (here $d = \gcd(a, b)$).

Quaternions

Here as he walked by
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Sir William Rowan Hamilton
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$$i^2 = j^2 = k^2 = ijk = -1$$

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$$\mathbb{R} \quad \rightsquigarrow \quad \mathbb{C} = \mathbb{R} + i\,\mathbb{R} \quad \rightsquigarrow \quad \mathbb{H} = \mathbb{R} + i\,\mathbb{R} + j\,\mathbb{R} + k\,\mathbb{R}$$

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$$ij = k, \quad jk = i, \quad ki = j, \quad ij = -ji, \quad ik = -ki, \quad jk = -kj$$

Skew-fields

Skew-field: $\mathbb{F}$ set with operations $+$ and $\cdot$ satisfying:

- Associativity: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$
  $x + (y + z) = (x + y) + z$
- Distributivity: $x \cdot (y + z) = x \cdot y + x \cdot z$
- Identity: $0 + x = x = x + 0$
  $1 \cdot x = x = x \cdot 1$
- Inverses: $\exists -x, \quad x + (-x) = (-x) + x = 0$
  $x \neq 0 \quad \Rightarrow \quad \exists\, x^{-1}, \quad x \cdot x^{-1} = x^{-1} \cdot x = 1$
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  $x \cdot y \neq y \cdot x$

Inverting Quaternions

$$(1 + i + j)\left(\frac{1}{3} - \frac{1}{3}i - \frac{1}{3}j\right) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 0 = 1$$

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Quaternion conjugation:

$$(x + yi + zj + wk)(x - yi - zj - wk) = x^2 + y^2 + z^2 + w^2$$

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Quaternion conjugation:

$$(x+yi+zj+wk)(x-yi-zj-wk)=x^2+y^2+z^2+w^2$$

$$\boxed{(x+yi+zj+wk)^{-1}=\frac{x-yi-zj-wk}{x^2+y^2+z^2+w^2}}$$

$$x^2+y^2+z^2+w^2=0 \quad \Leftrightarrow \quad x=y=z=w=0$$

Applications

Euclidean 3-space

Quantum Mechanics

Applications

Euclidean 3-space Imaginary quaternions

$$\mathbb{R}^3 \hookrightarrow \mathbb{H}$$

$$\vec{v} = (v_1, v_2, v_3) \mapsto v = v_1\, i + v_2\, j + v_3\, k$$

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$$i \leftrightarrow \sigma_1\sigma_2, \quad j \leftrightarrow \sigma_3\sigma_1, \quad k \leftrightarrow \sigma_2\sigma_3$$

$$S^3 = \{q \in \mathbb{H} \,:\, q\,\overline{q} = 1\} \to \mathbf{SO}(3)$$

$$q \mapsto v \mapsto q\, v\, q^{-1}$$

Quaternions over $\mathbb{Q}$

Theorem (Frobenius 1877): $\mathbb{F}$ a (skew-)field, $\mathbb{R} \subset \mathbb{F}$ center, then $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

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$$\mathbb{H}_{\mathbb{Q}} = \{x + yi + zj + wk \in \mathbb{H} \ : \ x, y, z, w \in \mathbb{Q}\}$$

$$\mathbb{H}_{2,3} = \left\{ x+yi+zj+wk \ : \ \begin{array}{l} x, y, z, w \in \mathbb{Q} \\ i^2 = 2, j^2 = 3, k^2 = -6, ij = k, \ldots \end{array} \right\}$$

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Check invertibility:

$$(x + yi + zj + wk)(x - yi - zj - wk) = x^2 - 2y^2 - 3z^2 + 6w^2$$

$$x^2 - 2y^2 - 3z^2 + 6w^2 \stackrel{?}{=} 0$$

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$$\mathbb{H}_{2,3} = \left\{ x+yi+zj+wk \ : \ \begin{array}{l} x, y, z, w \in \mathbb{Q} \\ i^2 = 2, j^2 = 3, k^2 = -6, ij = k, \ldots \end{array} \right\}$$

Check invertibility:

$$(x + yi + zj + wk)(x - yi - zj - wk) = x^2 - 2y^2 - 3z^2 + 6w^2$$

$$x^2 - 2y^2 - 3z^2 + 6w^2 = 0 \Leftrightarrow \ x = y = z = w = 0$$

Exercise!

Quaternions over $\mathbb{Q}$

Theorem (Frobenius 1877): $\mathbb{F}$ a (skew-)field, $\mathbb{R} \subset \mathbb{F}$ center, then $\mathbb{F}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

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Exercise! Hint $(x^2 - 2y^2) - 3(z^2 - 2w^2)$

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Theorem (Minkowski 1896, Merkurjev 1982): Every skew-field of dimension 4 over $\mathbb{Q}$ is a Hilbert symbol $\mathbb{H}_{a,b}$. Every skew-field of over $\mathbb{Q}$ is a tensor product Hilbert symbols $\mathbb{H}_{a,b}$.

Conics and Quaternions

Clifford–Hasse–Witt symbol $[a, b]$ and Hilbert symbol $\mathbb{H}_{a,b}$.

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Theorem: Conic sections and quaternion algebras over $\mathbb{Q}$ determine each other:

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Idea: Connection between 2-dimensional conic section

$$ax^2 + by^2 = 1$$

and 4-dimensional "quaternion invertibility" conic section

$$x^2 - ay^2 - bz^2 + abw^2 = 0$$