The existence of minimizers for an isoperimetric problem with Wasserstein penalty term in unbounded domains

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Abstract

We provide a partial answer to an open problem considered by [BCL18]: a double-minimization problem in an (possibly) unbounded domain Ω

$$\min\left\{P(E;\Omega)+\lambda W_p(E,F):\ E\subseteq\Omega,\ F\subseteq\mathbb{R}^d,\ |E\cap F|=0,\ |E|=|F|=1\right\},\tag{1}$$

for minimizing energy in a model of bi-layer membranes [LPR14].

- We first provide a method for existence results in unbounded domain in any dimension, under a restriction of parameters;
- We highlight a shape functional induced by the Wasserstein distance and provide useful properties.

Preliminary

Background

1. A Lebesgue measurable set E is a **set of finite perimeter** if
$$P(E;\Omega)=\sup\left\{\int_\Omega \chi_E(x)\operatorname{div}\phi(x)\,\mathrm{d}x:\ \phi\in C_c^1(\mathbb{R}^d;\mathbb{R}^d),\ \|\phi\|_\infty\leqslant 1\right\}.$$
Minimizing the perimeter under a volume constraint in an unbounded domain Ω leads to the classical Euclidean isoperimetric problem. The first term $P(E;\Omega)$ is thus an **attractive term.**
2. Given two Lebesgue measures of equal mass μ,ν, the p-**Wasserstein distance** ($p\geqslant 1$) between μ and ν is given by
$$W_p(\mu,\nu)=\inf_{\gamma\in\Gamma(\mu,\nu)}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^p\,\mathrm{d}\gamma(x,y)\right)^{1/p},$$
where $\Gamma(\mu,\nu)$ is the collection of *transport plans* whose marginals are μ,ν. One can generalize it to define the distance between two Lebesgue measurable sets E,F of equal volume:
$$W_p(E,F)=W_p(\mathcal{L}^d\llcorner E,\mathcal{L}^d\llcorner F).$$
We can well-define the Wasserstein shape functional on any *bounded* Lebesgue measurable set E of volume m:
$$\mathcal{W}_p(E):=\min\{W_p(E,\bar F):\ (E,\bar F)\in\mathcal{F}_m\}.$$
among the class $\mathcal{F}_m$ of disjoint pairs (E,F) of equal volume,
$$\mathcal{F}_m:=\left\{(E,F):\ E,F\subseteq\mathbb{R}^d,\ |E\cap F|=0,\ |E|=|F|=m\right\}.$$
Fixing the bounded set E, we can regard the minimizer F being the **optimal coating**.
Minimizing $\mathcal{W}_p(E)$ favors splitting the set E but not necessarily disperses the mass into vanishing components diverging infinitely apart. In the meanwhile, it favors extending perimeter so that lowering the averaged transport distance from E to its coating. The second term $\mathcal{W}_p(E)$ thus a (weak) **repulsive term**.

Some Classical Problems

Here is some literature to deal with an isoperimetric problem in unbounded domain with volume constraints.

- Euclidean isoperimetric problem: symmetrization principles.
- Sessile liquid drops problem: symmetrization principles.
- Minimizing clusters problems: cover-pack first at the cost of loss of volume, second fix volume.
- Gamow liquid drops problems: comparison criterion.

Our results

Strategy of Proof

1. **Equivalent formulation in a volume parameter m.** We apply scaling arguments to obtain an equivalent formulation (2) in terms of a volume parameter m, instead of original weight parameter λ:
$$\text{Minimize}\qquad P(E)+W_p(E,F)\quad\text{among all }(E,F)\in\mathcal{F}_m.\tag{2}$$
2. **Existence of a minimizing sequence of bounded sets.** We prove that there exists a minimizing sequence of *bounded* sets to the problem (2), inspired by [Alm76]. Here we use the so-called *Nucleation Lemma* in [Mag12]. Now, by using the known result for $\mathcal{W}_p(E)$ on any *bounded* set E, we express the double minimizing problem (2) into a shape problem
$$\text{Minimize}\qquad T(E)=P(E)+\mathcal{W}_p(E)\quad\text{among all bounded }E\text{ with }|E|=m.\tag{3}$$
3. **Existence of a minimizing sequence of uniformly bounded sets for small volume.** When m is small and $\frac{1}{p}+\frac{2}{d}>1$, we are able to find a minimizing sequence of uniformly bounded sets to the problem (3), through a non-optimality criterion, inspired by [KM14].

Step 1: Scaling Arguments

Noting $\qquad P(rE)=r^{d-1}P(E)\quad\text{and}\quad W_p(rE,rF)=r^{1+\frac{d}{p}}W_p(E,F),$

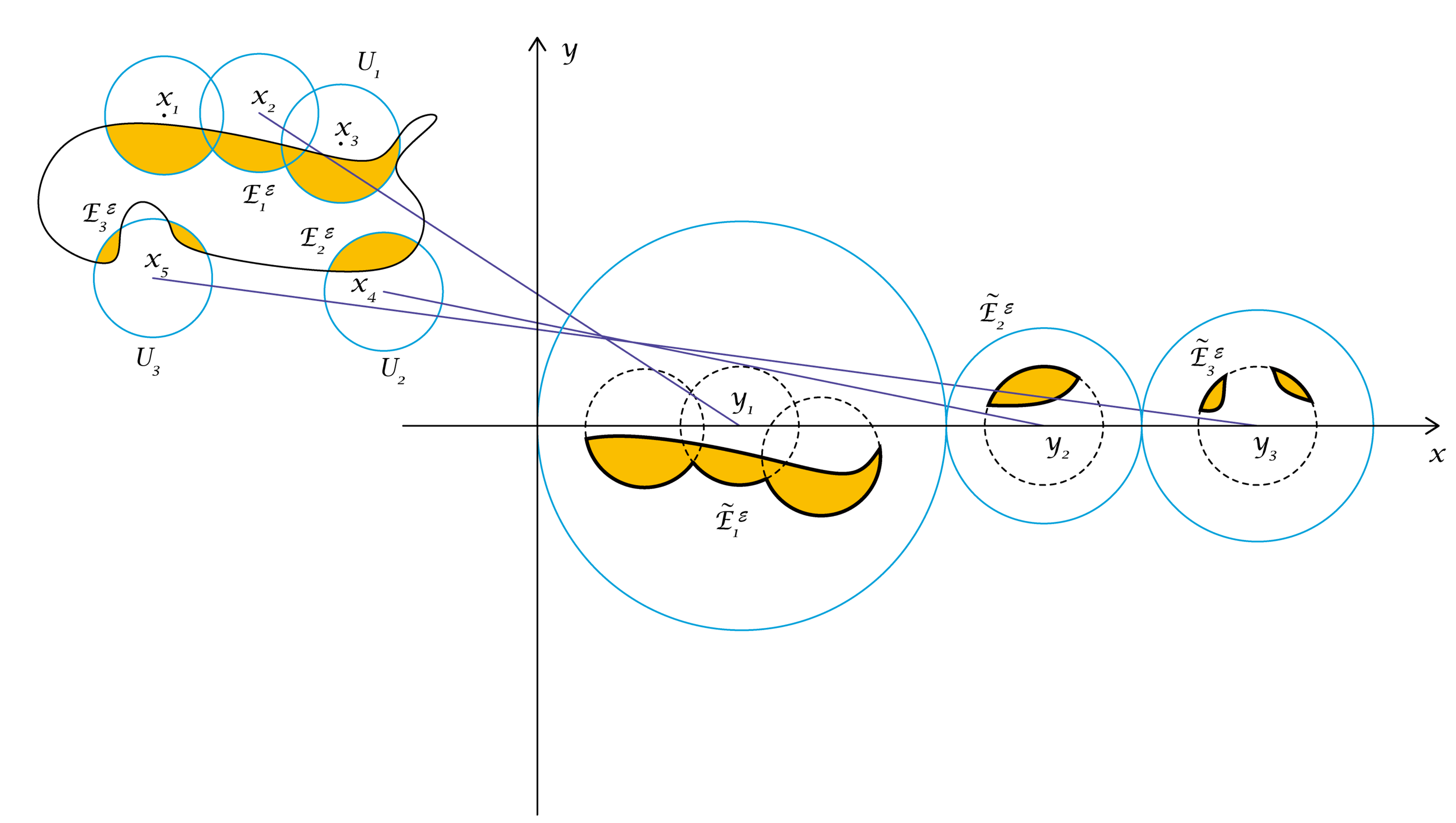
by setting parameters $\qquad \lambda=m^{\frac{1}{p}+\frac{2}{d}-1}\quad\text{and}\quad r=m^{\frac{1}{d}}.$

Step 2: Covering and Packing

For any $m>0$, $(E,F)\in\mathcal{F}_m$, and $0<\varepsilon\leqslant\min\left\{|E|,\ \frac{P(E)}{2dc(d)}\right\}$, there exists $(\bar E,\bar F)\in\mathcal{F}_m$ such that

$$P(\bar E)\leqslant P(E)+2\varepsilon,\qquad W_p(\bar E,\bar F)\leqslant W_p(E,F)+\left(\frac{2}{\omega_d}\right)^{1/d}\varepsilon^{\frac{1}{p}+\frac{1}{d}},$$

and $(\bar E,\bar F)\in\mathcal{F}$ are *bounded* sets inside the ball $B(O,R_\varepsilon)$ where $R_\varepsilon:=\left(6\left(\frac{P(E)}{c(d)\varepsilon}\right)^d+C_0(d)\left(\frac{P(E)}{c(d)\varepsilon}\right)^{d-1}|E|+\left(\frac{2\varepsilon}{\omega_d}\right)^{1/d}\right)$.



Step 3: Comparison

Suppose $p\geqslant 1$, $d\geqslant 1$ with $\frac{1}{p}+\frac{2}{d}>1$, there exists an $m_0>0$ such that for every bounded set $G\subseteq\mathbb{R}^d$ of finite perimeter with $|G|\leqslant m_0$, there exists a bounded set $E\subseteq\mathbb{R}^d$ of finite perimeter with

$$|E|=|G|,\qquad T(E)\leqslant T(G)\quad\text{and}\quad E\subseteq B_2.$$

Recent Developments

Remarks

- $\frac{1}{p}+\frac{2}{d}>1$ comes from the comparison criterion.
However, we realized that small λ (weak repulsive effect) corresponds to either (small m and $\frac{1}{p}+\frac{2}{d}>1$) or (large m and $\frac{1}{p}+\frac{2}{d}<1$).
Two groups [NTV21] and [CTG21] extend our results to the full range of parameters.
- General OT theory may deal with sets other than bounded sets.
- Properties of $\mathcal{W}_p$ has its own interest [XZ20, NTV21, CTG21]:
  - $\mathcal{W}_p(E)\leqslant C(d)|E|^{\frac{1}{p}+\frac{1}{d}}$;
  - $\mathcal{W}_p(E)$ is lower semi-continuity in the sets of finite perimeter;
  - $\mathcal{W}_p^p(E)$ is super-additive with equality when sets are far enough;
  - $|\mathcal{W}_p^p(E)-\mathcal{W}_p^p(E')|\leqslant C\max\left\{|E|^{\frac{p}{d}},|E'|^{\frac{p}{d}}\right\}|E\Delta E'|$

Recent results

1. Existence results:
   - [NTV21] follows the original ideas in [Alm76] via the nucleation lemma and volume-fixing techniques. A key estimate is the Lipschitz property of $\mathcal{W}_p^p$.
   - [CTG21] follows a concentration-compactness arguments. Because the loss of compactness can only come from splitting of the mass (rather than loss of mass in the infinity), they naturally work on the generalized minimizers and prove that they are $\Lambda-$minimizers of the perimeter. They avoid the use of Almgren nucleation lemma.
2. Characterization results:
   - [CTG21] follows a Fuglede-type argument. They use the regularity theory for $\Lambda-$minimizers of the perimeter together with a Taylor expansion of the energy around balls, to show that balls are the only minimizers when λ small.

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