

An isoperimetric problem with Wasserstein penalty term in unbounded domains

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joint work with
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Analysis Seminars

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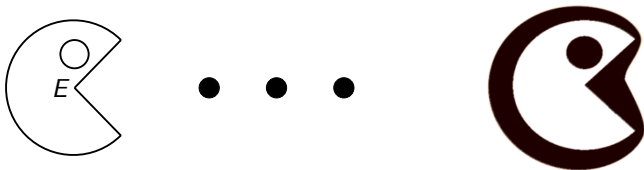


Figure: Source: [[Buttazzo-Carlier-Laborde 17'](#)]



Outline

Introduction

A double minimization problem

Preliminary

Strategy and Inspiration

Our strategy

Inspiration from Almgren's work

Main theorem and Proof

Reformulation into isoperimetric problem

Main theorem

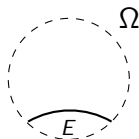
Future work and open problems

Remark

$P(E; \Omega)$: perimeter(surface area...), **attraction**.
 (Relative) isoperimetric sets in Ω .

$W_p(E; F)$: (Length ^{p} Mass) ^{$1-p$} , **repulsion**.

Coefficient is not essential, especially for $\Omega = \mathbb{R}^d$.



$$P(E; \Omega) = H^{d-1}(\Omega \setminus @E)$$

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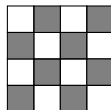
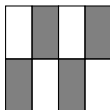
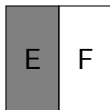
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Past literature

The double minimization problem:

arbitrary d , bounded Ω , **existence**.

$d = 2$ and $\Omega = \mathbb{R}^2$, **existence**. Isodiametric inequality.

$d > 3$ and Ω unbounded, **OPEN**. ! **AIM**: $\Omega = \mathbb{R}^d$.

F is fixed \Rightarrow an isoperimetric problem with an additional penalty:

[Xia 05'] $P(E; \Omega) + W_p^p(E; \Omega)$, bounded $\Omega \subset \mathbb{R}^d$.

! existence, regularity.

[Milakis 06'] $P(E; \Omega) + W_2^2(E; F)$ for smooth bounded Ω
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Existence of isoperimetric problem is subtle

Perron (a colleague of Steiner) criticized Steiner symmetrization on the Euclidean isoperimetric problem:

Theorem

Among all curves of a given length, the circle encloses the greatest area.


Proof.

For any curve that is not a circle, there is a method (given by Steiner) by which one finds a curve that encloses greater area. Therefore the circle has the greatest area. 

Theorem

Among all positive integers, the integer 1 is the largest.

Proof.

For any integer that is not 1, there is a method (“to take the square”) by which one finds a larger positive integer. Therefore 1 is the largest integer. 

Sets of finite perimeter

Definition

Let E be a Lebesgue-measurable set in \mathbb{R}^d , for any open set $\Omega \subset \mathbb{R}^d$, the **perimeter** of E in Ω , is

$$P(E; \Omega) := \sup_{T \in \mathcal{T}} \int_{\Omega} \operatorname{div} T(x) \, dx : T \in C_c^1(\Omega; \mathbb{R}^d); \|T\| \leq 1$$

Let ν_E be the distributional derivative of χ_E , then

$$P(E; \Omega) = \int_{\Omega} |\nu_E| = \operatorname{Var}(\chi_E; \Omega)$$

Property of sets of finite perimeter

We say $E_n \rightarrow E$ in Ω , if $\lim_{n \rightarrow \infty} \int_{\Omega} \chi_{E_n} - \chi_E = 0$:

$E \mapsto P(E; \Omega)$ is lower semi-continuous w.r.t convergence in measure.

A sequence of sets of finite perimeter $\{E_n\}$ in \mathbb{R}^d with $\sup_n P(E_n) < \infty$ and $E_n \subset B_R$, then up to subsequences, there exists a set E of finite perimeter with

$$E_n \rightarrow E; \quad E \subset B_R;$$

De Giorgi's structure theorem:

$$P(E) = H^{d-1} \llcorner E.$$

Optimal Transport

Given a compact domain Ω , W_p is a distance on $\mathcal{P}_p(\Omega)$ and metrizes its weak topology.

W_p is widely used in image processing, machine learning, fluid mechanism.

Brenier's theorem:

For $p > 1$, given $\mu, \nu \in \mathcal{P}_p(\Omega)$ for some compact domain Ω , and \mathcal{L}^d , then there exists an optimal transport map Φ such that

$$W_p(\mu; \nu) = \left(\int_{\Omega} |\Phi(x)|^p d\mu(x) \right)^{1/p} :$$

That is, $\nu = (\mu \circ \Phi)_\#$.

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Denote

$$F_m := \{(E; F) : E; F \subset \mathbb{R}^d; jE \setminus Fj = 0; jEj = jFj = m\}$$

(BCL Problem):

$$\text{Minimize } P(E) + W_p(E; F) \text{ among all } (E; F) \in F_m$$

(Volume constrained Problem):

$$\text{Minimize } P(E) + W_p(E; F) \text{ among all } (E; F) \in F_m$$

(Isoperimetric Problem): Let $W_p(E) = \min_F W_p(E; F)$

$$\text{Minimize } T(E) := P(E) + W_p(E)$$

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The direct method of Calculus of Variations

Two recipes:

Compactness of arbitrary minimizing sequence.

Lower semi-continuity of the functional.

Step 1: Volume Constraint

Scaling rules:

$$r^d = m; \quad P(rE) = r^{d-1} P(E) \quad \text{and} \quad W_p(rE; rF) = r^{1+\frac{d}{p}} W_p(E; F):$$

Volume constraint problem, for $m = m^{\frac{1}{p} + \frac{2}{d} - 1}$:

Minimize $P(E) + W_p(E; F)$ among all $(E; F) \in F_m$:

Lack of compactness in **unbounded** domain leads to the failure of volume constraint.

Target: the uniform boundedness.

Loss of compactness in unbounded domains

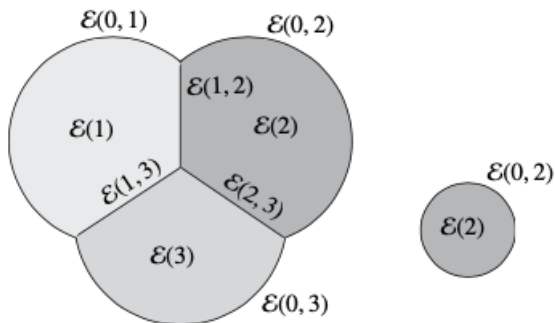
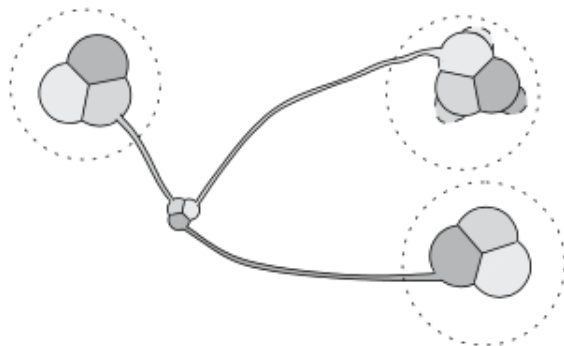


Figure: Diverging components. Source: [Maggi 12']

Almgren's seminal work

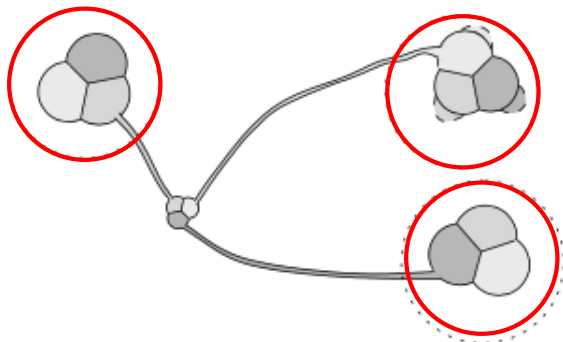
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Almgren's seminal work

Given minimizing sequence $\{E_k\}$,

$$P(E_k^0) \subset P(E_k) \quad \frac{d(E_k; E_k^0)}{C(d)^{n-d}};$$

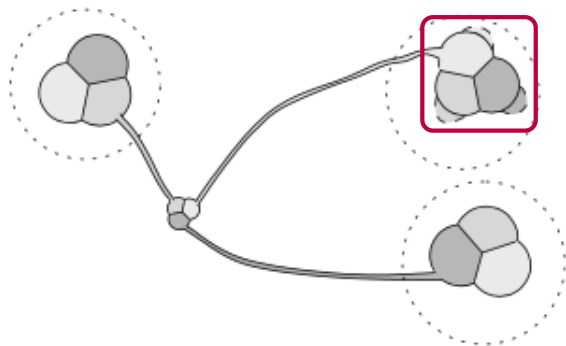


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Given minimizing sequence $\{E_k\}$,

$$P(E_k^0) \leq P(E_k) \leq \frac{d(E_k; E_k^0)}{C(d)^{1-d}}$$

$$P(E_k^{00}) \leq P(E_k^0) + C \cdot d(E_k; E_k^0):$$



Almgren's seminal work

$$\begin{array}{l}
 P(E_k^\emptyset) \subset P(E_k) \quad \frac{d(E_k; E_k^\emptyset)}{C(d)^{1-d}}; \quad \geq \quad P(E_k^{\emptyset\emptyset}) \subset P(E_k) \\
 P(E_k^{\emptyset\emptyset}) \subset P(E_k^\emptyset) + C \quad d(E_k; E_k^\emptyset); \quad \neq \quad m(E_k^{\emptyset\emptyset}) = m(E_k) \\
 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad E_k^{\emptyset\emptyset} \quad B(R(\emptyset_0))
 \end{array}$$

Wasserstein term incurs additional obstacle on analysis.

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(BCL Problem):

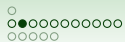
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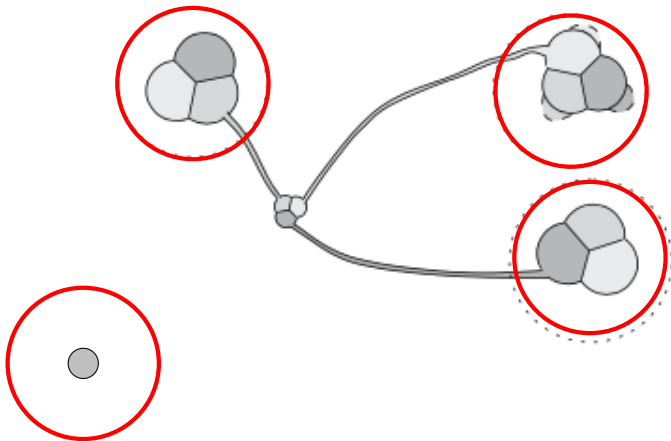
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Our strategy



Nucleation lemma

Lemma ([Almgren 76', Maggi 12'])

For every $d > 2$, there exists a positive constant $c(d)$ with the following property: given any set $E \subset \mathbb{R}^d$ of finite perimeter with $0 < |E| < 1$, and any positive number ϵ with $\epsilon \leq \min\{|E|; \frac{P(E)}{2dc(d)}\}$, there exists a finite family of points $I \subset \mathbb{R}^d$ such that:

$$\sum_{x \in I} |E \cap B(x; 2)| < \epsilon$$

$$|E \setminus \bigcup_{x \in I} B(x; 1)| > c(d) \frac{\epsilon^d}{P(E)}; \quad \#I \leq \frac{1}{c(d)} \frac{P(E)}{\epsilon^d}$$

$$\#I \leq \frac{1}{c(d)} \frac{P(E)}{\epsilon^d}$$

Step 2: Covering and Packing

Proposition ([Xia, Z. 20'])

Let $E \subset \mathbb{R}^d$ be a set of finite perimeter with $|E| < 1$ and $d > 2$.
 For any number $0 < \epsilon < \min\{|E|; \frac{P(E)}{2dc(d)}\}$, there exists a finite
 subset $I \subset \mathbb{R}^d$ with

$$\#I \leq |E| \frac{P(E)}{c(d)^\epsilon}$$

such that for some number $r \geq 2$ [2;3], the set

$$U := \bigcup_{x \in I} B(x; r)$$

satisfies

$$|E \cap U| < \epsilon \quad \text{and} \quad \boxed{H^{d-1}(E \setminus U) \leq \epsilon}:$$

Step 2: Covering and Packing (cont.)

Theorem ([Xia, Z. 20'])

For any $m > 0$, $(E; F) \in F_m$, and $0 < \epsilon \leq \min_{j \in E} |j|$; $\frac{P(E)}{2dc(d)}$,
there exists $(\tilde{E}; \tilde{F}) \in F_m$ such that

$$P(\tilde{E}) \leq P(E) + 2\epsilon; \quad W_p(\tilde{E}; \tilde{F}) \leq W_p(E; F) + \frac{2}{d!} \epsilon^{1+d};$$

and $(\tilde{E}; \tilde{F}) \in F$ are bounded sets inside the ball $B(O; R^\epsilon)$ where $O = (0; \dots; 0)$ is the origin in \mathbb{R}^d ,

$$R^\epsilon := \left(\frac{P(E)}{c(d)} \right)^d + C_0(d) \left(\frac{P(E)}{c(d)} \right)^{d-1} \min_{j \in E} |j| + \frac{2}{d!} \epsilon^{1+d};$$

Step 2: Covering and Packing (cont.)

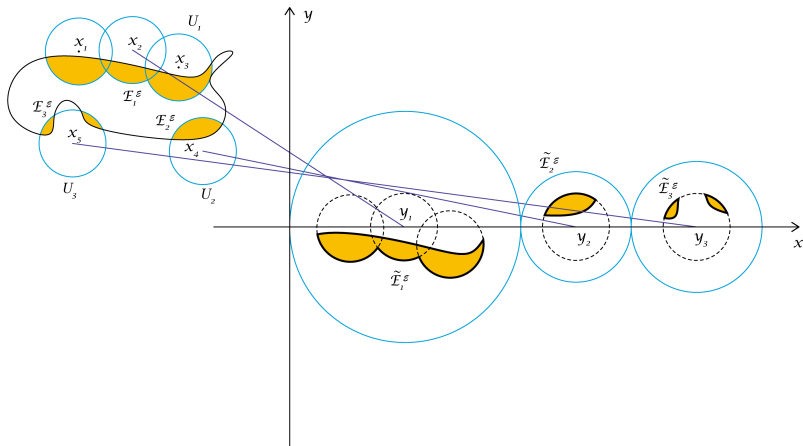


Figure: We use balls of fixed radius r to cover the majority of E . For each connected part E_j^ϵ combined with \hat{F}_j^ϵ , we pack each pair $(E_j^\epsilon; \hat{F}_j^\epsilon)$ into a ball and then align these balls together inside $B(O, R_\epsilon)$.

What we do not gain: the uniform bound

Minimize $P(E) + W_p(E; F)$ among all $(E; F) \in F_m$:

Given a minimizing sequence $f(E_n; F_n)g$, we obtain an alternative sequence $f(\hat{E}_n; \hat{F}_n)g \in B(R(\epsilon_n))$ with

$$P(\hat{E}_n) + W_p(\hat{E}_n; \hat{F}_n) \leq P(E_n) + W_p(E_n; F_n) + O(\epsilon_n):$$

To make $f(\hat{E}_n; \hat{F}_n)g$ being minimizing sequence, let $\epsilon_n \leq \frac{1}{n}$.
However, unlike minimizing clusters, $B(R(\epsilon_n))$ is **NO** more a uniformly bounded domain. ! Loss of compactness

$$\#I \leq |E| \frac{P(E)}{c(d)^n}^d$$

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$$\#I \leq |E| \frac{P(E)}{c(d)^d}$$

What we gain: a better minimizing sequence

$f(\tilde{E}_n; \tilde{F}_n)g$ is a minimizing sequence of bounded set.

(Volume constrained Problem):

Minimize $P(E) + W_p(E; F)$ among all **bounded** sets $(E; F) \in F_m$:

(Isoperimetric Problem):

Minimize $T(E) := P(E) + W_p(E)$

among all **bounded** set $E \subset \mathbb{R}^d$ of finite perimeter with $|E| = m$.

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Wasserstein functional

Definition ([Xia, Z. 20'])

For any bounded Lebesgue measurable set $E \subset \mathbb{R}^d$ and $p > 1$, let $m := |E|$ and define the Wasserstein functional on E by

$$W_p(E) := \min_{\mathbb{F}} W_p(E; \mathbb{F}) : |E \setminus \mathbb{F}| = 0; |E| = |\mathbb{F}| :$$

We call F as the W_p -minimizer of E if $W_p(E) = W_p(E; F)$.



Figure: Source: [Buttazzo-Carlier-Laborde 17']

Property of Wasserstein functional

Lemma ([Xia, Z. 20'])

For any bounded Lebesgue measurable set $E \subset \mathbb{R}^d$ and $p > 1$, let F denote a W_p -minimizer of E and Φ denote an optimal transport map that transports E to F . Then there is a constant

$C_0(d) = (3^{1-d} + 2)^{1/d}$ such that

1. For a.e. $x \in E$

$$\|\Phi(x) - x\| \leq C_0(d) |E|^{1-d} ;$$

- 2.

$$W_p(E) \leq C_0(d) |E|^{1/p + 1/d} ;$$

- 3.

$$\exists y \in \mathbb{R}^d : \text{dist}(y; E) \leq C_0(d) |E|^{1-d} = 0 ;$$

Property of Wasserstein functional (cont.)

Lemma (Lower semi-continuity of W_p , [Xia, Z. 20'])

Suppose $\{E_n\}$ is any sequence of sets of finite perimeter in \mathbb{R}^d with

$$\sup_n P(E_n) < 1 \quad \text{and} \quad E_n \subset B_R$$

for each n and some $R > 0$. If E_n converges to E , then we have

$$W_p(E) \leq \liminf_n W_p(E_n):$$

Main theorem

Theorem ([Xia, Z. 20'])

Suppose $d > 1$; $p > 1$ with $\frac{1}{p} + \frac{2}{d} > 1$, there exists an $m_0 > 0$ such that for any $m \in m_0$, the isoperimetric problem with Wasserstein penalty has a minimizer. Moreover, the minimizer is bounded. (Thus regularity results can be applied.)

Observation: Take $E = B_r$ with $|B_r| = m$.

$$P(B_r) = r^{d-1} \text{ and } W_p(B_r) = r^p r^{d-1} = r^{1+\frac{d}{p}};$$

$$m = |B_r| = r^d;$$

$$P(B_r) = W_p(B_r).$$

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Step 3: Uniform bound

Theorem ([Xia, Z. 20'])

Suppose $p > 1$, $d > 1$ with $\frac{1}{p} + \frac{2}{d} > 1$, there exists an $m_0 > 0$ such that for every bounded set $G \subset \mathbb{R}^d$ of finite perimeter with $|G| \leq m_0$, there exists a bounded set $E \subset \mathbb{R}^d$ of finite perimeter with

$$|E| = |G|; \quad T(E) \leq T(G) \quad \text{and} \quad E \subset B_2; \quad (1)$$

Recipes of proof:

[Figalli, Maggi, Pratelli 10'] Quantitative isoperimetric inequality;

Non-optimality criteria;

Gronwall's inequality.

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Quantitative isoperimetric inequality

Theorem ([Figalli, Maggi, Pratelli 10'])

There exists a constant $C(d)$ such that for any set $E \subset \mathbb{R}^d$ of finite perimeter, we have

$$4 \chi(E; B_r) \leq C(d) \frac{P(E) - P(B_r)}{P(B_r)};$$

where B_r is a d -ball with $|B_r| = |E|$, the Fraenkel asymmetry is given by

$$\chi(E_1; E_2) := \min_{x \in \mathbb{R}^d} \frac{|E_1 \Delta (E_2 + x)|}{|E_1|};$$

Non-optimality criteria

Lemma (Nonoptimality criteria, [Xia, Z. 20¹⁴])

Suppose $d > 1$; $p > 1$ with $\frac{1}{p} + \frac{2}{d} > 1$, let $G \subset \mathbb{R}^d$ be a bounded set of finite perimeter with $|G| = m < \min\{1, \frac{1}{d}\}$. Suppose there is a partition of G into two disjoint sets of finite perimeter G_1 and G_2 with positive volumes such that

$$P(G_1) + P(G_2) \leq P(G) - \frac{1}{2} T(G_2); \quad (2)$$

Then there is an $\epsilon = \epsilon(m; d) > 0$ such that if

$$|G_2| \leq \epsilon |G_1|;$$

there exists a bounded set $E \subset \mathbb{R}^d$ such that $|E| = |G|$ and $T(E) < T(G)$.

¹Inspired by [Knapfer and Muratov 14¹]

Open problems

Could $\frac{1}{p} + \frac{2}{d} > 1$ be removed? Could small volume assumption be removed?

Regularity of minimizers $(\mathbb{E}; F)$.

What are the minimizers? Must the minimizer be given by a ball?

Jordan-Kinderlehrer-Otto Scheme:

$$x_{k+1} = 2 \operatorname{argmin} F(x) + \frac{d^2(x; x_k)}{2}:$$

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Attraction: $P(E)$; Repulsion: $W(E; F)$.

Jordan-Kinderlehrer-Otto Scheme:

$$k_{+1} = 2 \operatorname{argmin} F(\cdot) + \frac{d^2(\cdot; k)}{2}$$

Open problems

Could $\frac{1}{p} + \frac{2}{d} > 1$ be removed? Could small volume assumption be removed?

Regularity of minimizers $(E; F)$.

What are the minimizers? Must the minimizer be given by a ball?

Attraction: $P(E)$; Repulsion: $R(E) = \int_E \int_E \frac{1}{|x-y|} dx dy$

[Knopfer and Muratov 14]: $36 \leq d \leq 7$, $\frac{1}{2} \leq \frac{1}{p} \leq 1$, small m .

Jordan-Kinderlehrer-Otto Scheme:

$$k_{+1} = 2 \operatorname{argmin} F(\cdot) + \frac{d^2(\cdot; k)}{2}$$

Open problems

Could $\frac{1}{p} + \frac{2}{d} > 1$ be removed? Could small volume assumption be removed?

Regularity of minimizers $\mathbb{E}; F$.

What are the minimizers? Must the minimizer be given by a ball?

Attraction: $A(m) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-y|} dx dy$;

Repulsion: $R(E) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-y|} dx dy$;

subject to $\int_{\mathbb{R}^d} dx = m; 0 < m < 1$.

[Frank, Lieb 20']: $d > 0, 2 < d < 1$, large m .

Jordan-Kinderlehrer-Otto Scheme:

$$k_{+1} = 2 \operatorname{argmin} F(\cdot) + \frac{d^2(\cdot; k)}{2}$$

Open problems

Could $\frac{1}{p} + \frac{2}{d} > 1$ be removed? Could small volume assumption be removed?

Regularity of minimizers $\mathbb{E}; F$.

What are the minimizers? Must the minimizer be given by a ball?

Jordan-Kinderlehrer-Otto Scheme:

$$x_{k+1} = 2 \operatorname{argmin} F(x) + \frac{d^2(x; x_k)}{2}:$$



Open problems

