An isoperimetric problem with Wasserstein penalty term in unbounded domains

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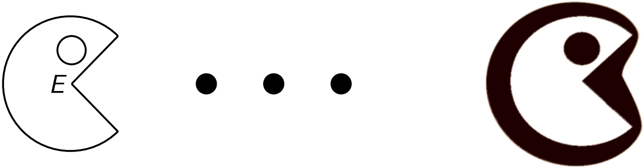


Figure: Source: [Buttazzo-Carlier-Laborde 17']

Outline

A double minimization problem

[Buttazzo, Carlier, Laborde 17'] consider

Minimize

$$P(E;\Omega) + \lambda W_p(E,F)$$

among sets (E,F):

$$E \subseteq \Omega,\ F \subseteq \mathbb{R}^d,\ |E\cap F| = 0,\ |E| = |F| = 1.$$

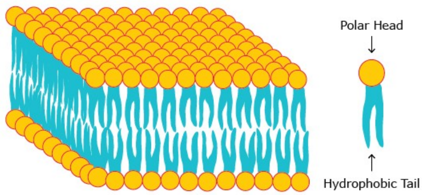
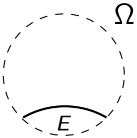


Figure: [Peletier, Röger 09'] model lipid bilayer membranes

Remark

- $P(E;\Omega)$: perimeter(surface area...), **attraction**. (Relative) isoperimetric sets in Ω.
- $W_p(E,F)$: $(\mathrm{Length}^p \times \mathrm{Mass})^{1/p}$, **repulsion**.
- Coefficient λ is not essential, especially for $\Omega = \mathbb{R}^d$.



$$P(E;\Omega) = \mathcal{H}^{d-1}(\Omega \cap \partial E)$$

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$$W_p(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \, \mathrm{d}\gamma(x,y)\right)^{1/p}.$$

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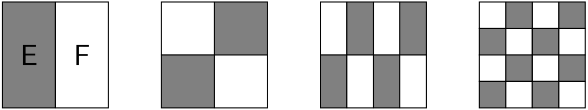

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Past literature

The double minimization problem:

- arbitrary d, bounded Ω, **existence**.
- $d = 2$ and $\Omega = \mathbb{R}^2$, **existence**. $\leftarrow$ Isodiametric inequality.
- $d \geqslant 3$ and Ω unbounded, **OPEN**. $\rightarrow$ **AIM:** $\Omega = \mathbb{R}^d$.

F is fixed $\Longrightarrow$ an isopermetric problem with an additional penalty:

- [Xia 05'] $P(E; \Omega) + \lambda W_p^p(E, \sigma\Omega)$, bounded $\Omega \subseteq \mathbb{R}^d$.

  $\rightarrow$ **existence, regularity**.
- [Milakis 06'] $P(E; \Omega) + \lambda W_2^2(E, F)$ for smooth bounded Ω and any fixed F in $\mathbb{R}^d$.

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Existence of isoperimetric problem is subtle

Perron (a colleague of Steiner) criticized Steiner symmetrization on the Euclidean isoperimetric problem:

Theorem

Among all curves of a given length, the circle encloses the greatest area.

Proof.

For any curve that is not a circle, there is a method (given by Steiner) by which one finds a curve that encloses greater area. Therefore the circle has the greatest area. □

Theorem

Among all positive integers, the integer 1 is the largest.

Proof.

For any integer that is not 1, there is a method ("to take the square") by which on finds a larger positive integer. Therefore 1 is the largest integer. □

Sets of finite perimeter

Definition

Let E be a Lebesgue-measurable set in $\mathbb{R}^d$, for any open set $\Omega \subseteq \mathbb{R}^d$, the **perimeter** of E in Ω, is

$$P(E;\Omega) := \sup\left\{\int_E \operatorname{div} T(x)\,\mathrm{d}x : T \in C_c^1(\Omega;\mathbb{R}^d), \|T\| \leqslant 1\right\}.$$

Let μ_E be the distributional derivative of $\mathbb{1}_E$, then

$$P(E;\Omega) = |\mu_E|(\Omega) = Var(\mathbb{1}_E, \Omega).$$

Property of sets of finite perimeter

- We say $E_n \to E$ in Ω, if $\lim_{n\to\infty} \left|\Omega \cap (E\Delta E_n)\right| = 0$.
- $E \mapsto P(E;\Omega)$ is lower semi-continuous w.r.t convergence in measure.
- A sequence of sets of finite perimeter $\{E_n\}$ in $\mathbb{R}^d$ with $\sup_n P(E_n) < \infty$ and $E_n \subseteq B_R$,
  then up to subsequences, there exists a set E of finite perimeter with
  $$E_n \to E, \qquad E \subseteq B_R.$$
- De Giorgi's structure theorem:
  $P(E) = \mathcal{H}^{d-1}(\partial^* E)$.

Optimal Transport

- Given a compact domain Ω, W_p is a distance on $\mathbb{P}_p(\Omega)$ and metrizes its weak topology.
- W_p is widely used in image processing, machine learning, fluid mechanism.
- Brenier's theorem:
  For $p \geqslant 1$, given $\mu, \nu \in \mathbb{P}_p(\Omega)$ for some compact domain Ω, and $\mu \ll \mathscr{L}^d$, then there exists an optimal transport map Φ such that

$$W_p(\mu, \nu) = \left(\int_\Omega \left| x - \Phi(x) \right|^p \mathrm{d}\mu(x) \right)^{1/p}.$$

  That is, $\gamma = (\mathbb{1} \times \Phi)_{\#}\mu$.

Outline

Our strategy

Denote

$$\mathcal{F}_m := \Big\{(E,F): \; E,F \subseteq \mathbb{R}^d, \; |E\cap F| = 0, \; |E| = |F| = m\Big\}.$$

(BCL Problem):

$$\text{Minimize} \qquad P(E) + \lambda W_p(E,F) \qquad \text{among all } (E,F) \in \mathcal{F}_1.$$

$$\Updownarrow$$

(Volume constrained Problem):

$$\text{Minimize} \qquad P(E) + W_p(E,F) \quad \text{among all } (E,F) \in \mathcal{F}_m.$$

$$\Updownarrow$$

(Isoperimetric Problem): Let $\mathcal{W}_p(E) = \min_F W_p(E,F)$

$$\text{Minimize} \qquad T(E) := P(E) + \mathcal{W}_p(E)$$

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The direct method of Calculus of Variations

Two recipes:

- Compactness of arbitrary minimizing sequence.
- Lower semi-continuity of the functional.

Step 1: Volume Constraint

Scaling rules:

$$r^d = m, \quad P(rE) = r^{d-1}P(E) \quad \text{and} \quad W_p(rE, rF) = r^{1+\frac{d}{p}}W_p(E, F).$$

Volume constraint problem, for $\lambda = m^{\frac{1}{p}+\frac{2}{d}-1}$:

$$\text{Minimize} \qquad P(E) + W_p(E, F) \quad \text{among all } (E, F) \in \mathcal{F}_m.$$

Lack of compactness in unbounded domain leads to the failure of volume constraint.

Target: the uniform boundedness.

Almgren's work on minimizing clusters

Minimize $\left\{P(\mathcal{E}):\left|\mathcal{E}(i)\right| = m_i,\ i = 1, 2, \cdots, N.\right\}$

- in unbounded domains;
- do NOT expect symmetry of minimizers.

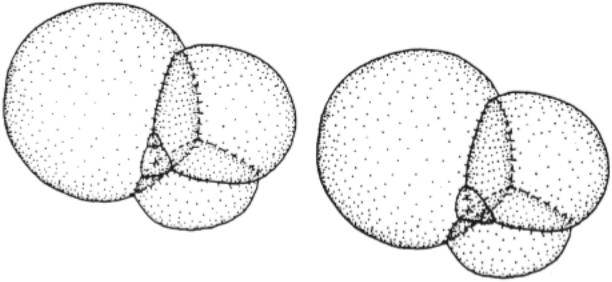


Figure 13.0.3. Soap bubble clusters are sometimes only relative minima for area. These two clusters enclose and separate the same five volumes, but the first has less surface area than the second.

Figure: Source: [Morgan 09']

Loss of compactness in unbounded domains

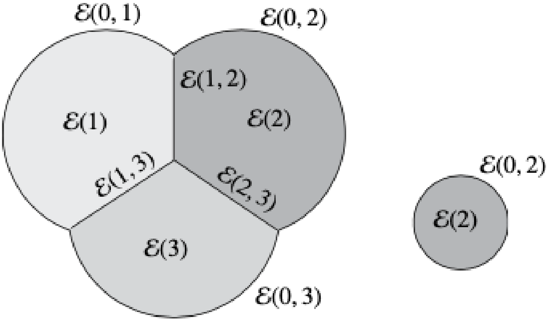
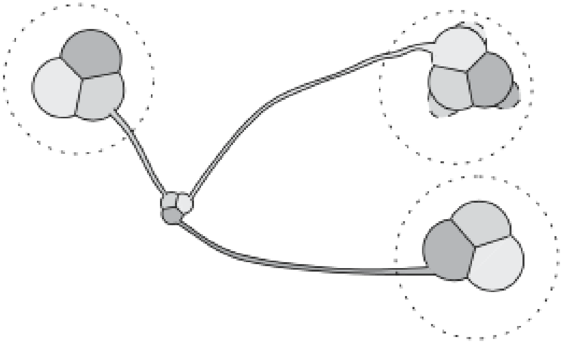


Figure: Diverging components. Source: [Maggi 12']

Almgren's seminal work

Given minimizing sequence $\{\mathcal{E}_k\}$,



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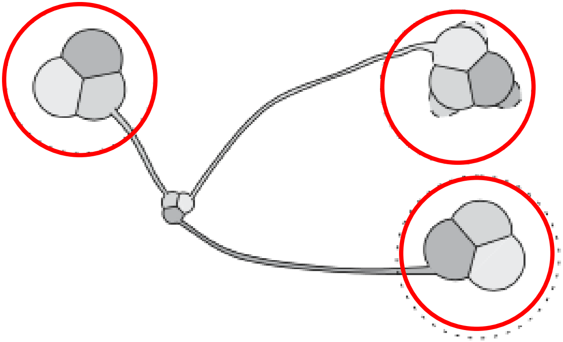
Given minimizing sequence $\{\mathcal{E}_k\}$,

$$P(\mathcal{E}_k') \leqslant P(\mathcal{E}_k) - \frac{d(\mathcal{E}_k, \mathcal{E}_k')}{C(d)\varepsilon^{1/d}};$$



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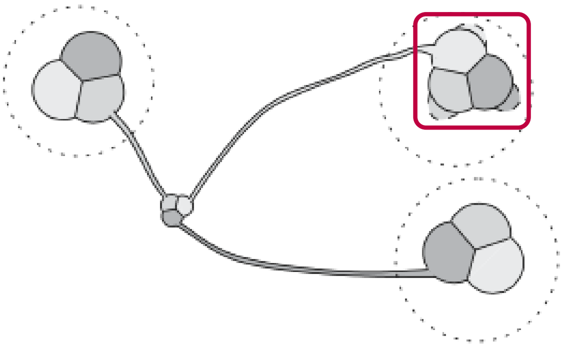

Almgren's seminal work

$$\left.\begin{aligned} P(\mathcal{E}_k') &\leqslant P(\mathcal{E}_k) - \frac{d(\mathcal{E}_k, \mathcal{E}_k')}{C(d)\varepsilon^{1/d}}; \\ P(\mathcal{E}_k'') &\leqslant P(\mathcal{E}_k') + C \cdot d(\mathcal{E}_k, \mathcal{E}_k'). \end{aligned}\right\} \overset{\varepsilon_0}{\Longrightarrow} \begin{aligned} P(\mathcal{E}_k'') &\leqslant P(\mathcal{E}_k) \\ m(\mathcal{E}_k'') &= m(\mathcal{E}_k) \\ \mathcal{E}_k'' &\subseteq B(R(\varepsilon_0)) \end{aligned}$$

Wasserstein term incurs additional obstacle on analysis.

Outline

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(BCL Problem):

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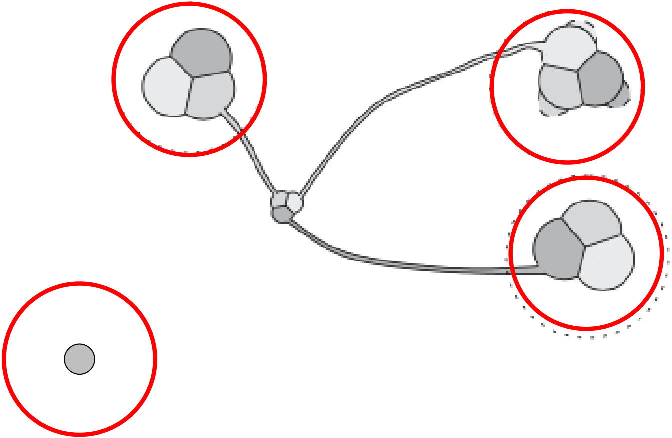

Nucleation lemma

Lemma ([Almgren 76', Maggi 12'])

For every $d \geqslant 2$, there exists a positive constant $c(d)$ with the following property: given any set $E \subseteq \mathbb{R}^d$ of finite perimeter with $0 < |E| < \infty$, and any positive number ε with $\varepsilon \leqslant \min\{|E|, \frac{P(E)}{2dc(d)}\}$, there exists a finite family of points $I \subseteq \mathbb{R}^d$ such that:

$$\left| E \setminus \bigcup_{x \in I} B(x, 2) \right| < \varepsilon$$

$$|E \cap B(x,1)| \geqslant \left(c(d) \frac{\varepsilon}{P(E)} \right)^d, \quad \forall\, x \in I.$$

$$\# I \leqslant |E| \left(\frac{P(E)}{c(d)\varepsilon} \right)^d$$

Step 2: Covering and Packing

Proposition ([Xia, Z. 20'])

Let $E \subseteq \mathbb{R}^d$ be a set of finite perimeter with $|E| < \infty$ and $d \geqslant 2$. For any number $0 < \varepsilon \leqslant \min\{|E|, \frac{P(E)}{2dc(d)}\}$, there exists a finite subset $I \subseteq \mathbb{R}^d$ with

$$\# I \leqslant |E| \left(\frac{P(E)}{c(d)\varepsilon} \right)^d$$

such that for some number $r \in [2, 3]$, the set

$$U := \bigcup_{x \in I} B(x, r)$$

satisfies

$$\left| E \setminus U \right| < \varepsilon \quad and \quad \boxed{\mathcal{H}^{d-1}(E \cap \partial U) \leqslant \varepsilon}.$$

Step 2: Covering and Packing (cont.)

Theorem ([Xia, Z. 20'])

For any $m > 0$, $(E, F) \in \mathcal{F}_m$, *and* $0 < \varepsilon \leqslant \min\left\{|E|,\ \frac{P(E)}{2dc(d)}\right\}$, *there exists* $(\widetilde{E}, \widetilde{F}) \in \mathcal{F}_m$ *such that*

$$P(\widetilde{E}) \leqslant P(E) + 2\varepsilon, \qquad W_p(\widetilde{E}, \widetilde{F}) \leqslant W_p(E, F) + \left(\frac{2}{\omega_d}\right)^{1/d} \varepsilon^{\frac{1}{p}+\frac{1}{d}},$$

and $(\widetilde{E}, \widetilde{F}) \in \mathcal{F}$ *are bounded sets inside the ball* $B(O, R_\varepsilon)$ *where* $O = (0, \cdots, 0)$ *is the origin in* $\mathbb{R}^d$,

$$R_\varepsilon := \left(6\left(\frac{P(E)}{c(d)\varepsilon}\right)^d + C_0(d)\left(\frac{P(E)}{c(d)\varepsilon}\right)^{d-1}\right)|E| + \left(\frac{2\varepsilon}{\omega_d}\right)^{1/d}.$$

Step 2: Covering and Packing (cont.)

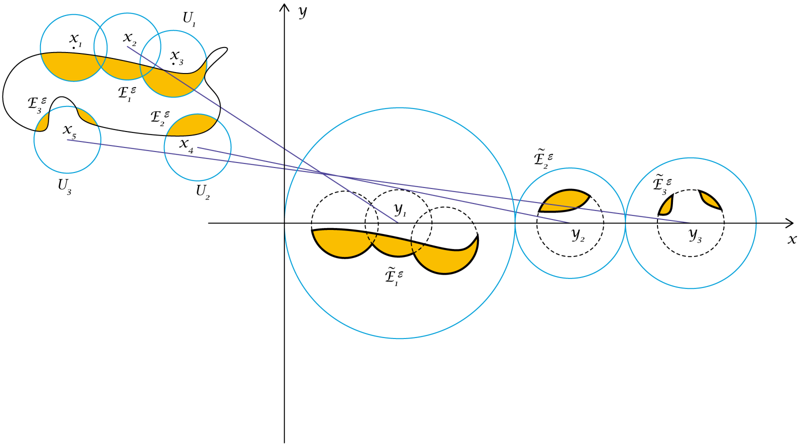

Figure: We use balls of fixed radius r to cover the majority of E. For each connected part E_j^ε combined with $\hat{F}_j^\varepsilon$, we pack each pair $(E_j^\varepsilon, \hat{F}_j^\varepsilon)$ into a ball and then align these balls together inside $B(O, R_\varepsilon)$.

What we do not gain: the uniform bound

$$\text{Minimize} \qquad P(E) + W_p(E,F) \quad \text{among all } (E,F) \in \mathcal{F}_m.$$

Given a minimizing sequence $\{(E_n, F_n)\}$, we obtain an alternative sequence $\{(\widetilde{E_n}, \widetilde{F_n})\} \subseteq B(R(\varepsilon_n))$ with

$$P(\widetilde{E_n}) + W_p(\widetilde{E_n}, \widetilde{F_n}) \leqslant P(E_n) + W_p(E_n, F_n) + \mathcal{O}(\varepsilon_n).$$

To make $\{(\widetilde{E_n}, \widetilde{F_n})\}$ being minimizing sequence, let $\varepsilon_n \leqslant \frac{1}{n}$. However, unlike minimizing clusters, $B(R(\varepsilon_n))$ is NO more a uniformly bounded domain. $\to$ Loss of compactness

$$\#I \leqslant |E| \left(\frac{P(E)}{c(d)\varepsilon} \right)^d$$

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What we gain: a better minimizing sequence

$\{(\widetilde{E_n}, \widetilde{F_n})\}$ is a minimizing sequence of bounded set.

(Volume constrained Problem):

Minimize $\quad P(E)+W_p(E,F) \quad$ among all bounded sets $(E,F) \in \mathcal{F}_m$.

(Isoperimetric Problem):

$$\text{Minimize} \qquad T(E) := P(E) + \mathcal{W}_p(E)$$

among all bounded set $E \subseteq \mathbb{R}^d$ of finite perimeter with $|E| = m$.

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Wasserstein functional

Definition ([Xia, Z. 20'])

For any bounded Lebesgue measurable set $E \subseteq \mathbb{R}^d$ and $p \geqslant 1$, let $m := |E|$ and define the Wasserstein functional on E by

$$\mathcal{W}_p(E) := \min \left\{ W_p(E, \widetilde{F}) : \ \left| E \cap \widetilde{F} \right| = 0, |E| = \left| \widetilde{F} \right| \right\}.$$

We call F as the $\mathcal{W}_p$-minimizer of E if $\mathcal{W}_p(E) = W_p(E, F)$.

Figure: Source: [Buttazzo-Carlier-Laborde 17']

Property of Wasserstein functional

Lemma ([Xia, Z. 20'])

For any bounded Lebesgue measurable set $E \subseteq \mathbb{R}^d$ and $p \geqslant 1$, let F denote a $\mathcal{W}_p$-minimizer of E and Φ denote an optimal transport map that transports E to F. Then there is a constant $C_0(d) = (3^{1/d} + 2)\ell_d$ such that

1. *For a.e. $x \in E$*

$$\left|\Phi(x) - x\right| \leqslant C_0(d)|E|^{1/d}.$$

2.

$$\mathcal{W}_p(E) \leqslant C_0(d)|E|^{\frac{1}{p}+\frac{1}{d}}.$$

3.

$$\left|F \setminus \left\{y \in \mathbb{R}^d : \mathrm{dist}(y, E) \leqslant C_0(d)|E|^{1/d}\right\}\right| = 0.$$

Property of Wasserstein functional (cont.)

Lemma (Lower semi-continuity of $\mathcal{W}_p$, [Xia, Z. 20'])

Suppose $\{E_n\}$ is any sequence of sets of finite perimeter in $\mathbb{R}^d$ with

$$\sup_n P(E_n) < \infty \qquad \text{and} \qquad E_n \subseteq B_R$$

for each n and some $R > 0$. If E_n converges to E, then we have

$$\mathcal{W}_p(E) \leqslant \liminf_{n\to\infty} \mathcal{W}_p(E_n).$$

Main theorem

Theorem ([Xia, Z. 20'])

Suppose $d \geqslant 1, p \geqslant 1$ *with* $\frac{1}{p} + \frac{2}{d} > 1$, *there exists an* $m_0 > 0$ *such that for any* $m \leqslant m_0$, *the isoperimetric problem with Wasserstein penalty has a minimizer. Moreover, the minimizer is bounded. (Thus regularity results can be applied.)*

Observation: Take $E = B_r$ with $|B_r| = m$.

$P(B_r) \approx r^{d-1}$ and $\mathcal{W}_p(B_r) \approx \left(r^p r^d\right)^{1/p} = r^{1+\frac{d}{p}}$;

$m \ll 1 \Rightarrow r \ll 1$;

$P(B_r) \gg \mathcal{W}_p(B_r)$.

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Step 3: Uniform bound

Theorem ([Xia, Z. 20'])

Suppose $p \geqslant 1$, $d \geqslant 1$ with $\frac{1}{p} + \frac{2}{d} > 1$, there exists an $m_0 > 0$ such that for every bounded set $G \subseteq \mathbb{R}^d$ of finite perimeter with $|G| \leqslant m_0$, there exists a bounded set $E \subseteq \mathbb{R}^d$ of finite perimeter with

$$|E| = |G|, \qquad T(E) \leqslant T(G) \quad \textit{and} \quad E \subseteq B_2. \tag{1}$$

Recipes of proof:

- [Figalli, Maggi, Pratelli 10'] Quantitative isoperimetric inequality;
- Non-optimality criteria;
- Gronwall's inequality.

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Quantitative isoperimetric inequality

Theorem ([Figalli, Maggi, Pratelli 10'])

There exists a constant $C(d)$ such that for any set $E \subseteq \mathbb{R}^d$ of finite perimeter, we have

$$\triangle(E, B_r) \leqslant C(d)\sqrt{\frac{P(E) - P(B_r)}{P(B_r)}},$$

where B_r is a d-ball with $|B_r| = |E|$, the Fraenkel asymmetry is given by

$$\triangle(E_1, E_2) := \min_{x \in \mathbb{R}^d} \frac{\left|E_1 \triangle (E_2 + x)\right|}{|E_1|},$$

Non-optimality criteria

Lemma (Nonoptimality criteria, [Xia, Z. 20'][1])

Suppose $d \geqslant 1, p \geqslant 1$ *with* $\frac{1}{p} + \frac{2}{d} > 1$, *let* $G \subseteq \mathbb{R}^d$ *be a bounded set of finite perimeter with* $|G| = m < \min\{1, \omega_d\}$. *Suppose there is a partition of* G *into two disjoint sets of finite perimeter* G_1 *and* G_2 *with positive volumes such that*

$$P(G_1) + P(G_2) - P(G) \leqslant \frac{1}{2} T(G_2). \tag{2}$$

Then there is an $\varepsilon = \varepsilon(m, d) > 0$ *such that if*

$$|G_2| \leqslant \varepsilon |G_1|,$$

there exists a bounded set $E \subseteq \mathbb{R}^d$ *such that* $|E| = |G|$ *and* $T(E) < T(G)$.

[1] Inspired by [Knüpfer and Muratov 14']

Non-optimality criteria (cont.)

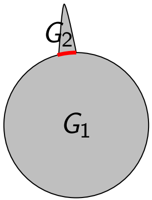

Figure: If a set G can be split into a dominated part G_1 and a remainder part G_2, with a small slicing surface area bounded by $\frac{1}{4}T(G_2)$, then G may not be a T-minimizer.

Outline

Open problems

- Could $\frac{1}{p} + \frac{2}{d} > 1$ be removed? Could small volume assumption be removed?
- Regularity of minimizers (E, F).
- What are the minimizers? Must the minimizer be given by a ball?
- Jordan-Kinderlehrer-Otto Scheme:

$$\rho_{k+1}^{\tau} \in \text{argmin } F(\rho) + \frac{d^2(\rho, \rho_k^{\tau})}{2\tau}.$$

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  Attraction: $P(E)$; Repulsion: $W(E, F)$.
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  Attraction: $P(E)$; Repulsion: $R(E) = \int_E \int_E \frac{1}{|x-y|^\alpha} \,\mathrm{d}x \,\mathrm{d}y$
  [Knüpfer and Muratov 14']: $3 \leqslant d \leqslant 7$, $\alpha \in (0, d-1)$, small m.
- Jordan-Kinderlehrer-Otto Scheme:

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  Attraction: $A(m) = \int\int \rho(x)\rho(y)|x-y|^{\lambda}\,\mathrm{d}x\,\mathrm{d}y$;
  Repulsion: $R(E) = \int\int \rho(x)\rho(y)\frac{1}{|x-y|^{\alpha}}\,\mathrm{d}x\,\mathrm{d}y$;
  subject to $\int \rho(x)\,\mathrm{d}x = m, 0 \leqslant \rho \leqslant 1$.
  [Frank, Lieb 20']: $\lambda > 0$, $\alpha \in (0, d-1)$, large m.
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