MAA Northeastern Section Meeting

Virtual, June 5, 2021

Is 73 the best number?

Carl Pomerance

Dartmouth College

Figure 1. Sheldon always knew 73 was the best.
PHOTO CREDIT: Michael Yarish/©2019 Warner Bros. Entertainment Inc.

Theorem. (Folk Lore) *Every positive integer is interesting.*

Proof. The number 1 is interesting, since it's the least positive integer. The number 2 is interesting, since it's the first prime number. One could go on. But to cut to the chase, suppose there is at least one uninteresting positive integer, and let n be the least such. Well then! That is indeed an interesting property for n to have! To resolve the contradiction, it must be that every number is interesting. $\square$

Theorem. (Folk Lore) *Every positive integer is interesting.*

Proof. The number 1 is interesting, since it's the least positive integer. The number 2 is interesting, since it's the first prime number. One could go on. But to cut to the chase, suppose there is at least one uninteresting positive integer, and let n be the least such. Well then! That is indeed an interesting property for n to have! To resolve the contradiction, it must be that every number is interesting. $\square$

(Every talk is supposed to have at least one theorem and one joke. This slide does both!)

From the 73rd episode of **The Big Bang Theory**:

Sheldon: What is the best number? By the way there's only one correct answer.

Raj: Five million, three hundred eighteen thousand, eight?

Sheldon: Wrong. **The best number is 73.** You're probably wondering why.

Leonard: No.

Howard: Uh-uh.

Raj: We're good.

Sheldon: **73 is the 21-st prime number. Its mirror, 37, is the 12-th, and its mirror, 21, is the product of multiplying, hang on to your hats, 7 and 3.** Eh? Eh? Did I lie?

Leonard: We get it. 73 is the Chuck Norris of numbers.

Sheldon: Chuck Norris wishes. In binary, 73 is a palindrome one zero zero one zero zero one, which backwards is one zero zero one zero zero one, exactly the same. All Chuck Norris backwards gets you is Sirron Kcuhc.

Leaving out the binary palindrome bit, Sheldon is basing his claim on 73 being the best number on two properties:

(1) The **Mirror Property**: A number has the mirror property if it and its mirror (i.e., reverse the digits) are both prime, and their indices in the sequence of primes are also mirrors of each other.

(2) The **Product Property**: The n-th prime p has the product property if the product of its (base-10) digits is n.

The **Sheldon Assertion, i.e., Conjecture**: *The only number with both the mirror property and the product property is* 73.

Some examples:

Let p_n denote the n-th prime.

Then $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, and $p_4 = 7$ all have the mirror property, for trivial reasons, since the primes and the subscripts are each 1-digit numbers, and reversing a 1-digit number leaves it fixed.

Slightly less trivially, $p_5 = 11$ has the mirror property. And so does

$$p_{8114118} = 143787341.$$

Heuristically, there are infinitely many palindromic primes with index in the sequence of primes also a palindrome.

The prime 73 and its index 21 are not palindromes. Heuristically, there are infinitely many such mirror primes, but they're not as common as the palindromic variety. (Coincidentally, both 73 and its index 21 *are* binary palindromes, one of these already noted by Sheldon.)

Let's look at the product property. In addition to $p_{21} = 73$, we have $p_7 = 17$ and

$$p_{181440} = 2475989, \text{ where } 2 \cdot 4 \cdot 7 \cdot 5 \cdot 9 \cdot 8 \cdot 9 = 181440.$$

Are there infinitely many primes with the product property? (To be discussed....)

All of these examples appeared in “The Sheldon Conjecture” by **Jessie Byrnes, Chris Spicer, and Alyssa Turnquist**, published in Math Horizons, November, 2015.

Chris Spicer Jessie Byrnes Alyssa Turnquist

So Sheldon’s Conjecture must be resting on the marriage of the mirror property and the product property.

Are there heuristics for there being infinitely many primes with the product property?

Lets begin with the fact that if a prime has k digits, then the product of those digits is $< 9^k$. So, if $p = p_n$ has the product property, then

$$n < 9^k = 9^{\lceil \log_{10} p_n \rceil} \approx p_n^{\ln 9 / \ln 10}.$$

But on the other hand, the **Prime Number Theorem** says that the number of primes up to p_n is approximately $p_n / \ln p_n$. But it's *exactly* n. So

$$n \approx p_n / \ln p_n.$$

Since $\ln 9 / \ln 10 = 0.9542 \cdots < 1$, it follows that the above two approximations are *incompatible* for large n.

That is, we cannot have $n < p_n^{\ln 9/\ln 10}$ when n is large, and so p_n cannot have the product property.

Perhaps a reasonable question for a calculus class, or even pre-calc:

Consider the two functions

$$f(x) = \frac{x}{\ln x}, \qquad g(x) = x^{0.9542}.$$

As $x \to \infty$, which one wins the race? And when does it finally take the lead for good?

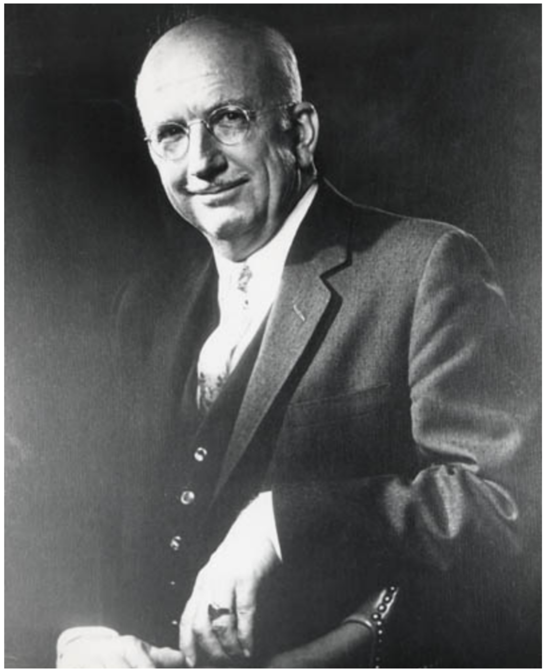
So, the Prime Number Theorem implies there are at most finitely many primes with the product property. And so, there is some hope that the Sheldon Conjecture (that 73 is the unique prime with both the product property and the mirror property) holds!

Can we actually find a numerical bound above which no prime can have the product property? For this we need to deal with "$\approx$" in the above argument. A rigorous statement of the Prime Number Theorem is that $\pi(x)$, the exact number of primes in $[1, x]$, satisfies

$$\lim_{x \to \infty} \frac{\pi(x)}{x/\ln x} = 1.$$

A limit at infinity is not going to allow us to get a numerical bound. We need something a little different.

Rosser, Schoenfeld (1962): *For* $x \geq 17$, $\pi(x) > x/\ln x$.



J. Barkely Rosser

Lowell Schoenfeld

Say $p_n \geq 17$ has the product property, has k digits, and the leading digit is a. Then $n \leq a \cdot 9^{k-1}$ and $p_n > a \cdot 10^{k-1}$. Note that n is **a lot** smaller than p_n.

By the **Rosser–Schoenfeld** theorem,

$$a \cdot 9^{k-1} \geq n = \pi(p_n) > \frac{p_n}{\ln p_n} > \frac{a \cdot 10^{k-1}}{\ln(a \cdot 10^{k-1})},$$

so that

$$\ln(a \cdot 10^{k-1}) > \left(\frac{10}{9}\right)^{k-1}.$$

The left side grows linearly in k while the right side grows exponentially. The biggest that a can be is 9, and we see even then, this last inequality fails for all values of $k \geq 46$.

We conclude that any prime with the product property must be $< 10^{45}$. We've "reduced" the problem to a finite search!

By the **Rosser–Schoenfeld** theorem, the number of primes below 10^{45} is greater than 9.6×10^{42}, so it's a very large finite search.

Is it feasible? Am I crazy to attempt this?

Well, I teamed up with **Spicer**, and we actually did it.

Pomerance, Spicer (2019): *The only prime with both the product property and the mirror property is* 73.

This article appeared in the October, 2019 issue of the American Mathematical Monthly.

Though they hadn't reduced the problem to a finite search, already in the Math Horizons article of **Spicer** and his students, **Byrnes** and **Turnquist**, they had a great idea for greatly speeding up the search:

Search over subscripts n rather than over primes p_n.

Why is that better, there are the same number of both?

Well, if p_n has the product property, then the subscript n, being the product of the digits of p_n, must have no prime factor > 7. It is a "7-smooth" number. That is, we can immediately reject any prime p_n whose subscript is not 7-smooth.

And, being 7-smooth is very special. There are only about 2,000,000 of them in the search range, and it is not hard to enumerate them.

So, given a 7-smooth number n, how does one go about deciding whether p_n has the product property or the mirror property?

In particular, how does one go about finding p_n?

Here's a modest example:

$$n = 276{,}468{,}770{,}930{,}688 = 2^{17} \cdot 3^{16} \cdot 7^2.$$

If we solve $x/\ln x = n$, we find that $x \approx 1.017 \times 10^{16}$. Are 1,0 the first two significant digits of p_n? (If so, it definitely does *not* have the product property!)

The question is, how good an approximation is $x/\ln x$ to $\pi(x)$?

The quick answer: not that good. In the same paper of **Rosser–Schoenfeld** from 1962, we find that for $x \geq 67$,

$$\frac{x}{\ln x - 1/2} < \pi(x) < \frac{x}{\ln x - 3/2}.$$

Using this, we see that if

$$n = 276{,}468{,}770{,}930{,}688,$$

then $9.747 \times 10^{15} < p_n < 1.004 \times 10^{16}$.

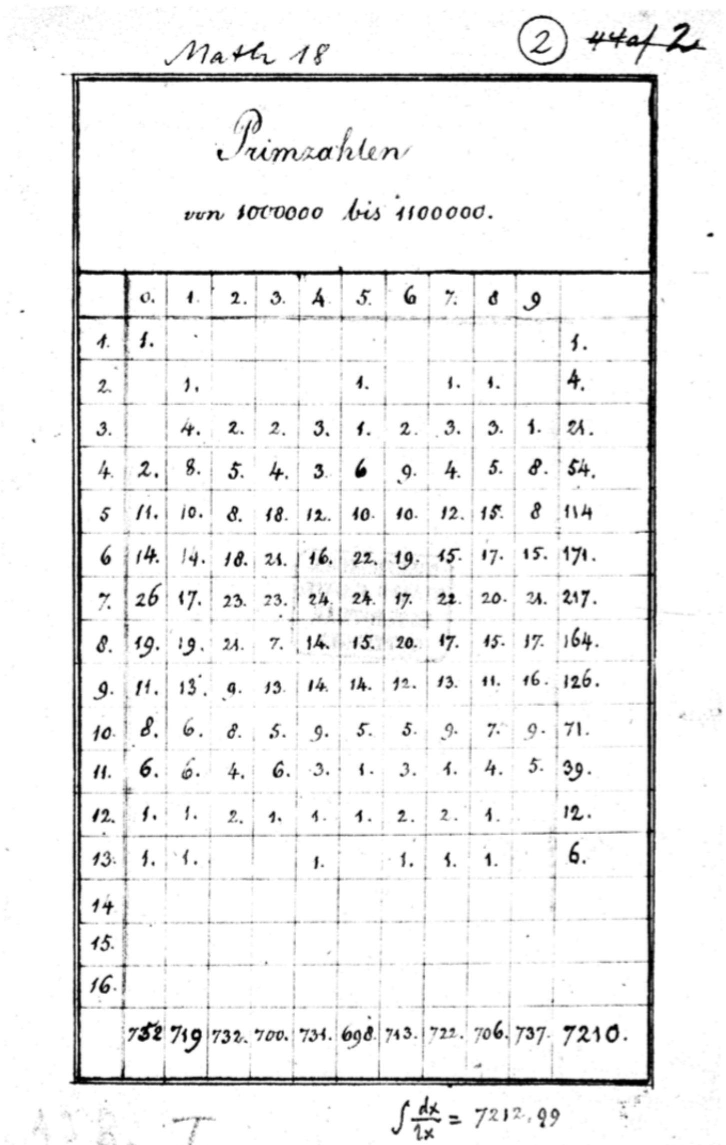
So even this finer approximation to $\pi(x)$ is not sufficient to resolve even the first digit of p_n, even in this modest example. (It's possible that if p_n is *very* close to 10^{16}, then even a finer approximation might not be able to distinguish the first digit.)

In any event, what's known about approximations to $\pi(x)$?

Around 1750, **Euler** showed that $\sum_{p\le x} 1/p$, the sum being over primes at most x, diverges to infinity like $\ln\ln x$. Since $\sum_{1<n\le x} 1/(n\ln n)$ also diverges to infinity like $\ln\ln x$, it suggests that maybe p_n is about $n\ln n$, though **Euler** did not make this leap.

Around 50 years later, **Gauss** began looking at primes statistically, creating tables by hand going up to the millions counting the number of primes in various intervals. He noticed that the primes tend to thin out and that "near" x, the chance that a random number is prime is very close to $1/\ln x$. This suggests that a good approximation to $\pi(x)$ might be

$$\mathrm{li}(x) := \int_2^x \frac{\mathrm{d}t}{\ln t}.$$



Math 18 ② ~~44a~~ 2

Primzahlen

von 1000000 bis 1100000.

	0.	1.	2.	3.	4	5.	6	7.	8	9	
1.	1.										1.
2.		1.				1.		1.	1.		4.
3.		4.	2.	2.	3.	1.	2.	3.	3.	1.	21.
4.	2.	8.	5.	4.	3.	6	9.	4.	5.	8.	54.
5	11.	10.	8.	18.	12.	10.	10.	12.	15.	8	114
6	14.	14.	18.	21.	16.	22.	19.	15.	17.	15.	171.
7.	26	17.	23.	23.	24.	24.	17.	22.	20.	21.	217.
8.	19.	19.	21.	7.	14.	15.	20.	17.	15.	17.	164.
9.	11.	13.	9.	13.	14.	14.	12.	13.	11.	16.	126.
10.	8.	6.	8.	5.	9.	5.	5.	9.	7.	9.	71.
11.	6.	6.	4.	6.	3.	1.	3.	1.	4.	5.	39.
12.	1.	1.	2.	1.	1.	1.	2.	2.	1.		12.
13.	1.	1.			1.		1.	1.	1.		6.
14											
15.											
16.											
	752	719	732.	700.	731.	698.	713.	722.	706.	737.	7210.

$\int \frac{dx}{lx} = 7212.99$

(Courtesy of Yuri Tschinkel and Brian Conrey)

Let's try out the **Gauss** approximation at 10^{20}:

$$\pi(10^{20}) = 2220819602560918840$$
$$\text{li}(10^{20}) = \int_2^{10^{20}} \frac{dt}{\ln t} = 2220819602783663482.4\ldots.$$

Not too bad!

About 50 years after **Gauss**, **Riemann** came up with a plan for proving the **Gauss** conjecture, which has still not been fully completed. The sticking point is the **Riemann Hypothesis**, an equivalent formulation being:

$$|\pi(x) - \text{li}(x)| < x^{1/2} \ln x \text{ for } x \geq 3.$$

(Prove this and win \$1,000,000 from the Clay Foundation.)

Though still not proved, the Riemann Hypothesis (in the form where the claim is that all of the non-real zeros of the Riemann zeta function have real part 1/2) has been checked up to high levels, and there are very nice consequences of this for the distribution of primes. For example, a new result of **Büthe** implies that for $10^{10} < p_n < 10^{19}$ we have

$$0 < p_n - \mathrm{li}^{-1}(n) < 2.16\sqrt{p_n}.$$

Here li^{-1} is the inverse function of li. Applying this inequality to our modest example $n = 276{,}468{,}770{,}930{,}688$, we see that

$$9{,}897{,}979{,}324{,}865{,}422 < p_n < 9{,}897{,}979{,}539{,}760{,}756.$$

This gives us unambiguously the top 7 digits of p_n and tells us that p_n has 16 digits. For this particular example, this is still not enough information to rule out p_n having the product property, and it may well have this property.

But we can rule out p_n nevertheless by considering p_m, where m is the mirror of n. Using the same tool with li^{-1} we find that p_m has 17 digits, and so cannot be the mirror of p_n.

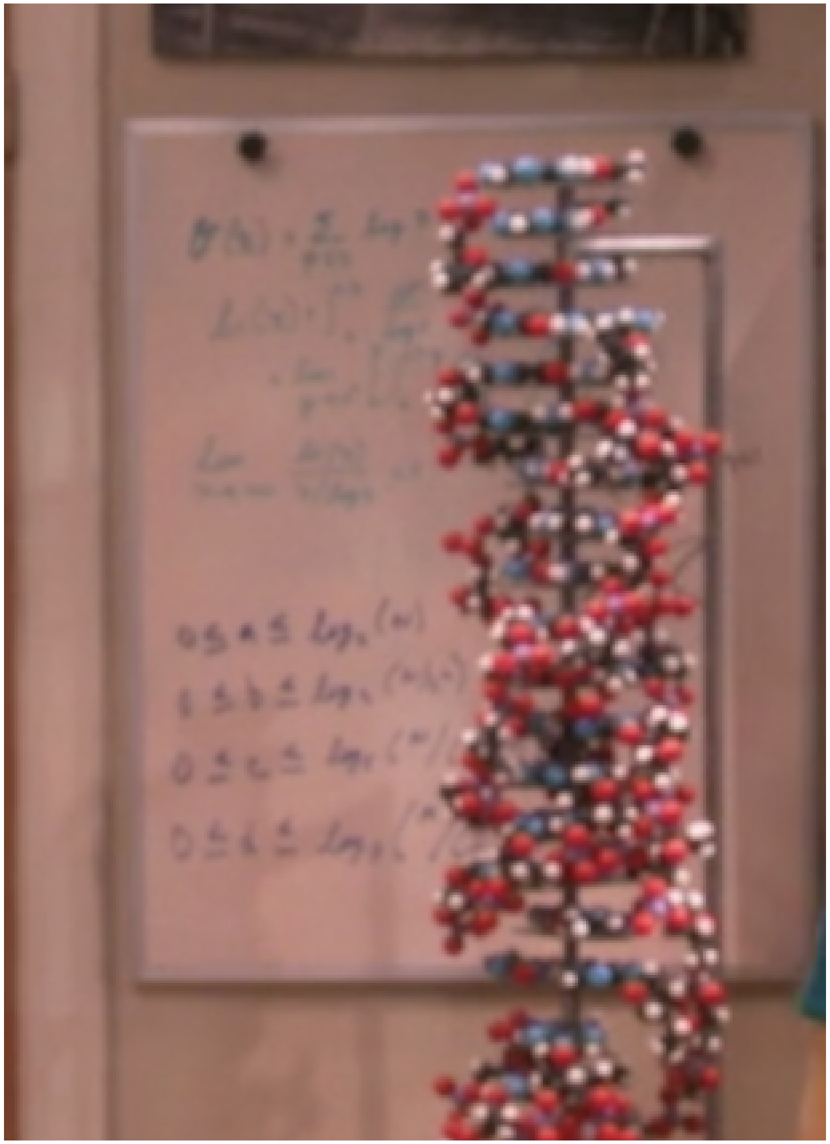
We would like some easily applied filters that let us quickly rule out most candidates, reserving more time-consuming methods for the few remaining numbers.

Here are some quick filters:
(1) The leading digit of p_n must be 1, 3, 7, or 9. (This rules out about 50% of candidates n.)
(2) $100 \nmid n$. (Else, the mirror of n is too short to give a prime being the mirror of p_n; this kills about 75% of the candidates.)
(3) When computing the top few digits of p_n, there cannot be a digit 0 appearing nor an interior digit 1. (Else the product property fails.)

We found some additional filters that ruled out many more candidates, and in the end we were able to prove that 73 is indeed the only number with both the mirror and product properties.

David Saltzberg, a physics professor at UCLA, was the science advisor for The Big Bang Theory. He's the one who does the whiteboards that appear in almost every episode. They are often in the background, not completely in focus, and only fleetingly shown, yet if one looks at them, they often have some interesting scientific or mathematical content.

Here's a sample from the episode that aired on April 18, 2019.




THE AMERICAN MATHEMATICAL
MONTHLY MAA

The Seventy-Ninth William Lowell Putnam Mathematical Competition
Mark Krusemeyer and Daniel H. Ullman 675
Proof of the Sheldon Conjecture
Carl Pomerance and Chris Spicer 688
Consensus and Fragmentation of Opinions with a Focus on Bounded Confidence
Rainer Hegselmann and Ulrich Krause 700
Fourier Transform Inversion Using an Elementary Differential Equation and a Contour Integral
Erik Talvila 717
Perfect Squares as Concatenation of Consecutive Integers
Florian Luca and Pantelimon Stănică 728

NOTES
Integer Triangles of Given Perimeter: A New Approach via Group Theory
James East and Ron Niles 735
A Fresh Look at Highly Composite Numbers
Harold N. Ellingsen, Jr. 740
A Golden Ratio Inequality for Vertex Degrees of Graphs
Fiachra Knox, Bojan Mohar, and David R. Wood 742
On the Probability That an Automorphism Fixes a Group Element
Ashish Goyal, Hemant Kalra, and Deepak Gumber 748
PROBLEMS AND SOLUTIONS 755
REVIEWS
Doing the Scholarship of Teaching and Learning in Mathematics edited by Jacqueline Dewar and Curtis Bennett and *The Scholarship of Teaching and Learning: A Guide for Scientists, Engineers, and Mathematicians* by Jacqueline Dewar, Curtis Bennett, and Matthew Fisher
Priscilla Bremser 764
MATHBITS
699, A Euclidean Algorithm for Integer Matrices; **747**, Count to Six in Six Languages; **754**, A Straightforward Approach to the Product Rule

An Official Publication of the Mathematical Association of America

Proof of the Sheldon Conjecture

Carl Pomerance and Chris Spicer

Abstract. In [3], the authors introduce the concept of a Sheldon prime, based on a conversation between several characters in the CBS television situation comedy *The Big Bang Theory*. The authors of [3] leave open the question of whether 73 is the unique Sheldon prime. This article answers this question in the affirmative.

1. INTRODUCTION. A Sheldon prime was first defined in [3] as an homage to Sheldon Cooper, a fictional theoretical physicist, see Figure 1, on the television show *The Big Bang Theory*, who claimed 73 is the best number because it has some seemingly unusual properties. First note that not only is 73 a prime number, its index in the sequence of primes is the product of its digits, namely 21: it is the 21st prime. In addition, reversing the digits of 73, we obtain the prime 37, which is the 12th prime, and 12 is the reverse of 21.

We give a more formal definition. For a positive integer n, let p_n denote the nth prime number. We say p_n has the *product property* if the product of its base-10 digits is precisely n. For any positive integer x, we define $\mathrm{rev}(x)$ to be the integer whose sequence of base-10 digits is the reverse of the digits of x. For example, $\mathrm{rev}(1234) = 4321$ and $\mathrm{rev}(310) = 13$. We say p_n satisfies the *mirror property* if $\mathrm{rev}(p_n) = p_{\mathrm{rev}(n)}$.

Definition. The prime p_n is a Sheldon prime if it satisfies both the product property and the mirror property.

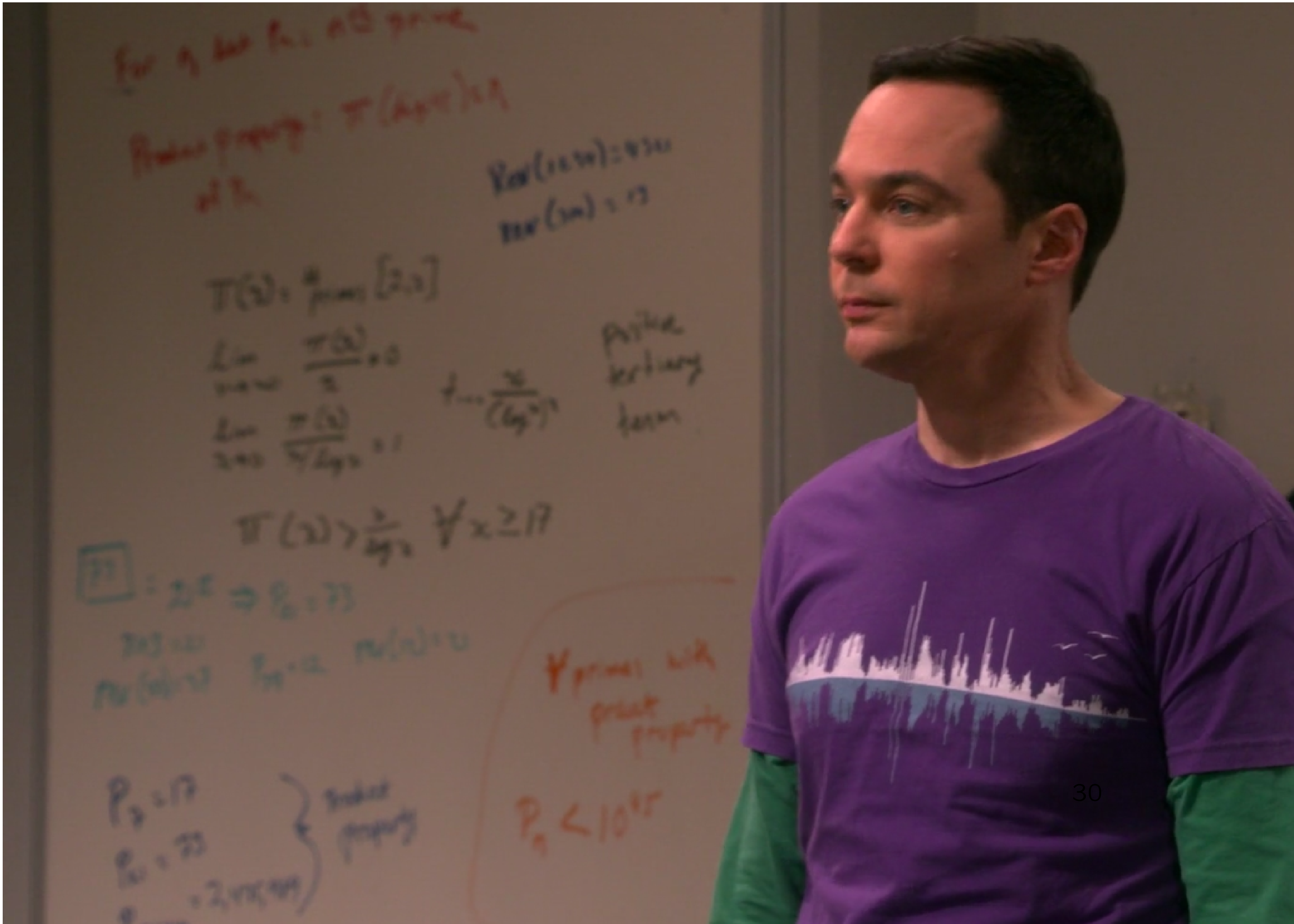
In [3], the "Sheldon Conjecture" was posed that 73 is the only Sheldon prime. In Section 5, we prove the following result.

Theorem 1. *The Sheldon conjecture holds: 73 is the unique Sheldon prime.*

5. PROOF OF THEOREM 1. We first search over any primes less than 10^{19}. By Lemma 7, if $p_n < 10^{19}$ then $n \leq N := 2.341 \times 10^{17}$. So we begin our search by creating a list of all 7-smooth numbers up to N. This is quickly computed by creating a list of numbers of the form $2^a 3^b 5^c 7^d$, with

$$
\begin{aligned}
0 &\leq a \leq \log_2(N),\\
0 &\leq b \leq \log_3(N/2^a),\\
0 &\leq c \leq \log_5(N/(2^a 3^b)),\\
0 &\leq d \leq \log_7(N/(2^a 3^b 5^c)).
\end{aligned}
$$

In particular, there are 57,776 integers of this form. We remove the 7,575 members of the table that are at most 10^{10} since we have previously searched over these numbers n to recover the 3 primes satisfying the product property.



A mystery: Where did the 73 problem come from?

A mystery: Where did the 73 problem come from?

I thought it was Saltzberg, and asked him. But he gave the credit to the episode's writers:

Lee Aronsohn, Jim Reynolds, and Maria Ferrari.

Thank you