COUNTING SOLVABLE $\mathcal{S}$-UNIT EQUATIONS AND LINEAR RECURRENCE SEQUENCES WITH ZEROS

ABSTRACT. We show that only a rather small proportion of linear equations are solvable in elements of a fixed finitely generated subgroup of a multiplicative group of a number field. The argument is based on modular techniques combined with a classical idea of P. Erdős (1935). We then use similar ideas to get a tight upper bound on the number of linear recurrence sequences which attain a zero value.

1. MOTIVATION AND SET-UP

Recently, there has been several works counting soluble (globally or locally) polynomial Diophantine equations in various families, see [1, 4–6, 8, 15–17] and references therein.

Here we address a similar question for families of linear equations in elements of finitely generated groups, which are also known as $\mathcal{S}$-*unit equations*, we refer to [11] for background.

Namely, let $\Gamma \subseteq \mathbb{K}^*$ be a finitely generated multiplicative subgroup of $\mathbb{K}^*$, where $\mathbb{K}$ is a number field of degree $d = [\mathbb{K} : \mathbb{Q}]$ over $\mathbb{Q}$.

We also fix an integral basis $\omega_1, \ldots, \omega_d$ of the ring of integers $\mathbb{Z}_\mathbb{K}$ of $\mathbb{K}$, and for an integer $H \geqslant 0$ we consider the set

$$\mathcal{A}(H) = \{\alpha = u_1\omega_1 + \ldots + u_d\omega_d : \ u_i \in [-H, H] \cap \mathbb{Z}, \ i = 1, \ldots, d\}.$$

Clearly, $\mathcal{A}(H)$ is of cardinality $\#\mathcal{A}(H) = (2H+1)^d$.

Finally, we denote by $Z_k(\Gamma, H)$ the number of k-tuples of coefficients $(\alpha_1, \ldots, \alpha_k) \in \mathcal{A}(H)^k$, such that the equation

$$\alpha_1\vartheta_1 + \ldots + \alpha_k\vartheta_k = 0, \qquad \vartheta_1, \ldots, \vartheta_k \in \Gamma, \tag{1.1}$$

has a solution. Our first main result, Theorem 2.1, estimates $Z_k(\Gamma, H)$ with a power savings.

We note that the question of estimating $Z_k(\Gamma, H)$ is somewhat dual to the scenario of [19] where, for $\mathbb{K} = \mathbb{Q}$ and $k = 3$, the coefficients are fixed but Γ varies among groups generated by r primes in a given interval.

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We also use similar ideas to bound the number of linear recurrence sequences which have a zero in their value set.

Let

$$f(X) = X^k - c_{k-1}X^{k-1} - \ldots - c_0 \in \mathbb{Z}[X], \quad c_0 \neq 0, \tag{1.2}$$

and let $\mathcal{L}_f$ denote the set of all linear recurrence sequences

$$\mathbf{u} = (u(j))_{j=1}^{\infty}$$

with f as the characteristic polynomial, that is, with

$$u(j+k) = c_{k-1}u(j+k-1) + \ldots + c_0u(j), \qquad j = 1, 2, \ldots, \tag{1.3}$$

and integer initial values $u(1), \ldots, u(k)$ not all zero.

If there are no roots of unity among the ratios of distinct roots of its characteristic polynomial f, then all sequences $\mathbf{u} \in \mathcal{L}_f$ are called *non-degenerate*.

By the classical Skolem–Mahler–Lech theorem, any non-degenerate linear recurrence sequence contains only finitely many zeros (see [2] for the strongest known bound). Hence, there is an integer $n_0 > 0$, depending only on $\mathbf{u}$, such that $u(n) \neq 0$ for all $n \geqslant n_0$.

It is also easy to see that "typical" polynomials f correspond to non-degenerate linear recurrence sequences, thus having a zero is a rare event. Our second main result, Theorem 2.3, implies that in fact typically linear recurrence sequences $\mathbf{u} \in \mathcal{L}_f$ (whether degenerate or not) do not have zeros at all.

For $U \geqslant 1$, we give an upper bound on the number $Z_f(U)$ of linear recurrence sequences $\mathbf{u} \in \mathcal{L}_f$ with integer initial values $(u(1), \ldots, u(k)) \in [-U, U]^k$ for which $u(n) = 0$ for some n.

Our approach to bounding $Z_k(\Gamma, H)$ and $Z_f(U)$ is based on a modular technique and also on generating a reasonably dense sequence of integers with small values of the Carmichael λ-function and composed from arbitrary sets of primes of positive relative density, see Lemma 3.1 below. (The Carmichael λ-function at a positive integer n returns the exponent of the group $(\mathbb{Z}/n\mathbb{Z})^*$.)

The argument we use dates back to work of Erdős [9]; it has also been used in various modifications in a number of other works, see, for example, [10].

We also note that in the case of $Z_f(U)$, surprisingly enough, the modular approach gives an essentially tight bound.

2. Main results

We first give an upper bound on $Z_k(\Gamma, H)$ with a power savings.

We always assume that d, k, the subgroup Γ and the characteristic polynomial $f \in \mathbb{Z}[X]$ are fixed. In particular all implied constants and the functions denoted by the o-symbol may depend on them.

Theorem 2.1. *Let $\mathbb{K}$ be a number field of degree $d = [\mathbb{K} : \mathbb{Q}]$ over $\mathbb{Q}$ and let $\Gamma \subseteq \mathbb{K}^*$ be a finitely generated group. Then, as $H \to \infty$,*

$$Z_k(\Gamma, H) \leqslant H^{dk-1+o(1)}.$$

Remark 2.2. Examining the proof of Theorem 2.1 one can notice that similar ideas can allow us to investigate equations with coefficients which are arbitrary algebraic numbers of the form α/β with $\alpha, \beta \in \mathcal{A}(H)$, or of the form α/b with $\alpha \in \mathcal{A}(H)$ and $b \in \{1, \ldots, H\}$.

A variation of the argument used in the proof of Theorem 2.1 also gives the following tight bounds.

Theorem 2.3. *Let $f \in \mathbb{Z}[X]$ be defined by (1.2). If f is separable, then, as $U \to \infty$,*

$$U^{k-1} \leqslant Z_f(U) \leqslant U^{k-1+o(1)}.$$

Remark 2.4. It is easy to see that our argument also applies to inhomogeneous versions of the equations (1.1) with some fixed $\rho \in \mathbb{K}$ on the right hand side and to counting linear recurrence sequences which contain a prescribed value $b \in \mathbb{Z}$ and leads to the same upper bounds (uniformly in ρ and b).

3. Small values of the Carmichael λ-function

We recall that for an integer $n \geqslant 2$, the Carmichael λ-function $\lambda(n)$ is the smallest positive integer m such that $a^m \equiv 1 \pmod{n}$ for all a coprime to n.

We say that a set of primes $\mathcal{P}$ is of relative density δ if

$$\#\left(\mathcal{P} \cap [1, x]\right) \sim \delta\pi(x), \qquad \text{as } x \to \infty,$$

where, as usual, $\pi(x)$ is the number of primes up to x. Let x be large, and let

$$y = \log x / \log\log x, \quad M = \operatorname{lcm}[1, 2, \ldots, \lfloor y \rfloor],$$

so that $M = x^{(1+o(1))/\log\log x}$ as $x \to \infty$. Recall that if $n = p_1 \ldots p_k$ where $p_1, \ldots, p_k$ are distinct primes, then

$$\lambda(n) = \operatorname{lcm}[p_1 - 1, \ldots, p_k - 1].$$

Thus, if each $p_i - 1 \mid M$, then $\lambda(n) \mid M$ and $\lambda(n) \leqslant x^{(1+o(1))/\log\log x}$.

Below, we also allow all constants and o-functions to depend on the real positive parameter ε and the set of primes $\mathcal{P}$.

Lemma 3.1. *Let $\varepsilon > 0$ be arbitrarily small and suppose $\mathcal{P}$ is a set of primes of relative density $\delta > 0$. There is a number x_0 (depending on ε and $\mathcal{P}$) such that if $x > x_0$, there is a squarefree integer $n \in ((1-\varepsilon)x, x]$ composed solely of primes p from $\mathcal{P}$ and such that $p - 1 \mid M$. In particular, $\lambda(n) \leqslant x^{(1+o(1))/\log\log x}$.*

Proof. Let $\mathcal{Q} = \{p \in \mathcal{P} : \ p - 1 \mid M\}$. First note that $\mathcal{P}$ and $\mathcal{Q}$ agree up to y. Thus, if x_0 is large enough (depending on ε and $\mathcal{P}$) and $\log\log x < t < y$, then the number of elements $p \in \mathcal{Q}$ such that $p \leqslant t$ is in the interval $((1-\varepsilon)\delta t/\log t, (1+\varepsilon)\delta t/\log t)$. We first show that this continues for t up to

$$z = \log x \log\log x.$$

Indeed, if $p \in \mathcal{P}\backslash\mathcal{Q}$, then $p-1$ is divisible either by a prime $q > y$ or by a prime power $\ell^j > y$, for a prime ℓ and integer $j \geqslant 2$. The number of primes $p \leqslant t$ satisfying the second condition is at most

$$\sum_{\substack{\ell^j > y \\ \ell \text{ prime} \\ j \geqslant 2}} t/\ell^j \leqslant \sum_{\substack{m^j > y \\ m \in \mathbb{N} \\ j \geqslant 2}} t/m^j \ll t/y^{1/2} = o\left(\pi(t)\right)$$

for $t \leqslant z$.

The same is true for the first condition as we now show. If $q \mid p-1$, write $p - 1 = aq$, so if $p \leqslant t$ and $q > y$, then $a < t/y$. Assume that $y < t \leqslant z$, fix an integer $a < t/y$, and count primes $q \leqslant t/a$ with $aq + 1$ prime. By Brun's sieve, the number of such primes q is $O\left((t/\varphi(a))(\log(t/a))^{-2}\right)$, where $\varphi(a)$ is the Euler function, see, for example, [13, Proposition 6.22] for a much more general and precise statement. Since $y < t \leqslant z$, we have $a \leqslant (\log\log x)^2$ and $\log(t/a) \sim \log t \sim \log\log x$. Since

$$\sum_{a < t/y} 1/\varphi(a) \ll \log\log\log x \sim \log\log t,$$

we have

$$\#\{p \in \mathcal{P}\backslash\mathcal{Q} : \ p \leqslant t\} \ll \pi(t)\log\log t/\log t = o(\pi(t)).$$

Let n_1 be the product of all of the primes in $\mathcal{Q} \cap [1, z]$, so that $\lambda(n_1) \mid M$ and $n_1 \geqslant x^{(1-c_0\varepsilon)\delta\log\log x}$, for some absolute constant c_0. Thus, assuming that ε is small enough, we see that n_1 is quite a bit larger than x. Remove the top primes from n_1 stopping just before removing the next one would drop the number below $x(\log x)^{1/2}$, and denote this number by n_2. Thus, $x(\log x)^{1/2} < n_2 < x(\log x)^{1/2}z$. Let $g = n_2/x$ so that $(\log x)^{1/2} < g < (\log x)^{1/2}z$.

Since $\mathcal{P}$ has a positive relative density in the primes, there are members p_1, p_2 in $\mathcal{P}$ with $p_1 \sim p_2 \sim g^{1/2}$, and in particular, we can take $p_1, p_2 \in \left((1-\varepsilon/2)g^{1/2}, g^{1/2}\right]$. Also, since $g^{1/2} < y$, we have $p_1, p_2 \in \mathcal{Q}$. Since

$$(\log x)^{1/4} < g^{1/2} < (\log x)^{1/4} z^{1/2} < y,$$

we have $p_1 p_2 \mid n_2$. Let $n = n_2/p_1p_2$. Then $n \in ((1-\varepsilon)x, x]$, which completes the proof. □

4. Proof of Theorem 2.1

We fix the basis elements $\omega_1, \ldots, \omega_d$ of $\mathbb{Z}_{\mathbb{K}} = \mathbb{Z}[\omega_1, \ldots, \omega_d]$ and let r be the rank of Γ.

We first observe that, if the prime p splits completely in $\mathbb{K}$, then the residue ring $\mathbb{Z}_{\mathbb{K}}/\mathfrak{P}$ modulo a prime ideal $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ lying over p is isomorphic to the finite field $\mathbb{F}_p$ of p elements. This means that for any $\alpha \in \mathbb{Z}_{\mathbb{K}}$, there is an integer $a_{\mathfrak{P}} \in \mathbb{Z}$ with

$$\alpha \equiv a_{\mathfrak{P}} \pmod{\mathfrak{P}}.$$

Let $\mathcal{P}$ be the set of primes which split completely in $\mathbb{K}$ and also are relatively prime (as ideals in $\mathbb{Z}_{\mathbb{K}}$) to the basis elements $\omega_1, \ldots, \omega_d$ of $\mathbb{Z}_{\mathbb{K}}$ and to the prime ideals appearing in the factorisation of the generators $\gamma_1, \ldots, \gamma_r$ of Γ, seen as fractional ideals in $\mathbb{K}$.

Therefore, for each $p \in \mathcal{P}$ and prime ideal $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ lying over p there are integers $w_{i,\mathfrak{P}} \in \mathbb{Z}$, $i = 1, \ldots, d$, with

$$\omega_i \equiv w_{i,\mathfrak{P}} \pmod{\mathfrak{P}}, \quad i = 1, \ldots, d, \tag{4.1}$$

and the equation (1.1) implies that

$$a_{1,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{1j}} + \ldots + a_{k,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{k,j}} \equiv 0 \pmod{\mathfrak{P}}, \tag{4.2}$$

with some integers s_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, r$, and some integers $a_{i,\mathfrak{P}} \equiv \alpha_i \pmod{\mathfrak{P}}$, $i = 1, \ldots, k$, and integers $g_{j,\mathfrak{P}} \equiv \gamma_j \pmod{\mathfrak{P}}$, $j = 1, \ldots, r$.

Since the left hand side of (4.2) is an integer, this also implies that

$$a_{1,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{1j}} + \ldots + a_{k,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{kj}} \equiv 0 \pmod{p}. \tag{4.3}$$

Since a prime p splits completely in $\mathbb{K}$ if and only if it splits completely in the Galois closure of $\mathbb{K}$, see [18, Corollary, Page 108], by the Chebotarev Density Theorem applied to the Galois closure of $\mathbb{K}$, see [14, Theorem 21.2], the set $\mathcal{P}$ is of positive relative density.

We choose now n as in Lemma 3.1 applied with $x = H$, and since the congruence (4.3) holds for each $p \in \mathcal{P}$, by the Chinese Reminder Theorem we obtain

$$a_1 \prod_{j=1}^{r} g_j^{s_{1j}} + \ldots + a_k \prod_{j=1}^{r} g_j^{s_{kj}} \equiv 0 \pmod{n}, \tag{4.4}$$

for some integers a_i, $i = 1, \ldots, k$, and g_j, $j = 1, \ldots, r$, such that

$$a_i \equiv a_{i,\mathfrak{P}} \pmod{\mathfrak{P}} \qquad \text{and} \qquad g_j \equiv g_{j,\mathfrak{P}} \pmod{\mathfrak{P}}$$

for any prime ideal $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ lying over a prime $p \mid n$.

Hence the integer vector $(a_1, \ldots, a_k)$ satisfies at least one of at most $\lambda(n)^{kr}$ possible nontrivial linear congruences (4.4), and thus takes at most $\lambda(n)^{kr} n^{k-1}$ possible values modulo n.

For a given $(a_1, \ldots, a_k)$ as above we are left to count the number of possibilities $(\alpha_1, \ldots, \alpha_k) \in \mathcal{A}(H)^k$ such that

$$\alpha_i \equiv a_{i,\mathfrak{P}} \pmod{\mathfrak{P}}$$

for all prime ideals $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ dividing n.

Let $\alpha \in \mathcal{A}(H)$, that is, $\alpha = u_1\omega_1 + \ldots + u_d\omega_d$, $u_i \in \mathbb{Z} \cap [-H, H]$, $i = 1, \ldots, d$. Let $\mathfrak{P}$ be a prime ideal of $\mathbb{Z}_{\mathbb{K}}$ lying over a prime $p \in \mathcal{P}$ and let $a_{\mathfrak{P}} \in \mathbb{Z}$ satisfy

$$\alpha \equiv a_{\mathfrak{P}} \pmod{\mathfrak{P}}. \tag{4.5}$$

From (4.5) and recalling the notation (4.1), we obtain

$$u_1 w_{1,\mathfrak{P}} + \ldots + u_d w_{d,\mathfrak{P}} \equiv a_{\mathfrak{P}} \pmod{\mathfrak{P}}.$$

Hence, as above, this congruence holds modulo p and thus modulo n chosen above, that is, we have

$$u_1 w_1 + \ldots + u_d w_d \equiv a_{\mathfrak{P}} \pmod{n},$$

such that $w_i \equiv w_{i,\mathfrak{P}} \pmod{\mathfrak{P}}$, $i = 1, \ldots, d$, and where by our definition of $\mathcal{P}$ we have $\gcd(w_1 \cdots w_d, n) = 1$.

We now see that for $n \leqslant H$ there are $O(H^d/n)$ elements $\alpha \in \mathcal{A}(H)$, which satisfy (4.5).

Therefore, recalling that there are at most $\lambda(n)^{kr} n^{k-1}$ possibilities for $(a_1, \ldots, a_k)$, we obtain

$$Z_k(\Gamma, H) = O\left(\lambda(n)^{kr} n^{k-1} \left(H^d/n\right)^k\right).$$

Since $\lambda(n) = n^{o(1)} = H^{o(1)}$, and by Lemma 3.1, we have $n > (1 - \varepsilon)H$, for $\varepsilon > 0$ arbitrarily small, we conclude the proof.

5. Proof of Theorem 2.3

The lower bound is obvious from considering initial values with, for example, $u(1) = 0$.

To establish the upper bound, we choose the set $\mathcal{P}$ of all primes p, such that $f(X)$ splits completely modulo each $p \in \mathcal{P}$. By the Chebotarev Density Theorem [14, Theorem 21.2] applied to the splitting field of f, the set $\mathcal{P}$ is of relative density $\delta \geqslant 1/k!$.

By removing at most finitely many members of $\mathcal{P}$ we may assume that any $p \in \mathcal{P}$ is relatively prime to $f(0)$ and the discriminant of f. This means that any linear recurrence sequence $\mathbf{u} = (u(j))_{j=1}^{\infty}$ with f as the characteristic polynomial, taken modulo p, is a simple linear recurrence and thus can be written as

$$u(j) \equiv \sum_{\nu=1}^{k} a_{\nu,p} g_{\nu,p}^{j} \pmod{p}, \qquad j = 1, 2, \ldots, \tag{5.1}$$

for some integers $a_{\nu,p}$ and distinct modulo p integers $g_{\nu,p}$ such that $\gcd(g_{\nu,p}, p) = 1$, see [12, Chapter 3] for more details.

We now take n as in Lemma 3.1 applied with $x = U$.

By the Chinese Remainder Theorem, we derive from (5.1) that

$$u(j) \equiv \sum_{\nu=1}^{k} A_\nu G_\nu^j \pmod{n}, \qquad j = 1, 2, \ldots,$$

for some integers A_ν and distinct modulo n integers G_ν such that $\gcd(G_\nu, n) = 1$. Therefore, $u(j)$, $j = 1, 2, \ldots$, is purely periodic modulo n with period

$$t \leqslant \lambda(n). \tag{5.2}$$

To represent $\mathbf{u}$ using the initial values $u(1), \ldots, u(k)$, we define the sequences $\mathbf{w}_i \in \mathcal{L}_f$, $i = 1, \ldots, k$, with initial values

$$w_i(j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}, \qquad j = 1, \ldots, k.$$

It is now obvious that for any $\mathbf{u} \in \mathcal{L}_f$ we have

$$u(j) = \sum_{i=1}^{k} u(i) w_i(j), \qquad j = 1, 2, \ldots. \tag{5.3}$$

Indeed, both the left and the right hand-sides of the equation (5.3) belong to $\mathcal{L}_f$ and have the same initial values; hence they coincide for all j.

In particular (5.3) implies that for any integer $m \geqslant 1$ we have

$$\gcd(w_1(m), \ldots, w_k(m), p) = 1 \tag{5.4}$$

for $p \in \mathcal{P}$. Indeed, writing (5.3) for shifts of say $\mathbf{w}_1$, that is, writing

$$w_1(j+h) = \sum_{i=1}^{k} w_1(i+h) w_i(j), \qquad h = 0, \ldots, k-1,$$

we see that if (5.4) fails then for some m we have

$$p \mid w_1(m+h), \qquad h = 0, \ldots, k-1.$$

Next, the recurrence relation (1.3) implies that $p \mid w_1(m+k)$, and similarly $p \mid w_1(j)$ for all $j \geqslant m$. Recalling that $\mathbf{w}_1$ is periodic, we conclude that $p \mid w_1(1)$, which is a contradiction.

If $\mathbf{u} \in \mathcal{L}_f$ has a zero, then, by periodicity, for some positive integer $j \leqslant t$, the representation (5.3) implies

$$\sum_{i=1}^{k} u(i) w_i(j) \equiv 0 \pmod{n}.$$

Recalling (5.4), we see that, since by our construction $n \leqslant U$, this is possible for at most $O(U^k/n)$ initial values $(u(1), \ldots, u(k)) \in [-U, U]^k$. Hence, by (5.2),

$$Z_f(U) = O\left(tU^k/n\right) = O\left(\lambda(n) U^k/n\right)$$

and since, as before, by Lemma 3.1, we have $n > (1-\varepsilon)U$, for $\varepsilon > 0$ arbitrarily small, we conclude the proof.

6. Comments and open questions

We note that our approach can be used for several other similar problems. For example, given two (not necessary distinct) separable, monic polynomials $f, g \in \mathbb{Z}[X]$ one can ask for how many pairs of linear recurrence sequences $(u, v) \in \mathcal{L}_f \times \mathcal{L}_g$, with all initial values in $[-U, U]$, have their images intersect, that is, $u(j) = v(h)$ for some positive integer j and h. It is easy to see that our argument yields an optimal bound $U^{k+\ell-1+o(1)}$ for the number of such pairs, where $k = \deg f$, $\ell = \deg g$.

On the other hand, our argument does not apply to sequences whose characteristic polynomial has multiple roots. So we ask:

Open Question 6.1. *Obtain a version of Theorem 2.3 for arbitrary monic polynomials $f \in \mathbb{Z}[X]$.*

There are some other interesting counting questions on linear recurrence sequences, which apparently require new arguments. For example, one can ask about squares or arbitrary perfect powers in their value set, the topic which has been actively investigated for a fixed linear recurrence sequence, see [7] and references therein.

Open Question 6.2. *Obtain upper bounds on the number of linear recurrence sequences $\mathbf{u} \in \mathcal{L}_f$ with integer initial values $(u(1), \ldots, u(k)) \in [-U, U]^k$ for which $u(n) = r^2$ for some integers $n \geqslant 1$ and r. Similarly, one can seek upper bounds for sequences as above for which $u(n) = r^\nu$ for some integers $n \geqslant 1$, r and $\nu \geqslant 2$.*

Instead of squares and powers, one can also ask analogues of Question 6.2 for other sparse sequences of arithmetic nature, such as palindromes or Piatetski-Shapiro sequences (that is, sequences of integer parts $\lfloor r^\gamma \rfloor$, $r = 1, 2, \ldots$, with some $\gamma > 1$).

Furthermore, motivated by the results from [3], which show that multiplicative dependence in the values of various classes of linear recurrence sequences is a rare event, we also ask the following:

Open Question 6.3. *For a fixed integer $s \geqslant 2$, obtain an upper bound on the number of linear recurrence sequences $\mathbf{u} \in \mathcal{L}_f$ with integer initial values $(u(1), \ldots, u(k)) \in [-U, U]^k$ for which*

$$u(n_1)^{e_1} \ldots u(n_s)^{e_s} = 1$$

for some integers $1 \leqslant n_1 < \ldots < n_s$ and a non-zero vector $(e_1, \ldots, e_s) \in \mathbb{Z}^s$.

Finally, we note that the bound on $Z_f(U)$ in Theorem 2.3 is not uniform with respect to f. Hence, it is natural to ask what happens if we vary the polynomial as well.

Open Question 6.4. *Obtain nontrivial bounds on the number of linear recurrence sequences $\mathbf{u} \in \mathcal{L}_f$ with integer initial values*

$$(u(1), \ldots, u(d)) \in [-U, U]^d$$

and characteristic polynomials

$$f(X) = X^d - c_{d-1}X^{d-1} - \ldots - c_0 \in \mathbb{Z}[X], \quad (c_0, \ldots, c_{d-1}) \in [-H, H]^d,$$

for which $u(n) = 0$ for some n.

One can also blend Questions 6.2 and 6.3 with the setting of Question 6.4.

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