COUNTING SOLVABLE $\mathcal{S}$-UNIT EQUATIONS AND LINEAR RECURRENCE SEQUENCES WITH ZEROS

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ABSTRACT. We show that only a rather small proportion of linear equations are solvable in elements of a fixed finitely generated subgroup of a multiplicative group of a number field. The argument is based on modular techniques combined with a classical idea of P. Erdős (1935). We then use similar ideas to get a tight upper bound on the number of linear recurrence sequences which attain a zero value.

1. MOTIVATION AND SET-UP

Recently, there has been several works counting soluble (globally or locally) polynomial Diophantine equations in various families, see [1, 3–6, 13–15] and references therein.

Here we address a similar question for families of linear equations in elements of finitely generated groups, which are also known as $\mathcal{S}$-*unit equations*, we refer to [9] for background.

Namely, let $\Gamma \subseteq \mathbb{K}^*$ be a finitely generated multiplicative subgroup of $\mathbb{K}^*$, where $\mathbb{K}$ is a number field of degree $d = [\mathbb{K} : \mathbb{Q}]$ over $\mathbb{Q}$.

We also fix an integral basis $\omega_1, \ldots, \omega_d$ of the ring of integers $\mathbb{Z}_\mathbb{K}$ of $\mathbb{K}$, and for an integer $H \geqslant 0$ we consider the set

$$\mathcal{A}(H) = \{\alpha = u_1\omega_1 + \ldots + u_d\omega_d : \ u_i \in [-H, H] \cap \mathbb{Z}, \ i = 1, \ldots, d\}.$$

Clearly, $\mathcal{A}(H)$ is of cardinality $\#\mathcal{A}(H) = (2H+1)^d$.

Finally, we denote by $Z_k(\Gamma, H)$ the number of k-tuples of coefficients $(\alpha_1, \ldots, \alpha_k) \in \mathcal{A}(H)^k$, such that the equation

$$\alpha_1\vartheta_1 + \ldots + \alpha_k\vartheta_k = 0, \qquad \vartheta_1, \ldots, \vartheta_k \in \Gamma, \tag{1.1}$$

has a solution. Our first main result, Theorem 2.1, estimates $Z_k(\Gamma, H)$ with a power savings.

We note that the question of estimating $Z_k(\Gamma, H)$ is somewhat dual to the scenario of [17] where, for $\mathbb{K} = \mathbb{Q}$ and $k = 3$, the coefficients

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are fixed but Γ varies among groups generated by r primes in a given interval.

We also use similar ideas to bound the number of linear recurrence sequences, which have a zero in their value set.

Let

$$f(X) = X^k - c_{k-1}X^{k-1} - \ldots - c_0 \in \mathbb{Z}[X], \quad c_0 \neq 0, \tag{1.2}$$

and let $\mathcal{L}_f$ denote the set of all linear recurrence sequences

$$\mathbf{u} = (u(j))_{j=1}^{\infty}$$

with the characteristic polynomial f, that is, with

$$u(j+k) = c_{k-1}u(j+k-1) + \ldots + c_0 u(j), \qquad j = 1, 2, \ldots, \tag{1.3}$$

and integer initial values $u(1), \ldots, u(k)$ not all zero.

If there are no roots of unity among the ratios of distinct roots of its characteristic polynomial f, then all sequences $\mathbf{u} \in \mathcal{L}_f$ are called *non-degenerate*.

By the classical Skolem–Mahler–Lech theorem, any non-degenerate linear recurrence sequence contains only finitely many zeros (see [2] for the strongest known bound). Hence, there is an integer $n_0 > 0$, depending only on $\mathbf{u}$, such that $u(n) \neq 0$ for all $n \geqslant n_0$.

It is also easy to see that "typical" polynomials f correspond to non-degenerate linear recurrence sequences, thus having a zero is a rare event. Our second main result, Theorem 2.3, implies that in fact typically linear recurrence sequences $\mathbf{u} \in \mathcal{L}_f$ (whether degenerate or not) do not have zeros at all.

For $U \geqslant 1$, we give an upper bound on the number $Z_f(U)$ of linear recurrence sequences $\mathbf{u} \in \mathcal{L}_f$ with integer initial values $(u(1), \ldots, u(k)) \in [-U, U]^k$ for which $u(n) = 0$ for some n.

Our approach to bounding $Z_k(\Gamma, H)$ and $Z_f(U)$ is based on a modular technique and also on generating a reasonably dense sequence of integers with small values of the Carmichael λ-function and composed from arbitrary sets of primes of positive relative density, see Lemma 3.1 below. (The Carmichael λ-function at a positive integer n returns the exponent of the group $(\mathbb{Z}/n\mathbb{Z})^*$.)

The argument we use dates back to work of Erdős [7]; it has also been used in various modifications in a number of other works, see, for example, [8].

We also note that in the case of $Z_f(U)$, surprisingly enough, the modular approach gives an essentially tight bound.

2. Main results

We first give an upper bound on $Z_k(\Gamma, H)$ with a power savings.

We always assume that d, k, the subgroup Γ and the characteristic polynomial $f \in \mathbb{Z}[X]$ are fixed. In particular all implied constants and the functions denoted by the o-symbol may depend on them.

Theorem 2.1. *Let $\mathbb{K}$ be a number field of degree $d = [\mathbb{K} : \mathbb{Q}]$ over $\mathbb{Q}$ and let $\Gamma \subseteq \mathbb{K}^*$ be a finitely generated group. Then, as $H \to \infty$,*

$$Z_k(\Gamma, H) \leqslant H^{dk-1+o(1)}.$$

Remark 2.2. Examining the proof of Theorem 2.1 one can notice that similar ideas can allow us to investigate equations with coefficients which are arbitrary algebraic numbers of the form α/β with $\alpha, \beta \in \mathcal{A}(H)$, or of the form α/b with $\alpha \in \mathcal{A}(H)$ and $b \in \{1, \ldots, H\}$.

A variation of the argument used in the proof of Theorem 2.1 also gives the following tight bounds.

Theorem 2.3. *Let $f \in \mathbb{Z}[X]$ be defined by (1.2). If f is separable, then, as $U \to \infty$,*

$$U^{k-1} \leqslant Z_f(U) \leqslant U^{k-1+o(1)}.$$

Remark 2.4. It is easy to see that our argument also applies to inhomogeneous versions of the equations (1.1) with some fixed $\rho \in \mathbb{K}$ on the right hand side and to counting linear recurrence sequences which contain a prescribed value $b \in \mathbb{Z}$ and leads to the same upper bounds (uniformly in ρ and b).

3. Small values of the Carmichael λ-function

We recall that for an integer $n \geqslant 2$, the Carmichael λ-function $\lambda(n)$ is the smallest positive integer m such that $a^m \equiv 1 \pmod{n}$ for all a coprime to n.

We say that a set of primes $\mathcal{P}$ is of relative density δ if

$$\#\left(\mathcal{P} \cap [1, x]\right) \sim \delta\pi(x), \qquad \text{as } x \to \infty,$$

where, as usual, $\pi(x)$ is the number of primes up to x. Let x be large, and let

$$y = \log x / \log\log x, \quad M = \mathrm{lcm}[1, 2, \ldots, \lfloor y \rfloor],$$

so that $M = x^{(1+o(1))/\log\log x}$ as $x \to \infty$. Recall that if $n = p_1 \ldots p_k$ where $p_1, \ldots, p_k$ are distinct primes, then

$$\lambda(n) = \mathrm{lcm}[p_1 - 1, \ldots, p_k - 1].$$

Thus, if each $p_i - 1 \mid M$, then $\lambda(n) \mid M$ and $\lambda(n) \leqslant x^{(1+o(1))/\log\log x}$.

Below, we also allow all constants and o-functions to depend on the real positive parameter ε and the set of primes $\mathcal{P}$.

Lemma 3.1. *Let $\varepsilon > 0$ be arbitrarily small and suppose $\mathcal{P}$ is a set of primes of relative density $\delta > 0$. There is a number x_0 (depending on ε and $\mathcal{P}$) such that if $x > x_0$, there is a squarefree integer $n \in ((1-\varepsilon)x, x]$ composed solely of primes p from $\mathcal{P}$ and such that $p - 1 \mid M$. In particular, $\lambda(n) \leqslant x^{(1+o(1))/\log\log x}$.*

Proof. Let $\mathcal{Q} = \{p \in \mathcal{P} : \ p - 1 \mid M\}$. First note that $\mathcal{P}$ and $\mathcal{Q}$ agree up to y. Thus, if x_0 is large enough (depending on ε and $\mathcal{P}$) and $\log\log x < t < y$, then the number of elements $p \in \mathcal{Q}$ such that $p \leqslant t$ is in the interval $((1-\varepsilon)\delta t/\log t, (1+\varepsilon)\delta t/\log t)$. We first show that this continues for t up to

$$z = \log x \log\log x.$$

Indeed, if $p \in \mathcal{P}\backslash\mathcal{Q}$, then $p - 1$ is divisible either by a prime $q > y$ or by a prime power $\ell^j > y$, for a prime ℓ and integer $j \geqslant 2$. The number of primes $p \leqslant t$ satisfying the second condition is at most

$$\sum_{\substack{\ell^j > y \\ \ell \text{ prime} \\ j \geqslant 2}} t/\ell^j \leqslant \sum_{\substack{m^j > y \\ m \in \mathbb{N} \\ j \geqslant 2}} t/m^j \ll t/y^{1/2} = o\left(\pi(t)\right)$$

for $t \leqslant z$.

The same is true for the first condition as we now show. If $q \mid p-1$, write $p - 1 = aq$, so if $p \leqslant t$ and $q > y$, then $a < t/y$. Assume that $y < t \leqslant z$, fix an integer $a < t/y$, and count primes $q \leqslant t/a$ with $aq + 1$ prime. By Brun's sieve, the number of such primes q is $O\left((t/\varphi(a))(\log(t/a))^{-2}\right)$, where $\varphi(a)$ is the Euler function, see, for example, [11, Proposition 6.22] for a much more general and precise statement. Since $y < t \leqslant z$, we have $a \leqslant (\log\log x)^2$ and $\log(t/a) \sim \log t \sim \log\log x$. Since

$$\sum_{a < t/y} 1/\varphi(a) \ll \log\log\log x \sim \log\log t,$$

we have

$$\#\{p \in \mathcal{P}\backslash\mathcal{Q} : \ p \leqslant t\} \ll \pi(t)\log\log t/\log t = o(\pi(t)).$$

Let n_1 be the product of all of the primes in $\mathcal{Q} \cap [1, z]$, so that $\lambda(n_1) \mid M$ and $n_1 \geqslant x^{(1-c_0\varepsilon)\delta\log\log x}$, for some absolute constant c_0. Thus, assuming that ε is small enough, we see that n_1 is quite a bit larger than x. Remove the top primes from n_1 stopping just before removing the next one would drop the number below $x(\log x)^{1/2}$, and denote this number by n_2. Thus, $x(\log x)^{1/2} < n_2 < x(\log x)^{1/2} z$. Let $g = n_2/x$ so that $(\log x)^{1/2} < g < (\log x)^{1/2} z$.

Since $\mathcal{P}$ has a positive relative density in the primes, there are members p_1, p_2 in $\mathcal{P}$ with $p_1 \sim p_2 \sim g^{1/2}$, and in particular, we can take $p_1, p_2 \in \big((1-\varepsilon/2)g^{1/2}, g^{1/2}\big]$. Also, since $g^{1/2} < y$, we have $p_1, p_2 \in \mathcal{Q}$. Since

$$(\log x)^{1/4} < g^{1/2} < (\log x)^{1/4} z^{1/2} < y,$$

we have $p_1 p_2 \mid n_2$. Let $n = n_2/p_1 p_2$. Then $n \in ((1-\varepsilon)x, x]$, which completes the proof. □

4. Proof of Theorem 2.1

We fix the basis elements $\omega_1, \ldots, \omega_d$ of $\mathbb{Z}_{\mathbb{K}} = \mathbb{Z}[\omega_1, \ldots, \omega_d]$ and let r be the rank of Γ.

We first observe that if the prime p splits completely in $\mathbb{K}$ then the residue ring $\mathbb{Z}_{\mathbb{K}}/\mathfrak{P}$ modulo a prime ideal $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ lying over p is isomorphic to the finite field $\mathbb{F}_p$ of p elements. This means that for any $\alpha \in \mathbb{Z}_{\mathbb{K}}$, there is an integer $a_{\mathfrak{P}} \in \mathbb{Z}$ with

$$\alpha \equiv a_{\mathfrak{P}} \pmod{\mathfrak{P}}.$$

Let $\mathcal{P}$ be the set of primes which split completely in $\mathbb{K}$ and also are relatively prime (as ideals in $\mathbb{Z}_{\mathbb{K}}$) to the basis elements $\omega_1, \ldots, \omega_d$ of $\mathbb{Z}_{\mathbb{K}}$ and to the prime ideals appearing in the factorisation of the generators $\gamma_1, \ldots, \gamma_r$ of Γ, seen as fractional ideals in $\mathbb{K}$.

Therefore, for each $p \in \mathcal{P}$ and prime ideal $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ lying over p there are integers $w_{i,\mathfrak{P}} \in \mathbb{Z}$, $i = 1, \ldots, d$, with

$$\omega_i \equiv w_{i,\mathfrak{P}} \pmod{\mathfrak{P}}, \quad i = 1, \ldots, d, \tag{4.1}$$

and the equation (1.1) implies that

$$a_{1,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{1j}} + \ldots + a_{k,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{k,j}} \equiv 0 \pmod{\mathfrak{P}}, \tag{4.2}$$

with some integers s_{ij}, $i = 1, \ldots, k$, $j = 1, \ldots, r$, and some integers $a_{i,\mathfrak{P}} \equiv \alpha_i \pmod{\mathfrak{P}}$, $i = 1, \ldots, k$, and integers $g_{j,\mathfrak{P}} \equiv \gamma_j \pmod{\mathfrak{P}}$, $j = 1, \ldots, r$.

Since the left hand side of (4.2) is an integer, this also implies that

$$a_{1,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{1j}} + \ldots + a_{k,\mathfrak{P}} \prod_{j=1}^{r} g_{j,\mathfrak{P}}^{s_{kj}} \equiv 0 \pmod{p}. \tag{4.3}$$

Since a prime p splits completely in $\mathbb{K}$ if and only if it splits completely in the Galois closure of $\mathbb{K}$, see [16, Corollary, Page 108], by the Chebotarev Density Theorem applied to the Galois closure of $\mathbb{K}$, see [12, Theorem 21.2], the set $\mathcal{P}$ is of positive relative density.

We choose now n as in Lemma 3.1 applied with $x = H$, and since the congruence (4.3) holds for each $p \in \mathcal{P}$, by the Chinese Reminder Theorem we obtain

$$a_1 \prod_{j=1}^{r} g_j^{s_{1j}} + \ldots + a_k \prod_{j=1}^{r} g_j^{s_{kj}} \equiv 0 \pmod{n}, \tag{4.4}$$

for some integers a_i, $i = 1, \ldots, k$, and g_j, $j = 1, \ldots, r$, such that

$$a_i \equiv a_{i,\mathfrak{P}} \pmod{\mathfrak{P}} \qquad \text{and} \qquad g_j \equiv g_{j,\mathfrak{P}} \pmod{\mathfrak{P}}$$

for any prime ideal $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ lying over a prime $p \mid n$.

Hence the integer vector $(a_1, \ldots, a_k)$ satisfies at least one of at most $\lambda(n)^{kr}$ possible nontrivial linear congruences (4.4), and thus takes at most $\lambda(n)^{kr} n^{k-1}$ possible values modulo n.

For a given $(a_1, \ldots, a_k)$ as above we are left to count the number of possibilities $(\alpha_1, \ldots, \alpha_k) \in \mathcal{A}(H)^k$ such that

$$\alpha_i \equiv a_{i,\mathfrak{P}} \pmod{\mathfrak{P}}$$

for all prime ideals $\mathfrak{P}$ of $\mathbb{Z}_{\mathbb{K}}$ dividing n.

Let $\alpha \in \mathcal{A}(H)$, that is, $\alpha = u_1\omega_1 + \ldots + u_d\omega_d$, $u_i \in \mathbb{Z} \cap [-H, H]$, $i = 1, \ldots, d$. Let $\mathfrak{P}$ be a prime ideal of $\mathbb{Z}_{\mathbb{K}}$ lying over a prime $p \in \mathcal{P}$ and let $a_{\mathfrak{P}} \in \mathbb{Z}$ satisfy

$$\alpha \equiv a_{\mathfrak{P}} \pmod{\mathfrak{P}}. \tag{4.5}$$

From (4.5) and recalling the notation (4.1), we obtain

$$u_1 w_{1,\mathfrak{P}} + \ldots + u_d w_{d,\mathfrak{P}} \equiv a_{\mathfrak{P}} \pmod{\mathfrak{P}}.$$

Hence, as above, this congruence holds modulo p and thus modulo n chosen above, that is, we have

$$u_1 w_1 + \ldots + u_d w_d \equiv a_{\mathfrak{P}} \pmod{n},$$

such that $w_i \equiv w_{i,\mathfrak{P}} \pmod{\mathfrak{P}}$, $i = 1, \ldots, d$, and where by our definition of $\mathcal{P}$ we have $\gcd(w_1 \cdots w_d, n) = 1$.

We now see that for $n \leqslant H$ there are $O(H^d/n)$ elements $\alpha \in \mathcal{A}(H)$, which satisfy (4.5).

Therefore, recalling that there are at most $\lambda(n)^{kr} n^{k-1}$ possibilities for $(a_1, \ldots, a_k)$, we obtain

$$Z_k(\Gamma, H) = O\left(\lambda(n)^{kr} n^{k-1} \left(H^d/n\right)^k\right).$$

Since $\lambda(n) = n^{o(1)} = H^{o(1)}$, and by Lemma 3.1, we have $n > (1 - \varepsilon)H$, for $\varepsilon > 0$ arbitrarily small, we conclude the proof.

5. Proof of Theorem 2.3

The lower bound is obvious from considering initial values with, for example, $u(1) = 0$.

To establish the upper bound, we choose the set $\mathcal{P}$ of all primes p, such that $f(X)$ splits completely modulo each $p \in \mathcal{P}$. By the Chebotarev Density Theorem [12, Theorem 21.2] applied to the splitting field of f, the set $\mathcal{P}$ is of relative density $\delta \geqslant 1/k!$.

By removing at most finitely many members of $\mathcal{P}$ we may assume that any $p \in \mathcal{P}$ is relatively prime to $f(0)$ and the discriminant of f. This means that any linear recurrence sequence $\mathbf{u} = (u(j))_{j=1}^{\infty}$ with the characteristic polynomial f, taken modulo p, is a simple linear recurrence and thus can be written as

$$u(j) \equiv \sum_{\nu=1}^{k} a_{\nu,p} g_{\nu,p}^{j} \pmod p, \qquad j = 1, 2, \ldots, \tag{5.1}$$

for some integers $a_{\nu,p}$ and distinct modulo p integers $g_{\nu,p}$ such that $\gcd(g_{\nu,p}, p) = 1$, see [10, Chapter 3] for more details.

We now take n as in Lemma 3.1 applied with $x = U$.

By the Chinese Remainder Theorem, we derive from (5.1) that

$$u(j) \equiv \sum_{\nu=1}^{k} A_{\nu} G_{\nu}^{j} \pmod n, \qquad j = 1, 2, \ldots,$$

for some integers A_ν and distinct modulo n integers G_ν such that $\gcd(G_\nu, n) = 1$. Therefore, $u(j)$, $j = 1, 2, \ldots$, is purely periodic modulo n with period

$$t \leqslant \lambda(n). \tag{5.2}$$

To represent $\mathbf{u}$ using the initial values $u(1), \ldots, u(k)$, we define the sequences $\mathbf{w}_i \in \mathcal{L}_f$, $i = 1, \ldots, k$, with initial values

$$w_i(j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}, \qquad j = 1, \ldots, k.$$

It is now obvious that for any $\mathbf{u} \in \mathcal{L}_f$ we have

$$u(j) = \sum_{i=1}^{k} u(i) w_i(j), \qquad j = 1, 2, \ldots. \tag{5.3}$$

Indeed, both the left and the right hand-sides of the equation (5.3) belong to $\mathcal{L}_f$ and have the same initial values; hence they coincide for all j.

In particular (5.3) implies that for any integer $m \geqslant 1$ we have

$$\gcd\left(w_1(m), \ldots, w_k(m), p\right) = 1 \tag{5.4}$$

for $p \in \mathcal{P}$. Indeed, writing (5.3) for shifts of say $\mathbf{w}_1$, that is, writing

$$w_1(j+h) = \sum_{i=1}^{k} w_1(i+h) w_i(j), \qquad h = 0, \ldots, k-1,$$

we see that if (5.4) fails then for some m we have

$$p \mid w_1(m+h), \qquad h = 0, \ldots, k-1.$$

Next, the recurrence relation (1.3) implies that $p \mid w_1(m+k)$, and similarly $p \mid w_1(j)$ for all $j \geqslant m$. Recalling that $\mathbf{w}_1$ is periodic, we conclude that $p \mid w_1(1)$, which is a contradiction.

If $\mathbf{u} \in \mathcal{L}_f$ has a zero, then, by periodicity, for some positive integer $j \leqslant t$, the representation (5.3) implies

$$\sum_{i=1}^{k} u(i) w_i(j) \equiv 0 \pmod{n}.$$

Recalling (5.4), we see that, since by our construction $n \leqslant U$, this is possible for at most $O(U^k/n)$ initial values $(u(1), \ldots, u(k)) \in [-U, U]^k$. Hence, by (5.2),

$$Z_f(U) = O\left(tU^k/n\right) = O\left(\lambda(n) U^k/n\right)$$

and since, as before, by Lemma 3.1, we have $n > (1-\varepsilon)U$, for $\varepsilon > 0$ arbitrarily small, we conclude the proof.

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