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SETS WITH PRESCRIBED ARITHMETIC DENSITIES

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ABSTRACT. Using concepts of generalized asymptotic and logarithmic densities based on weighted arithmetic means over an arithmetical semigroup $\mathbb{G}$ we prove that under some additional technical assumptions on the weighted counting function of its elements, a subset of $\mathbb{G}$ exists with all four generalized densities (upper and lower asymptotic and logarithmic) prescribed subject to the natural condition $0 \le \underline{d}(A) \le \underline{\ell}(A) \le \overline{\ell}(A) \le \overline{d}(A) \le 1$.

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Introduction

The content of the underlying paper grew out from several motivations. A simple consequence of the Prime Number Theorem (cf. [19, p. 155]) is that the set of all rational numbers p/q, where p, q are primes, is dense in $(0, \infty)$. Led by the concepts of difference and multiplicative bases [12] of the set of positive integers $\mathbb{N}$, Šalát defined [16, 17] the concept of a ratio set. The **ratio set** of a subset A of the set positive real numbers is defined by $R(A) = \{a/b : a, b \in A\}$. The set A is called (R)**-dense** if $R(A)$ is (topologically) dense in the set of positive real numbers $\mathbb{R}^+$. A natural question is to ask what arithmetic density relations imposed on $A \subset \mathbb{N}$ imply its (R)-denseness.

Motivated by the relationship between difference and multiplicative bases and their density properties [12, p. 24, 177–182], Šalát proved that if the asymptotic

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density $d(A)$ of $A \subset \mathbb{N}$ is positive, then A is an (R)-dense set. In the same paper, he also showed that only the condition that the lower asymptotic density $\underline{d}(A)$ is positive does not guarantee the (R)-density of A. Furthermore, he proved that if the upper asymptotic density $\overline{d}(A)$ of A equals 1, then A is an (R)-dense set, but on the other hand, for each $\varepsilon > 0$ there exists a set $A \subset \mathbb{N}$ such that its upper asymptotic density satisfies $\overline{d}(A) > 1 - \varepsilon$, yet A is not (R)-dense.

On a related note, we note that a sequence A of positive integers having positive lower asymptotic density, say d, does not necessarily contain a subsequence $B \subset A$ with asymptotic density d. The set of pairs $(\underline{d}(B), \overline{d}(B))$, where B runs over all subsequences of A, is described in [5, 6]. To construct examples in which such subsets exist, one may also use results from the theory of distribution functions ([20]).

Strauch and Tóth [21] refined Šalát's results by showing that if $\underline{d}(A) + \overline{d}(A) \geq 1$ (and, in particular, if $\underline{d}(A) \geq 1/2$), then A is (R)-dense. They also showed that for every $t \in (0, 1/2)$ there exists an $A \subset \mathbb{N}$ with $\underline{d}(A) = t$ such that A is not (R)-dense. They also proved that for every couple of real numbers α and β with $0 \leq \alpha \leq \beta \leq 1$ there exists an (R)-dense set such that

$$\underline{d}(A) = \alpha \quad \text{and} \quad \overline{d}(A) = \beta. \tag{1}$$

Mišík [11] showed that we can prescribe not only the values (1) but simultaneously also the values of the lower and upper logarithmic densities $\underline{\ell}(A)$ and $\overline{\ell}(A)$ as well. Since we always have

$$0 \leq \underline{d}(A) \leq \underline{\ell}(A) \leq \overline{\ell}(A) \leq \overline{d}(A) \leq 1, \tag{2}$$

he proved that given any quadruple $\alpha,\ \beta,\ \gamma,\ \delta$ of real numbers such that $0 \leq \alpha \leq \beta \leq \gamma \leq \delta \leq 1$, there exists a set $A \subseteq \mathbb{N}$ with $\underline{d}(A) = \alpha$, $\underline{\ell}(A) = \beta$, $\overline{\ell}(A) = \gamma$, and $\overline{d}(A) = \delta$ (see [11, Theorem 2]. In [10], a constructive proof of this result was given).

In another direction, let $d(S)$ and $\sigma(S)$ denote the asymptotic density (provided that it exists), and the Schnirelmann density, respectively, of the set of natural numbers not divisible by any element of a set S of natural numbers, and let $D(S) = d(S) - \sigma(S) \geq 0$. In [4], it is proved, among other results, that for finite subsets S of the set $\mathcal{P}$ of all primes the following hold:

(1) $\sup\{D(S) : S \subset \mathcal{P},\ S \text{ finite}\} = 1$;

(2) there exists S' with $S \subset S' \subset \mathcal{P}$ such that $\sigma(S') = \sigma(S)$ and $D(S') = 0$.

Mišík [11] also extended the validity of the inequality (1) in another direction, namely in a framework of generalized asymptotic densities defined through weighted-means. The weighted means approach to a density concept was for the first time used by van der Corput [2, p. 202] in connection with his investigations concerning the properties of the Schnirelmann density. This natural generalization of the density concept was later (probably independently) used by Alexander [1, Section 1] to model the relation between the asymptotic density and the logarithmic density with the aim to generalize some known results on primitive sequences. In [14, 15], it was shown that using the weighted arithmetic means approach to the density concept, a theory can be developed in which relations of type (2) remain true for a wider class of arithmetic densities (see, for instance, Lemma 8 below).

In the present paper, we show that Mišík's result related to (2) can be extended to a class of arithmetical densities even in a more general setting of the so-called arithmetical semigroups.

This paper is organized as follows. In the first section, we review the notion of an arithmetic semigroup. In the second section, we review the concept of a generalized asymptotic density and some of its properties, and we show following Strauch, Tóth and Mišík's ideas, that if the generalized asymptotic densities satisfy two mild natural conditions, then the semigroup contains (R)-dense subsets of asymptotic density zero. Consequently, we can concentrate ourselves merely to conditions of type (1) and (2). While to extend (1) requires only a technical adaptation of Mišík's ideas, the generalization of (2) is based on completely different ideas. To do this, in the third section, we first review the generalized logarithmic density. Finally, in the last section we prove our main result, namely that under some additional technical assumptions every arithmetical semigroup contains subsets having all four generalized densities (upper and lower asymptotic and logarithmic, respectively) prescribed. This is achieved by reducing the problem to the construction of suitable sequences of real numbers in the interval $(0, 1)$.

1. Arithmetical semigroups

Let $\mathbb{G}$ denote a free commutative semigroup relative to a multiplication operation denoted by juxtaposition, with identity element $1_{\mathbb{G}}$, and with at most countably many generators. Such a semigroup will be called **arithmetical** if in addition a real-valued **norm** $|\cdot|$ is defined on $\mathbb{G}$ such that

1. $|1_{\mathbb{G}}| = 1, |a| > 1$ for all $a \in \mathbb{G} \setminus \{1_{\mathbb{G}}\}$,
2. $|ab| = |a|.|b|$ for all $a, b \in \mathbb{G}$,
3. the total number
$$N_{\mathbb{G}}(x) = \sum_{\substack{|a| \le x \\ a \in \mathbb{G}}} 1$$
of elements $a \in \mathbb{G}$ of norm not exceeding x is finite for each real x.

We shall denote by $\mathcal{P}_{\mathbb{G}}$ the set of generators of $\mathbb{G}$ and its elements will be called **primes** of $\mathbb{G}$. In applications, a significant rôle is played by the arithmetical semigroups satisfying the so-called

Axiom A. *There exist positive constants A and δ and a constant η with $0 \le \eta < \delta$, such that*
$$N_{\mathbb{G}}(x) = Ax^{\delta} + \mathcal{O}(x^{\eta}).$$

More details on the abstract axiomatic approach to some arithmetical problems via the notion of arithmetical semigroup, and especially the theory of arithmetical semigroups satisfying Axiom A, can be found in [8, 9], where the interested reader can find many interesting and non-trivial instances of arithmetical semigroups satisfying Axiom A.

A more general class of arithmetical semigroups are the so called **δ-regular** semigroups whose definition is based on the notion of δ-regularly varying function (see [22]).

In most concrete applications, the norm mapping represents the size, or dimension, of the elements under consideration, and usually attains integral values. Moreover, the norm is often a single integer-valued function, a restriction which we shall not adopt here. Given an arithmetical semigroup $\mathbb{G}$, let
$$|\mathbb{G}| = \mathrm{Im}\{|\cdot| : \mathbb{G} \to \mathbb{R}\}$$
be its "shadow" image in the reals. The norm function on $|\mathbb{G}|$ is then the identity mapping.

In what follows, we shall be interested to extend some results proved for $\mathbb{G} = \mathbb{N}$. In this case the fact that the set of positive rationals $\mathbb{Q}^+ = R(\mathbb{N})$ is (topologically) dense in $\mathbb{R}^+$ played a significant rôle. To find some characterizations ensuring that $R(|\mathbb{G}|)$ is also dense in $\mathbb{R}^+$ for some arithmetical semigroups $\mathbb{G}$, we shall need the following result (for a proof, apply the exponential function to the corresponding result [3, p. 25] for $(\mathbb{R}, +)$):

Lemma 1. *Let S be a multiplicative subgroup of $\mathbb{R}^+ = (0, \infty)$. Then S is dense in $\mathbb{R}^+$, unless S is cyclic.*

Corollary 2. *Let A be any set of real numbers all ≥ 1 such that $A(x) = \#\{a \in A : a \leq x\}$ is finite for each $x > 1$. If $A(x)/\ln x$ is unbounded as x tends to infinity, then the set of all ratios of all products of elements from A is dense in $\mathbb{R}^+$.*

P r o o f. Due to Lemma 1, the only obstruction to density is that A is contained in some cyclic subgroup of $\mathbb{R}^+$. However, should that be so, then $A(x) = \mathcal{O}(\ln x)$. □

For a set $A \subset (1, \infty)$, recall that $R(A) = \{a/b : a \in A, b \in A\}$ denotes its **ratio set**.

Corollary 3. *If $\mathbb{G}$ is an arithmetical semigroup such that $N_{|\mathbb{G}|}(x)/\ln x$ is unbounded as x tends to infinity, then $R(|\mathbb{G}|)$ is dense in $\mathbb{R}^+$.*

If $\mathbb{G}$ is an arithmetical semigroup such that $R(|\mathbb{G}|)$ is dense in $\mathbb{R}^+$, then $\mathbb{G}$ is said to have **dense hull**.

Corollary 4. *An arithmetical semigroup $\mathbb{G}$ satisfying Axiom A has dense hull.*

2. Generalized asymptotic density

Let $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ be a function defined on an arithmetical semigroup $\mathbb{G}$ and taking positive real values. For $\mathcal{C} \subset \mathbb{G}$, let $N_{\mathcal{C}}(\mathfrak{m}, x) = \sum_{|a| \leq x} \mathfrak{m}(a)\chi_{\mathcal{C}}(a)$, and let

$$\sigma_x(\mathcal{C}, \mathfrak{m}) = \frac{N_{\mathcal{C}}(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)} = \frac{\sum_{|a| \leq x} \mathfrak{m}(a)\chi_{\mathcal{C}}(a)}{\sum_{|a| \leq x} \mathfrak{m}(a)}$$

denote the $\mathfrak{m}$-weighted arithmetic means of the indicator $\chi_{\mathcal{C}}$ of $\mathcal{C}$. The numbers

$$\underline{\sigma}(\mathcal{C}, \mathfrak{m}) = \liminf_{x \to \infty} \sigma_x(\mathcal{C}, \mathfrak{m}) \qquad \text{and} \qquad \overline{\sigma}(\mathcal{C}, \mathfrak{m}) = \limsup_{x \to \infty} \sigma_x(\mathcal{C}, \mathfrak{m})$$

are called the **lower $\mathfrak{m}$-density of $\mathcal{C}$** and the **upper $\mathfrak{m}$-density of $\mathcal{C}$**, respectively. If $\underline{\sigma}(\mathcal{C}, \mathfrak{m}) = \overline{\sigma}(\mathcal{C}, \mathfrak{m})$, this common value is called the **$\mathfrak{m}$-density of $\mathcal{C}$** and is denoted by $\sigma(C, \mathfrak{m})$. To guarantee that the expected properties of arithmetical densities also transfer to the above defined $\mathfrak{m}$-densities, it is necessary to impose some additional properties on $\mathfrak{m}$. To do so, write $N'_{\mathcal{C}}(\mathfrak{m}, x) = \sum_{|a| = x} \mathfrak{m}(a)\chi_{\mathcal{C}}(a)$. The conditions are[1]:

(I) the series $\sum_{a \in \mathbb{G}} \mathfrak{m}(a)$ diverges,

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[1]Under some circumstances (II) is a consequence of (I), cf. Lemma 9 and its proof. This is also the case when $\mathbb{G} = \mathbb{N}$ and $\mathfrak{m}(n) = 1$ identically. Condition (II) was omitted by mistake in [14].

(II) $\lim_{x\to\infty} \dfrac{N'_{\mathbb{G}}(\mathfrak{m},x)}{N_{\mathbb{G}}(\mathfrak{m},x)} = 0$.

Note that if $N_{\mathbb{G}}(\mathfrak{m},x)$ is a δ-regularly varying function (see [18]), where $\delta \in (-\infty,+\infty)$, then (II) is a consequence of (I) (cf. [15]). This is for instance the case when $\mathbb{G}$ satisfies Axiom A and $\mathfrak{m}(n) = 1$ for every $n \in \mathbb{G}$. For this choice of $\mathfrak{m}$, the lower $\mathfrak{m}$-density and the upper $\mathfrak{m}$-density of a set $\mathcal{C} \subset \mathbb{G}$ coincide with its lower and upper asymptotic densities $\underline{d}(\mathcal{C})$ and $\overline{d}(\mathcal{C})$, respectively.

A subset $\mathcal{C} \subset \mathbb{G}$ will be called **(*R*)-dense** if the set $R(|\mathcal{C}|)$ is topologically dense in $\mathbb{R}^+$. Since the zero density sets can be joined to other sets without changing their density relationships, the next result makes possible to ignore the (R)-density requirement, as mentioned in the introduction:

Theorem 5. *Let $\mathbb{G}$ be an arithmetical semigroup with dense hull. Let $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ satisfy conditions* (I) *and* (II). *Then there exists an (R)-dense set $\mathcal{C} \subset \mathbb{G}$ such that $\sigma(\mathcal{C},\mathfrak{m}) = 0$.*

Proof. The following proof uses the ideas of the proof of the corresponding [11, Lemma 1]. Since in the case of arithmetical semigroups the sets $\{a \in \mathbb{G} : |a| = x\}$ are not at most singletons and that the norm values are not integers in general, technical adjustments of that proof are necessary.

Let $\{\rho_n\}_{n=1}^{\infty} = (0,1) \cap R(|\mathbb{G}|)$. The proof will proceed by induction, where at the kth step we construct a subset $\mathcal{D}_k$ of $\mathbb{G}$ and a real number x_k, $k = 1,2,\dots$, such that:

(i) the ratio set $R(|\mathcal{D}_k|)$ of $\mathcal{D}_k$ contains ρ_k,

(ii) $\mathcal{D}_j \subset \{a \in \mathbb{G} : |a| \le x_j\}$ for $j = 1,\dots,k-1$,

(iii) $\mathcal{D}_{j+1} \cap \{a \in \mathbb{G} : |a| \le x_j\} = \mathcal{D}_j$ for $j = 1,\dots,k-1$,

(iv) $\sigma_x(\mathcal{D}_k,\mathfrak{m}) < 1/i$ for every $x > x_i$ and $i = 1,\dots,k$.

$k = 1$: By (II), there exists $x_1 \in \mathbb{R}^+$ with

$$\frac{N'_{\mathbb{G}}(\mathfrak{m},x_1)}{N_{\mathbb{G}}(\mathfrak{m},x_1)} < \frac{1}{3} \qquad \text{for } x > x_1.$$

Since the norm $|\cdot|$ is multiplicative, we can find two elements $b_1, c_1 \in \mathbb{G}$ such that $|c_1| > |b_1| > x_1$, and

$$\frac{|b_1|}{|c_1|} = \rho_1.$$

Let $\mathcal{D}_1 = \{a \in \mathbb{G} : |a| = |c_1|\} \cup \{a \in \mathbb{G} : |a| = |b_1|\}$. Then

- $\sigma_x(\mathcal{D}_1,\mathfrak{m}) = 0$ if $x < |b_1|$;

- if $|b_1| \le x < |c_1|$, $N_{\mathcal{D}_1}(\mathfrak{m}, x) = N'_{\mathcal{D}_1}(\mathfrak{m}, |b_1|)$, and therefore

$$\sigma_x(\mathcal{D}_1, \mathfrak{m}) \le \frac{N'_{\mathcal{D}_1}(\mathfrak{m}, |b_1|)}{N_{\mathbb{G}}(\mathfrak{m}, |b_1|)} < \frac{1}{3};$$

- if $x \ge |c_1| > |b_1|$, then similarly

$$\begin{aligned}\sigma_x(\mathcal{D}_1, \mathfrak{m}) &= \frac{N'_{\mathcal{D}_1}(\mathfrak{m}, |b_1|) + N'_{\mathcal{D}_1}(\mathfrak{m}, |c_1|)}{N_{\mathbb{G}}(\mathfrak{m}, x)} \le \frac{N'_{\mathcal{D}_1}(\mathfrak{m}, |c_1|)}{N_{\mathbb{G}}(\mathfrak{m}, |c_1|)} + \frac{N'_{\mathcal{D}_1}(\mathfrak{m}, |b_1|)}{N_{\mathbb{G}}(\mathfrak{m}, |b_1|)} \\ &< \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

Thus, (i)–(iv) hold for $k = 1$.

Knowing $\mathcal{D}_{k-1}$ and x_{k-1}, we can continue the induction process as follows. Let x_k be such that $x_k > \max\{|a| : a \in \mathcal{D}_{k-1}\}$, and

$$\sigma_{x_k}(\mathcal{D}_{k-1}, \mathfrak{m}) < \frac{1}{3k}, \tag{3}$$

$$\frac{N'_{\mathbb{G}}(\mathfrak{m}, x)}{N_{\mathbb{G}}(\mathfrak{m}, x)} < \frac{1}{3k} \quad \text{for } x > x_k. \tag{4}$$

Again, let b_k and c_k be elements of $\mathbb{G}$ such that $\frac{|b_k|}{|c_k|} = \rho_k$, and $|c_k| > |b_k| > x_k$. Then the set

$$\mathcal{D}_k = \mathcal{D}_{k-1} \cup \{a \in \mathbb{G} \ : \ |a| = |b_k|\} \cup \{a \in \mathbb{G} \ : \ |a| = |c_k|\},$$

fulfills conditions (i)–(iii). To verify (iv), consider the following cases:

- if $x_k \le x < |b_k|$, then

$$\sigma_x(\mathcal{D}_k, \mathfrak{m}) \le \frac{N_{\mathcal{D}_{k-1}}(\mathfrak{m}, x_k)}{N_{\mathbb{G}}(\mathfrak{m}, x_k)} = \sigma_{x_k}(\mathcal{D}_{k-1}, \mathfrak{m}) < \frac{1}{3k};$$

- if $|b_k| \le x < |c_k|$, then

$$\begin{aligned}\sigma_x(\mathcal{D}_k, \mathfrak{m}) &\le \frac{N_{\mathcal{D}_{k-1}}(\mathfrak{m}, x_k) + N'_{\mathbb{G}}(\mathfrak{m}, |b_k|)}{N_{\mathbb{G}}(\mathfrak{m}, x)} \\ &\le \sigma_{x_k}(\mathcal{D}_{k-1}, \mathfrak{m}) + \frac{N'_{\mathbb{G}}(\mathfrak{m}, |b_k|)}{N_{\mathbb{G}}(\mathfrak{m}, |b_k|)} < \frac{2}{3k};\end{aligned}$$

- if $x \ge |c_k|$, then

$$\begin{aligned}\sigma_x(\mathcal{D}_k, \mathfrak{m}) &= \frac{N_{\mathcal{D}_{k-1}}(\mathfrak{m}, x_k) + N'_{\mathbb{G}}(\mathfrak{m}, |b_k|) + N'_{\mathbb{G}}(\mathfrak{m}, |c_k|)}{N_{\mathbb{G}}(\mathfrak{m}, x)} \\ &\le \frac{N_{\mathcal{D}_{k-1}}(\mathfrak{m}, x_k)}{N_{\mathbb{G}}(\mathfrak{m}, x_k)} + \frac{N'_{\mathbb{G}}(\mathfrak{m}, |b_k|)}{N_{\mathbb{G}}(\mathfrak{m}, |b_k|)} + \frac{N'_{\mathbb{G}}(\mathfrak{m}, |c_k|)}{N_{\mathbb{G}}(\mathfrak{m}, |c_k|)} < \frac{1}{k},\end{aligned}$$

which together verify (iv). If $\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k$, then $\mathcal{D}$ is (R)-dense by (i), and of $\mathfrak{m}$-density zero by (iv) and the fact that $\{x_k\}_{k=1}^{\infty}$ is strictly increasing. □

Theorem 6. *Let $\mathbb{G}$ be an arithmetical semigroup with dense hull. Let $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ satisfy conditions* (I) *and* (II) *and $0 \le \alpha \le \beta \le 1$ be arbitrary real numbers. Then there exists an (R)-dense set $\mathcal{A} \subset \mathbb{G}$ such that $\underline{\sigma}(\mathcal{A}, \mathfrak{m}) = \alpha$ and $\overline{\sigma}(\mathcal{A}, \mathfrak{m}) = \beta$.*

Proof. If $\mathcal{C}$ is a subset from Theorem 5, then for an arbitrary $\mathcal{B} \subset \mathbb{G}$ the set $\mathcal{A} = \mathcal{B} \cup \mathcal{C}$ is (R)-dense, while $\underline{\sigma}(\mathcal{A}, \mathfrak{m}) = \underline{\sigma}(\mathcal{B}, \mathfrak{m})$ and $\overline{\sigma}(\mathcal{A}, \mathfrak{m}) = \overline{\sigma}(\mathcal{B}, \mathfrak{m})$. So, it is sufficient to construct a set $\mathcal{B} \subset \mathbb{G}$ with $\underline{\sigma}(\mathcal{B}, \mathfrak{m}) = \alpha$ and $\overline{\sigma}(\mathcal{B}, \mathfrak{m}) = \beta$. Theorem 5 shows that we can suppose that $\beta > 0$.

Let $|\mathbb{G}| = \{\gamma_n\}_{n=1}^{\infty}$ be the set of elements of $|\mathbb{G}|$ ordered increasingly by their magnitude. Clearly, $\gamma_1 = 1$. Put $x_1 = \gamma_1$, $y_1 = \gamma_2$, $\mathcal{B}_1 = \{1_{\mathbb{G}}\}$. Suppose that we had already constructed couples x_i, y_i such that

$$x_1 < y_1 < x_2 < y_2 < \cdots < x_{n-1} < y_{n-1}, \tag{5}$$

and the sets $\mathcal{B}_k = \bigcup_{i=1}^{k} \{a \in \mathbb{G} : x_i \le |a| < y_i\}$ for $k = 1, 2, \ldots, n-1$.

Then take for $x_n = \gamma_{k_n}$ the smallest element in $\{\gamma_n\}_{n=1}^{\infty}$ which is greater than y_{n-1} such that

$$\sigma_{x_n}\left(\bigcup_{i=1}^{n-1} \{a \in \mathbb{G} \ : \ x_i \le |a| < y_i\}\right) < \alpha + \frac{1}{n},$$

and y_n be such that

$$\sigma_{y_n}\left(\bigcup_{i=1}^{n} \{a \in \mathbb{G} \ : \ x_i \le |a| < y_i\}\right) > \beta - \frac{1}{n}.$$

It is possible, however, that some of the x's and y's are immediate successors in the sequence $\{\gamma_i\}_{i=1}^{\infty}$; e.g., if $\beta - \frac{1}{n} < \alpha + \frac{1}{n}$, or if $\alpha = \beta$. The existence of both x_n and y_n is guaranteed by (I). Consequently, $\underline{\sigma}(\mathcal{A}, \mathfrak{m}) \le \alpha$ and $\overline{\sigma}(\mathcal{A}, \mathfrak{m}) \ge \beta$.

To see the equalities, consider the first one. The left-hand term of the next inequality is non-increasing for $y_{n-1} \le x < x_n$, and

$$\sigma_{x}\left(\bigcup_{i=1}^{n-1} \{a \in \mathbb{G} \ : \ x_i \le |a| < y_i\}\right) \ge \alpha + \frac{1}{n},$$

while its value jumps at x_n by a quantity which due to (II) tends to 0 with $n \to \infty$. Consequently, $\underline{\sigma}(\mathcal{A}, \mathfrak{m}) = \alpha$, and a similar argument works for $\overline{\sigma}(\mathcal{A}, \mathfrak{m})$. □

Corollary 7. *Let $\mathbb{G}$ be an arithmetical semigroup satisfying Axiom A and $0 \le \alpha \le \beta \le 1$ be given real numbers. Then there exists an (R)-dense set $\mathcal{A} \subset \mathbb{G}$ such that $\underline{d}(\mathcal{A}) = \alpha$ and $\overline{d}(\mathcal{A}) = \beta$.*

3. Generalized logarithmic density

Asymptotic density is not the only way of measuring the sets of integers. In many cases, logarithmic density provides a more sensitive indicator of the properties of the integers endowed with certain multiplicative constraints (cf. [7]). The relationship between asymptotic and logarithmic density was analyzed in [14, 15]. It was shown there that starting with an $\mathfrak{m}$-density it is possible, under additional assumptions on $\mathfrak{m}$ (cf. for instance, Lemma 8 below), to construct a new $\widehat{\mathfrak{m}}$-density, such that

$$0 \le \underline{\sigma}(\mathcal{C}, \mathfrak{m}) \le \underline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}) \le \overline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}) \le \overline{\sigma}(\mathcal{C}, \mathfrak{m}) \le 1. \tag{6}$$

The condition imposed on $\mathfrak{m}$ in [14] for which the relations between both density concepts were studied is: Let $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ satisfy

(III) for every $a \in \mathbb{G}$, there exists a positive real number $\widehat{\mathfrak{m}}(a) < 1$, such that for every subset $\mathcal{C} \subset \mathbb{G}$ having $\mathfrak{m}$-density $\sigma(\mathcal{C}, \mathfrak{m})$, the set $a\mathcal{C} = \{ac : c \in \mathbb{G}\}$ has $\mathfrak{m}$-density and $\sigma(a\mathcal{C}, \mathfrak{m}) = \widehat{\mathfrak{m}}(a)\sigma(\mathcal{C}, \mathfrak{m})$.

One sufficient condition for (6) to hold is given in the next result:

Lemma 8 ([14, Proposition 1]). *Let $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ be a completely multiplicative function such that*

$$\sum_{|a| \le x} \mathfrak{m}(a) = Bx^{\Delta} + \mathcal{O}(x^{\Theta}), \quad 0 \le \Theta < \Delta, \qquad \text{as } x \to \infty. \tag{7}$$

Then,

(1) $\sum_{|a| \le x} \widehat{\mathfrak{m}}(a) = \Delta B \ln x + \psi_m + \mathcal{O}(x^{\Theta - \Delta})$ *holds with some suitable constant* ψ_m, *and*

(2) *condition* (6) *holds for every* $\mathcal{C} \subset \mathbb{G}$.

The proof uses partial summation and the fact that if $\mathfrak{m}$ is completely multiplicative, then relation (7) implies[2] that

$$\widehat{\mathfrak{m}}(a) = \mathfrak{m}(a)|a|^{-\Delta}. \tag{8}$$

Partial summation also gives that

$$\sum_{s < |a| \le t} \frac{\mathfrak{m}(a)}{|a|^{\Delta}} = B\Delta \ln(t/s) + \mathcal{O}(s^{\Theta - \Delta}), \tag{9}$$

[2]Relation (8) is also a consequence of the more general assumption that $\sum_{\substack{a \in \mathbb{G} \\ |a| \le x}} \mathfrak{m}(a) = x^{\Delta} L(x)$, where $L(x)$ is slowly oscillating.

or that

$$\sum_{s\le|a|<t}\frac{\mathfrak{m}(a)}{|a|^{\Delta}} = B\Delta\ln(t/s)+\mathcal{O}(s^{\Theta-\Delta}), \tag{10}$$

where both equalities hold uniformly in $t>s$.

The next result shows that under certain circumstances the means $\widehat{\mathfrak{m}}$ lead again to a density:

Lemma 9. *Under the assumptions of Lemma 8, the function $\widehat{\mathfrak{m}}$ satisfies conditions* (I) *and* (II)*.*

Proof. (I) follows from part 1 of Lemma 8. For the proof of (II), note that

$$N'_{\mathbb{G}}(\widehat{\mathfrak{m}},x)=\sum_{|a|=x}\widehat{\mathfrak{m}}(a)=\sum_{|a|=x}\frac{\mathfrak{m}(a)}{|a|^{\Delta}}=\frac{1}{x^{\Delta}}\sum_{|a|=x}\mathfrak{m}(a).$$

Lemma 8 again gives

$$\lim_{x\to\infty}\frac{N'_{\mathbb{G}}(\widehat{\mathfrak{m}},x)}{N_{\mathbb{G}}(\widehat{\mathfrak{m}},x)}\ll\lim_{x\to\infty}\frac{1}{x^{\Delta}}\cdot\frac{Bx^{\Delta}}{\Delta B\ln x}=0. \qquad \square$$

On the other side the optimism indicated above is not always justified, since a further descent starting with $\widehat{\mathfrak{m}}$ in place of $\mathfrak{m}$ gives nothing new:

Lemma 10 ([14, Proposition 2])**.** *Under the assumptions of Lemma 8, we have* $\widehat{\widehat{\mathfrak{m}}}=\widehat{\mathfrak{m}}$.

We saw that if $\mathbb{G}=\mathbb{N}$, then the lower and the upper $\widehat{\mathfrak{m}}$-densities coincide with the lower and the upper logarithmic densities, respectively, provided that $\mathfrak{m}$ is the constant function equal to 1. Consequently, a natural way to define the lower and upper **logarithmic densities** of a subset $\mathcal{C}$ of an arithmetical semigroup satisfying Axiom A is

$$\underline{l}(\mathcal{C})=\liminf_{x\to\infty}\frac{1}{\delta A\ln x}\sum_{\substack{a\in\mathcal{C}\\|a|\le x}}|a|^{-\delta},\qquad \overline{l}(\mathcal{C})=\limsup_{x\to\infty}\frac{1}{\delta A\ln x}\sum_{\substack{a\in\mathcal{C}\\|a|\le x}}|a|^{-\delta}.$$

If $\mathbb{G}$ is an arithmetical semigroup, and $\mathfrak{m}:\mathbb{G}\to\mathbb{R}^{+}$ is a completely multiplicative function such that (7) holds, then the lower and the upper $\widehat{\mathfrak{m}}$-density of some $\mathcal{C} \subset \mathbb{G}$ can be defined by

$$\underline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}) = \liminf_{x\to\infty} \frac{N_{\mathcal{C}}(\widehat{\mathfrak{m}}, x)}{N_{\mathbb{G}}(\widehat{\mathfrak{m}}, x)} = \liminf_{x\to\infty} \frac{1}{\Delta B \ln x} \sum_{\substack{a\in\mathcal{C}\\|a|\le x}} \frac{\mathfrak{m}(a)}{|a|^{\Delta}},$$

$$\overline{\sigma}(\mathcal{C}, \widehat{\mathfrak{m}}) = \limsup_{x\to\infty} \frac{N_{\mathcal{C}}(\widehat{\mathfrak{m}}, x)}{N_{\mathbb{G}}(\widehat{\mathfrak{m}}, x)} = \limsup_{x\to\infty} \frac{1}{\Delta B \ln x} \sum_{\substack{a\in\mathcal{C}\\|a|\le x}} \frac{\mathfrak{m}(a)}{|a|^{\Delta}}.$$

4. Sets with prescribed upper and lower densities

In Theorem 6, we saw that it is possible to prescribe the values of the lower and upper $\mathfrak{m}$-densities even with some additional requirements. The next result shows that we can prescribe lower and upper densities together with the lower and upper derived logarithmic densities. The only limitation beforehand for the given values is the necessary condition (6):

Theorem 11. *Let $\mathbb{G}$ be an arithmetical semigroup and $\mathfrak{m} : \mathbb{G} \to \mathbb{R}^+$ be such that*

$$\sum_{|a|\le x, a\in\mathbb{G}} \mathfrak{m}(a) \sim Bx^{\Delta}, \tag{11}$$

where B and $\Delta > 0$. Let $\mathfrak{m}' = \mathfrak{m}(a)/(|a|^{\Delta})$. Given numbers $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$, then there is a subset $\mathcal{A} \subset \mathbb{G}$ such that

$$\underline{\sigma}(\mathcal{A}, \mathfrak{m}) = \alpha, \quad \underline{\sigma}(\mathcal{A}, \mathfrak{m}') = \beta, \quad \overline{\sigma}(\mathcal{A}, \mathfrak{m}') = \gamma, \quad \overline{\sigma}(\mathcal{A}, \mathfrak{m}) = \delta. \tag{12}$$

P r o o f. Partial summation shows that

$$\sum_{|a|\le x} \mathfrak{m}'(a) = \int_1^x \Delta t^{-\Delta-1} \sum_{|a|\le t} \mathfrak{m}(a)dt + \mathcal{O}(1) \sim \Delta B \ln x.$$

The same type of argument shows that if $\mathcal{A} \subset \mathbb{G}$ and $\alpha = \underline{\sigma}(\mathcal{A}, \mathfrak{m})$ and $\delta = \overline{\sigma}(\mathcal{A}, \mathfrak{m})$, then

$$\alpha + o(1) \le \frac{\sum_{a\in\mathcal{A}, |a|\le x} \mathfrak{m}'(a)}{\Delta B \ln x} \le \delta + o(1).$$

Thus, if $\beta = \underline{\sigma}(\mathcal{A}, \mathfrak{m}')$ and $\gamma = \overline{\sigma}(\mathcal{A}, \mathfrak{m}')$, we then have $\alpha \le \beta \le \gamma \le \delta$.

The case $\alpha = \beta = \gamma = \delta$, namely showing for each $\eta \in [0, 1]$, there is a set $\mathcal{A} \subset \mathbb{G}$ with $\sigma(\mathcal{A}, \mathfrak{m}) = \eta$, follows from Theorem 6 (in fact, this is a bit easier than Theorem 6 since the hypothesis here is stronger, and we are proving less). For the general case, let $x_n = 2^{2^{2^n}}$, and let $y_n = nx_n$. Thus,

$$\sum_{|a|\le x_n} \mathfrak{m}(a) = o\left(\sum_{x_n<|a|\le y_n} \mathfrak{m}(a)\right), \sum_{|a|\le y_n} \mathfrak{m}'(a) = o\left(\sum_{y_n<|a|\le x_{n+1}} \mathfrak{m}'(a)\right). \quad (13)$$

Let $\mathcal{A}_\alpha, \mathcal{A}_\beta, \mathcal{A}_\gamma, \mathcal{A}_\delta$ be subsets of $\mathbb{G}$ with (m)-densities $\alpha, \beta, \gamma, \delta$, respectively. Now define a set $\mathcal{A} \subset \mathcal{A}_\alpha \cup \mathcal{A}_\beta \cup \mathcal{A}_\gamma \cup \mathcal{A}_\delta$ as $\bigcup_{n\ge1}(\mathcal{A}_n \cup \mathcal{A}'_n)$, where

$$\mathcal{A}_n = \begin{cases} \{a \in \mathcal{A}_\alpha : |a| \in (x_n, y_n]\}, & \text{if } n \text{ is odd,} \\ \{a \in \mathcal{A}_\delta : |a| \in (x_n, y_n]\}, & \text{if } n \text{ is even,} \end{cases}$$

$$\mathcal{A}'_n = \begin{cases} \{a \in \mathcal{A}_\beta : |a| \in (y_n, x_{n+1}]\}, & \text{if } n \text{ is odd,} \\ \{a \in \mathcal{A}_\gamma : |a| \in (y_n, x_{n+1}]\}, & \text{if } n \text{ is even.} \end{cases}$$

Relation (13) shows that $\mathcal{A}$ fulfills the conditions of Theorem 11. $\square$

Corollary 12. *Let $\mathbb{G}$ be an arithmetical semigroup satisfying Axiom A and $0 \le \alpha \le \beta \le \gamma \le \delta \le 1$ be given real numbers. Then there exists a set $\mathcal{A} \subset \mathbb{G}$ such that*

$$\underline{d}(\mathcal{A}) = \alpha, \quad \underline{\ell}(\mathcal{A}) = \beta, \quad \overline{\ell}(\mathcal{A}) = \gamma, \quad \overline{d}(\mathcal{A}) = \delta.$$

It is interesting to note that it seems that Mišík [11] was the first who posed the question about the existence of a set of positive integers with prescribed lower and upper asymptotic and logarithmic densities. Note that the generalization in terms of arithmetical semigroups allows us to transfer the classical results from integers to other objects such as as algebraic integers, ideals of number fields, finite Abelian groups, etc. (see [8, 9] for more non-standard examples). Moreover, there are simple ways to construct further arithmetical semigroups satisfying Axiom A from a given one. For instance [8, 4.1.3 Proposition], given any element $a \in \mathbb{G}$ of an arithmetical semigroup $\mathbb{G}$ satisfying Axiom A, then the set $\mathbb{G}\langle a\rangle$ of all elements $b \in \mathbb{G}$ that are coprime to a, also satisfies Axiom A. More generally, if $\mathbb{G}$ is δ-regular and a_i are pairwise coprime, then the set of all elements of $\mathbb{G}$ coprime to all of the a_i's is under certain conditions again a δ-regular arithmetical semigroup [22, Satz 2.2] (for the case $\mathbb{G} = \mathbb{N}$ see [13, p. 14]).

Theorem 5 also shows that there exist sets $\mathcal{C}$ of asymptotic $\mathfrak{m}$-density zero (hence, due to (6), of "logarithmic" density $\sigma(\mathcal{C}, \widehat{\mathfrak{m}})$ equal to zero as well) such that the set of ratios of all products of elements of $\mathcal{C}$ is dense in $\mathbb{R}^+$. It follows that our sets $\mathcal{A}$ can be endowed with the additional property that the set of ratios of all products of elements of $\mathcal{A}$ is also dense in $\mathbb{R}^+$.

Finally note that (11) may be weakened. For example, if we define $\mathfrak{m}''(a)$ as $\mathfrak{m}'(a)/\ln(|a|)$, then the same results should go through for this double-logarithmic density (in fact, all six upper and lower densities might be prescribed).

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