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Dedicated to Emil Grosswald on the occasion of his sixty-eighth birthday.

1. Introduction.

In [16], 0 . Ore introduced the arithmetic functions $A(n), G(n), H(n)$ which are respectively the arithmetic mean, the geometric mean, and the harmonic mean of the natural divisors of $n$. Thus one easily has

$$
A(n)=\sigma(n) / d(n), \quad G(n)=\sqrt{n}, \quad H(n)=n d(n) / \sigma(n),
$$

so that all three functions are seen to be multiplicative. We note that $G(n)$ is an integer if and only if $n$ is a square, so that the set of $n$ with $G(n)$ integral has density 0 . In [10], Kanold showed that the set of $n$ with $H(n)$ integral also has density 0 . The corresponding statement for $A(n)$ is certainly not true, for, as Ore pointed out (in [16]), if $n$ is odd and square-free, then $A(n)$ is integral. In fact, the set of $n$ for which $A(n)$ is integral has density 1 (cf. [18]). In our Theorem 2.1 we study the distribution of those exceptional $n$ for which $A(n)$ is not integral.

We show in Theorem 3.1 that the set of $n$ for which $d(n)^{2} \mid \sigma(n)$ has asymptotic density exactly $1 / 2$.

In our Theorem 4.1 we show that the mean value of $A(n)$ for $n \leq x$ is asymptotic to $c x / \sqrt{\log x}$ where $c$ is an explicit constant. Moreover, in Theorem 5.1, we show that the number of $n$ with $A(n) \leq x$ is asymptotic to a constant times $x \log x$. This last result is somewhat more difficurt to establish than, say, an asymptotic formula for the number of $n$ with $\sigma(n) \leq x$ (cf. Bateman [1]).

In [18], Pomerance asked what can be said about the distribution of the distinct integral values of $A(n)$. In Theorem 6.1 , we show that they have
density 0 and that, in fact, the non-integral values may be thrown in as well. At the end of the paper we briefly consider some further problems.

This paper is the long-term result of conversations which the first and fourth authors had with Herbert Wilf in the spring of 1962. We also take this opportunity to acknowledge several interesting conversations about the contents of this paper with Harold Diamond and Gabor Halasz. Finally, we mention that the research of the last two authors was supported by grants from the National Science Foundation.
2. The distribution of the $n$ for which $A(n)$ is not integral.

THEOREM 2.1. Let $N(x)$ denote the number of $n \leq x$ for which $A(n)=\sigma(n) / d(n)$ is not an integer. Then

$$
N(x)=x \cdot \exp \{-(1+o(1)) \cdot 2 \sqrt{\log 2} \sqrt{\log \log x}\}
$$

PROOF. We first show $N(x) \geq x \cdot \exp \{-(1+o(1)) \cdot 2 \sqrt{\log 2} \sqrt{\log \log x}\}$.
Let $p_{0}=p_{0}(x)$ be the closest prime to $\sqrt{(\log \log x) / \log 2 \text {. (If there is a tie, }}$ choose either prime). Then $p_{0}=(1+0(1)) \sqrt{(\log \log x) / \log 2}$. We-shall consider integers $n \leq x$ such that $2^{p_{0}^{-1}} \| n, p_{0} \nmid \sigma(n)$. For such $n, d(n) Y_{\sigma}(n)$. Let

$$
M(y)=\#\{m \leq y: 2\}_{1}, p_{0}\{\sigma(m)\} .
$$

The following lemma will enable us to estimate $N(x)$ from below.
LEMMA 2.1. There is an absolute constant $c>0$ such that for $x^{1 / 2}<y<x$,

$$
M(y) \geq c y /(\log y)^{1 /\left(p_{0}-1\right)}
$$

Assuming Lemma 2.1, we have at once

$$
\begin{aligned}
N(x) & \geq M\left(x / 2^{P_{0}-1}\right) \geqslant c \times /\left\{2^{P_{0}^{-1}} \cdot(\log x)^{1 /\left(p_{0}-1\right)}\right\} \\
& =x \cdot \exp \{-(1+o(1)) \cdot 2 \sqrt{\log 2}, \log \log x)
\end{aligned}
$$

To prove Lemma 2.1 (and several other results in this paper) we shall use the following corollary of the Siegel-Walfisz theorem.

THEOREM A. (Norton [15], Pomerance [17]). Let $0<2<k$ be integers with $(\ell, k)=1$. Then for all $x \geq 3$

$$
\begin{aligned}
& \sum_{p \leq x} 1 / p=\frac{1}{\phi(k)} \log \log x+e^{*}+0\left(\frac{1}{\phi(k)} \log k\right) \\
& p \equiv \ell(\bmod k)
\end{aligned}
$$

where $l^{*}=1 / \ell$ if $\ell$ is prime and 0 otherwise. The constant implied by the 0 -notation is absolute.

PROOF OF LEMMA 2.1. Let

$$
\begin{array}{ll}
W(z)=(1 / 2) & \pi \\
& q \equiv-1\left(\bmod p_{0}\right) \\
& q \leq z
\end{array}
$$

Then using Theorem $A$, we have, uniformly for $z \geq 3$,

$$
\begin{aligned}
W(z) & =(1 / 2) \cdot \exp \left\{-\sum_{k \geq 1}^{\sum_{2}} 1 /\left(k q^{k}\right)\right\} \\
& \equiv(1 / 2) \cdot \exp \left\{-\sum_{k \geq 2}^{1} \sum_{2} 1 /\left(k q^{k}\right)\right\} \cdot \exp \left\{-\sum^{\prime} 1 / q\right\} \\
& =\exp (0(1)) \cdot \exp \left(-(\log \log z) /\left(p_{0}-1\right)\right)
\end{aligned}
$$

where $\sum^{\prime}$ denotes the sum over primes $q \leq z ; q \equiv-7\left(\bmod p_{0}\right)$. Thus if

$$
z_{1}=y^{\exp \left(-(\log \log y)^{1 / 3}\right)}, z_{2}=y^{\exp \left(-(\log \log y)^{1 / 2}\right)}
$$

then

$$
\begin{equation*}
c_{1} y /(\log y)^{1 /\left(p_{0}-1\right)} \leq y W\left(z_{1}\right) \leq y W\left(z_{2}\right) \leq c_{2} y /(\log y)^{1 /\left(p_{0}-1\right)} \tag{2.1}
\end{equation*}
$$

where $0<c_{1} \leq c_{2}$ are absolute constants. For $1=1,2$ let $P_{i}=2 \underset{q \leq z_{i}}{\pi}$

Since

$$
M(y) \geq \#\left\{m \leq y:\left(m, P_{1}\right)=1, p_{0}\{\sigma(m)\},\right.
$$

it follows that

$$
M(y) \geq S_{1}-S_{2}-S_{3}-S_{4}-S_{5}-S_{6},
$$

where

$$
\begin{aligned}
& s_{1}=\#\left\{n \leq y:\left(n, p_{1}\right)=1\right\}, \\
& \left.s_{2}=\# i n \leq y:\left(n, p_{1}\right)=1, \exists q \equiv-1\left(p_{0}\right), z_{1}<q<y / z_{1}^{10}, q \mid n\right\} \\
& s_{3}=\#\left\{n \leq y:\left(n, p_{2}\right)=1, \exists q \equiv-1\left(p_{0}\right), y / z_{1}^{10} \leq q<y / z_{2}^{10}, q \mid n\right\} \\
& s_{4}=\#\left\{n \leq y: \exists q \geq y / z_{2}^{10}, q \mid n\right\} \\
& s_{5}=\# \cdot\left\{n \leq y:\left(n, p_{1}\right)=1, \exists q^{a}<y / z_{1}^{10}, a>1, q^{a+1} \equiv 1\left(p_{0}\right), q^{a} \mid n\right\} \\
& s_{6}=\#\left\{n \leq y: \exists q^{a} \geq y / z_{1}^{10}, a>1, q^{a} \mid n\right\} .
\end{aligned}
$$

By Theorem 2.5 in [7], we have

$$
c_{3} y W\left(z_{1}\right) \leq s_{1} \leq c_{4} y W\left(z_{1}\right),
$$

where $0<c_{3} \leq c_{4}$ are absolute constants. Thus by (2.1) the proof of Lemma 2.1 will be complete if we show that $S_{1}=0\left(S_{1}\right)$ for $2 \leq 1 \leq 6$. We have the following estimates (where we use Theorem $A$ in considering $S_{2}$ and upper bound sieve results in considering $S_{2}, S_{3}$, and $S_{5}$ )

$$
\begin{gathered}
S_{2} \leq \sum_{q \geq-1\left(p_{0}\right)} \quad \sum_{z_{1}<y / q} 1 \\
<q<y / z_{1}^{10} \quad\left(m, p_{1}\right)=1 \\
<y W\left(z_{1}\right) \quad \sum_{q}^{\sum}-1\left(p_{0}\right) \\
z_{1}<q<y
\end{gathered}
$$

$$
\begin{aligned}
& =y W\left(z_{1}\right) \cdot\left\{\frac{1}{P_{0}-T} \cdot \log \frac{\log y}{\log z_{1}}+O\left(\frac{\log P_{0}}{P_{0}}\right)\right\} \\
& \ll y W\left(z_{1}\right) /(\log \log y)^{1 / 6}=O\left(S_{1}\right) . \\
& S_{3} \leq q \equiv-1\left(p_{0}\right) \quad m \leq y / q 1 \\
& y / z_{1}^{10} \leq q<y / z_{2}^{10} \\
& \left(m, p_{2}\right)=1 \\
& \ll y W\left(z_{2}\right) \quad \sum_{y / z_{1} 10}^{\sum} 1 / q \\
& \ll y W\left(z_{2}\right) \frac{\log z_{1}}{\log y}=o\left(s_{1}\right) \text {. } \\
& S_{4} \leq y \quad \sum_{y / z_{2}^{10} \leq q \leq y} \quad 1 / q \ll y \frac{\log z_{2}}{\log y}=\frac{y}{\exp (\sqrt{\log \log y})}=0\left(s_{1}\right) . \\
& S_{5} \leq \sum_{q^{a}<y / z_{1}^{10}}^{\sum \sum} \quad \begin{array}{ll}
m \leq y / q^{a} \\
& a^{1}>1, q^{a+1} \equiv 1\left(p_{0}\right)
\end{array} \quad\left(m, p_{1}\right)=1 . \\
& \ll y W\left(z_{1}\right) \sum_{a>1} 1 / q^{a} \leq y W\left(z_{1}\right) \sum_{a>1} 1 / q^{a} \ll y W\left(z_{1}\right) p_{0}^{-1 / 4}=o\left(S_{1}\right) . \\
& q^{a+1} \equiv 1\left(p_{0}\right) \quad q^{a}>p_{0}^{1 / 2} \\
& S_{6} \leq y \sum_{q^{a} \geq y / z_{1}^{10}} 1 / q^{a} \ll y / \sqrt{y / z_{1}^{10}} \ll y^{2 / 3}=o\left(S_{1}\right) . \\
& a>1
\end{aligned}
$$

This completes the proof of Lemma 2.1 and thus of our estimate of $N(x)$ from below.

We now show that $N(x) \leq x \cdot \exp \{-(1+0(1)) 2 \sqrt{\log 2} \sqrt{\log \log x}\}$. For any integer $n$, we may write $n=s m$ where $4 s$ is square-full, $m$ is odd and square-free, and $(s, m)=1$. Suppose $d(n) \neq(n)$. Now $d(n)=d(s) d(m), \sigma(n)=\sigma(s) \sigma(m)$, and
$d(m) \mid \sigma(m)$. Thus $d(s) / \sigma(n) / d(m)$. We now divide the $n \leq x$ for which $d(n) / \sigma(n)$ into 3 classes:
(1) $s>\exp (4 \sqrt{\log 2} \sqrt{\log \log x})$,
(2) $s<\log \log x$,
(3) $\log \log x \leq s \leq \exp (4 \sqrt{\log 2} \sqrt{\log \log x})$.

Let $N_{1}(x), N_{2}(x), N_{3}(x)$ denote, respectively, the number of $n$ in classes $1,2,3$. Let $S(y)$ denote the number of $s \leq y$ for which $4 s$ is square-full. Then $S(y) \ll y^{1 / 2}$. Thus

$$
N_{1}(x) \ll x \cdot \exp (-2 \sqrt{\log 2} \sqrt{\log \log x})
$$

To consider the $n$ in the other two classes, we shall use the following lemma.

LEMMA 2.2. Let $u, v$ be integers with $(u, v)=1, v \geq 3$. Let $N(v, u, k, y)$ denote the number of integers $m \leq y$ which are not divisible by $k$ or more primes $q \equiv u(\bmod v)$. There are absolute constants $c$ and $c^{\prime}$ such that

$$
N(v, u, k, y) \leq c \sum_{i=0}^{k-1} y\left\{\frac{\log \log y^{1}}{\phi(v)^{1}} /\left\{1!(\log y)^{1 / \phi(v)}\right\},\right.
$$

provided $!\leq k<\frac{1}{\phi(y)} \cdot \log \log y-c^{\prime}$ and $y \geq 3$.
PROOF. This lenma follows by applying a result of Halasz [6] in conjunction with Theoreni A.

Now say $n=s m$ is in class 2. Since $d(s) \nmid \sigma(n) / d(m)$, it follows that $m$ is not divisible by any prime $q \equiv-1(\bmod 2 d(s))$. From Theorem 317 in [9] we have

$$
\begin{equation*}
d(t) \leq 2^{(1+o(1))(\log t) / \log \log t} . \tag{2.2}
\end{equation*}
$$

So, since $s<\log \log x$, we have $d(s)<\frac{1}{10} \sqrt{\log \log x}$ for large $x$. Applying Lenma 2.2, we have

$$
\begin{aligned}
& \leq c x(\log \log x) / \exp (5 \sqrt{\log \log x}) \text {, }
\end{aligned}
$$

so that class 2 is under control.
Thus to complete the proof of Theorem 2.1, it remains only to consider the $n$ in class 3. Suppose $n=s m$ is in class 3. Since $d(s) \psi_{\sigma}(n) / d(m)$, it follows that there is a prime power. $p^{k}$ with $p^{k} \mid d(s)$ but $p^{k} \gamma_{0}(n) / d(m)$. Thus it is not the case that there are $k$ primes in the class $-1(\bmod 2 p)$ which divide $m$. Thus

$$
\begin{equation*}
N_{3}(x) \leq \sum_{s} \sum_{p^{k} \mid d(s)} N(2 p,-1, k, x / s) \tag{2.3}
\end{equation*}
$$

where $s$ runs through the integers in $[\log \log x, \exp (4 \sqrt{\log 2} \sqrt{\log \log x})]$ for which $4 s$ is square-full. We shall require the following lemma.

LEMMA 2.3. Suppose t is an integer and $\mathrm{p}^{\mathrm{k}} \mid \mathrm{d}(\mathrm{t})$, where p is a prime. Then for all sufficiently large $t$ we have

$$
p<2 \log t, k<(3 / 2)(\log t) / \log \log t
$$

PROOF. Since $p \mid d(t)$, there is a prime power $q^{b}| | t$ with $p \mid b+1$. Then

$$
p \leq b+1 \leq \frac{\log t}{\log 2}+1<2 \log t
$$

for large $t$. From (2.2) we have

$$
p^{k} \leq d(t) \leq 2^{(1+o(1))(\log t) / \log \log t}
$$

so that

$$
k \leq(1+o(1))(\log t) / \log \log t<(3 / 2)(\log t) / \log \log t
$$

for large $t$. Thus Lemma 2.3 is proved.
In view of (2.3) and Lemma 2.3, we have for all large $x$

$$
N_{3}(x) \leq \sum_{\substack{p<\sqrt{8} \log \log x}} \sum^{k<\sqrt{6 \log \log x}} \mid d(s)<i(2 p,-1, k, x / s),
$$

where the inner sum is taken over those $s$ such that $4 s$ is square-full and $d(s) \equiv 0\left(\bmod p^{k}\right)$. Let $f\left(p^{k}\right)$ denote the inner sum. Thus

$$
N_{3}(x) \leq 48 \log \log x \cdot \max \left\{f\left(p^{k}\right): p<8 \sqrt{\log \log x}, k<6 \sqrt{\log \log x}\right\} .
$$

Thus to prove Theorem 1, it is enough to show that each

$$
\begin{equation*}
f\left(p^{k}\right) \leq x \cdot \exp \{-(1+o(1)) 2 \sqrt{\log 2} \sqrt{\log \log x}\} \tag{2.4}
\end{equation*}
$$

By Lemmas 2.2 and 2.3, we have

$$
\begin{equation*}
f\left(p^{k}\right) \leq \frac{c x}{(\log x)^{1 / \phi(2 p)}} \cdot \sum_{p^{k} \mid d(s)} \frac{1}{5} \sum_{i=0}^{k-1}\left\{\frac{1}{\phi(2 p)} \log \log x\right\}^{i} / i . . \tag{2.5}
\end{equation*}
$$

The following two lemmes will enable us to prove (2.4) and thus complete the proof of the upper estimate of $N(x)$.

LEMMA 2.4. Suppose the prime power $p^{k} \mid d(t)$ where $p-1>\frac{1}{12} \sqrt{\log \log x}$ and $t \leq \exp (4 \sqrt{\log 2} \sqrt{\log \log x})$. Then for all sufficiently large $x$, we have $k<60$.

PROOF, Say $q^{b}| | t$ where $p \mid b+1$. If $p^{2} \mid b+1$, then

$$
t \geq q^{b} \geq 2^{p^{2}-1}>2^{\frac{1}{44} \log \log x}>t
$$

for all large $x$. Thus we may assume $p \| \mid b+1$. Hence there are $k$ distinct primes $q_{\uparrow}, \ldots, q_{k}$ with $\left(q_{1} \ldots q_{k}\right)^{p-1} \mid t$. Thus $2^{k(p-1)} \leq t$, so that $k<48 / \sqrt{\log 2}<60$. Thus Lemma 2.4 is proved.

LEMMA 2.5. Let $P$ be a prime and let $S$ be a set of integers whose elements satisfy $\mathrm{pld}(\mathrm{s})$ and 4 s is square-full. Then there is an absolute constant c such that

$$
\sum_{s \in S} 1 / s \leq c \cdot 2^{-p}
$$

PROOF. We may assume $p$ is odd. If $p \mid d(s)$, there is a prime power $q^{b}| | s$ with $p \mid b+1$. Let $t$ be the product of $a l l$ such $q^{b}$ in $s$ and write $s=u t$. Then

$$
\sum_{\in S} 1 / s \leq\left(\sum 1 / u\right)(\Sigma 1 / t)
$$

where $u$ runs through all integers for which $4 u$ is square-full and $t$ runs through all integers $>1$ which are ( $p-1$ )-full. Thus

$$
\sum_{s S} 1 / s \ll \sum 1 / t \ll \zeta(p-1)-1 \ll 2^{-p} \text {, }
$$

where $\zeta$ is Riemann's function. Thus Lemma 2.5 is proved.
We now show that (2.4) holds if $p-1>\frac{1}{12} \sqrt{\log \log x}$. In this case, Lenma 2.4 implies we may assume $k<60$. Thus by (2.5) and Lerma 2.5,

$$
\begin{aligned}
f\left(p^{k}\right) & \leq c \cdot 60 \cdot \frac{x(\log \log x)^{59}}{(\log x)^{1 /(p-1)}} \cdot{ }_{p^{k} \mid d(s)} 1 / s \\
& \ll \frac{x(\log \log x)^{59}}{(\log x)^{1 /(p-1)} \cdot 2^{p-1}} .
\end{aligned}
$$

Applying the inequality of the arithmetic and geometric means to the logarithm of the last denominator, we have (2.4).

Now suppose $p-1 \leq \frac{1}{12} \sqrt{\log \log x}$. If $p^{k} \mid d(s)$, since $s \leq \exp (4 \sqrt{\log 2} \sqrt{\log \log x})$, Lemma 2.3 implies for all large $x$

$$
k<12 \sqrt{\log 2} \sqrt{\log \log x} / \log \log \log x<10 \sqrt{\log \log x} / \log \log \log x .
$$

Thus by (2.5) we have

$$
f\left(p^{k}\right) \leq e c x \cdot(\log \log x)^{10 \sqrt{\log \log x} / \log \log \log x}(\log x)^{-1 / \phi(2 p)} \sum_{p^{k} \mid d(s)} 1 / s
$$

$$
\ll x \exp (-2 \sqrt{\log \log x})
$$

so that a stronger result than (2.4) holds in this case. Thus Theorem 2.1 is completely proved.

REMARK. Although giving the approximate rate of growth of $N(x)$, the estimate in Theorem 2.1 is not an asymptotic formula. We believe an asymptotic formula for $N(x)$ could be established along the general lines of our proof, but it appears that certain strong conjectures about the distribution of prime numbers in short
intervals (specifically, in a short interval centered at $\sqrt{\log \log x} / \sqrt{\log 2}$ ) would have to be assumed.
3. The $n$ for which $d(n)^{2} \mid \sigma(n)$.

For every positive real number $\beta$, 1et $\left\langle n^{\beta}>=\pi p^{[a \beta]}\right.$. Thus if $\beta$ is $p^{a}| | n$
a positive integer, $\left\langle n^{\beta}\right\rangle=n^{\beta}$.
THEOREM 3.1. For any $\varepsilon$ in $(0,2)$, the set of $n$ for which $<d(n)^{2-\varepsilon}>\mid \sigma(n)$ has asymptotic density 1 , the set of $n$ for which $<d(n)^{2+\varepsilon}>\mid \sigma(n)$ has asymptotic density 0 , and the set of $n$ for which $d(n)^{2} / \sigma(n)$ has asymptotic density $1 / 2$.

PROOF. Write $d(n)=a(n) b(n)$, where $a(n)$ is odd and $b(n)$ is a power of 2 . We first note that the set of $n$ for which $\left\langle a(n)^{\beta}\right\rangle$ divides $\sigma(n)$ has asymptotic density 1, no matter what positive value we choose for $\beta$. This can be seen as follows. Leet $\varepsilon$ be an arbitrary positive number. If we write $n=s m$ where $(s, m)=1, s$ is square-full, and $m$ is square-free, then it is easy to see that there is a positive integer $K$ such that the set of $n$ whose square-full part $s$ exceeds $K$ has asymptotic density $<\varepsilon$. (Indeed, this follows at once from the fact that the sum of the reciprocals of the square-full numbers is convergent.) Hence we may consider only those $n$ whose square-full part $s$ does not exceed $K$. Since $a(n) \mid d(s)$, we accordingly know that $\left\langle a(n)^{\beta}\right\rangle \leq K^{\beta}$. If $p_{1}, p_{2}, \ldots$ are the primes congruent to -1 modulo $\left[K^{\beta}\right]$ :, let $P$ be the set of positive integers $n$ for which there is no $i$ such that $p_{i} \| n$; since $\sum 1 / p_{i}$ diverges (according to Dirichlet), the asymptotic density of the set $P$ is

$$
\prod_{i=1}^{\infty}\left(1-\frac{1}{p_{i}}+\frac{1}{p_{i}^{2}}\right)=0 .
$$

But if $n$ has square-full part $\leq K$ and if $p_{i} \| n$ for some $i$, then $\left\langle a(n)^{\beta}>\right|\left(p_{i}+1\right) \mid \sigma(n)$. Thus the set of $n$ for which $\left\langle a(n)^{\beta}>\chi \sigma(n)\right.$ is contained in the union of $P$ and the set of integers $n$ with $s(n)>K$; accordingly the set of $n$ for which $\left\langle a(n)^{\beta}>\operatorname{lo}(n)\right.$ has density less than $\varepsilon$. Since $\varepsilon$ is arbitrary, the set of $n$ for which $\left\langle a(n)^{\beta}>f \sigma(n)\right.$ has asymptotic density 1 .

Since $\left\langle a(n)^{\beta}\right\rangle$ and $\left\langle b(n)^{\beta}>\right.$ are relatively prime and their product is $\left.<d(n)^{\beta}\right\rangle$, we need only be concerned with the divisibility of $\sigma(n)$ by $\left\langle b(n)^{\beta}\right\rangle$ for $B=2-\varepsilon, 2$, and $2+\varepsilon$.

Let $\mathrm{v}_{2}(\mathrm{n})$ denote the exponent on 2 (possibly 0 ) in the prime factorization of $n$. Let $g_{\beta}(n)=v_{2}(\sigma(n))-\beta v_{2}(d(n))$. Note that $g_{\beta}$ is an additive function. Moreover $<b(n)^{\beta}>\mid \sigma(n)$ if and only if $g_{\beta}(n)>-1$. We shall require the following lemma.

LEMMA 3.1. If $B \neq 2$, the nomal value of $g_{B}(n)$ is $(2-\beta) \log \log n_{\text {; }}$ that is, if $\varepsilon>0$, the set of $n$ with $(2-\beta-\varepsilon) \log \log n<g_{\beta}(n)<(2-\beta+\varepsilon) \log \log n$ has asymptotic density 1 . For every real number $u$, the set of $n$ with $g_{2}(n) \leq u \sqrt{2 \log \log n}$ has asymptotic density $(2 \pi)^{-1 / 2} \int_{-\infty}^{u} e^{-v^{2} / 2} d v \stackrel{\text { def }}{=} G(u)$. We can see how the Theorem is a corollary of the Lemma. Indeed $g_{2-\varepsilon}(n)>-1$ for all $n$ but for a set of asymptotic density $0, g_{2+\varepsilon}(n) \leq-1$ for all $n$ but for a set of asymptotic density 0 , and $g_{2}(n)>-1$ for a set of $n$ of asymptotic density $G(0)=1 / 2$.

PROOF OF THE LEMMA. If $g(n)$ is any real valued additive function, let $A(x)=\sum_{p \leq x} g(p) / p$, and let $B^{2}(x)=\sum_{p \leq x} g^{2}(p) / p$. We shall use the following generalization of the Erdös-Kac Theorem (see Kubilius [12] or Shapiro [21]): If $B(x) \rightarrow \infty$ as $x+\infty$ and if for every $n>0$,

$$
\sum_{p \leq x}^{|g(p)|>n B(x)} \quad g^{2}(p) / p=o\left(B^{2}(x)\right) \text {, }
$$

then for every real number $u$,

$$
\lim _{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} 1 \quad=G(u) .
$$

We apply this theorem to the functions $g_{B}(n)$.
Using Theorem $A$, we have for any $B$,

$$
\begin{aligned}
& A_{B}(x) \stackrel{\text { def }}{=} \sum_{p \leq x} g_{B}(p) / p=\sum_{i=1}^{\infty} \sum_{p \leq x^{1 / p}} \sum_{p \equiv 2^{1}-1\left(\bmod 2^{i+1}\right)}^{-\beta} \sum_{p \leq x} 1 / p \\
&= \sum_{i=1}^{\infty} i \cdot 2^{-i} \log \log x+0\left(\sum_{i=1}^{\infty} i^{2} \cdot 2^{-1}\right)-\beta \log \log x+0(1) \\
&=(2-\beta) \log \log x+0(1) .
\end{aligned}
$$

## Moreover,

$$
\begin{aligned}
& B_{\beta}^{2}(x) \stackrel{\operatorname{def}}{=} \sum_{\leq x} g_{\beta}^{2}(p) / p=\sum_{i=1}^{\infty} \sum_{p \leq x}(i-\beta)^{2} / p \\
& p \equiv 2^{1}-1\left(\bmod 2^{i+1}\right) \\
&= \sum_{i=1}^{\infty}(i-\beta)^{2} \cdot 2^{-1} \log \log x+0\left(\sum_{i=1}^{\infty} i(i-\beta)^{2} \cdot 2^{-i}\right) \\
&=\left(6-4 \beta+\beta^{2}\right) \log \log x+0(1) .
\end{aligned}
$$

Now for every $n>0$, let $i_{B}=1_{\beta}(\eta, x)=\eta B_{\beta}(x)+\beta$. We have (if we assume $x$ large enough so that $B<{ }_{\eta} B_{B}(x)$ )

$$
\begin{aligned}
& =\sum_{i>i_{G}}(1-\beta)^{2} \cdot 2^{-1} \log \log x+0\left(\sum_{i>i_{\beta}} i(i-\beta)^{2} \cdot 2^{-i}\right) \\
& =O\left(\frac{n^{2} B_{\beta}^{2}(x) \log \log x}{2^{n B_{B}(x)}}\right)=O(1)=O\left(B_{B}^{2}(x)\right)
\end{aligned}
$$

by our estimate for $B_{B}^{2}(x)$.
Hence the generalization of the Erdös-Kac Theorem quoted above is applicable. Thus the normal value of $g_{\beta}(n)$ for $\beta \neq 2$ is $(2-\beta) \log \log n$ and, if $\rho(n)$ tends to infinity with $n$, we have for all $n$

$$
-\rho(n)(\log \log n)^{1 / 2}<g_{\beta}(n)-(2-\beta) \log \log n<\rho(n)(\log \log n)^{1 / 2}
$$

except for a set of asymptotic density zero. Moreover, since $A_{2}(x)=0(1)$ and
since $B_{2}(x)$ is indistinguishable from $B_{2}(n)$ for $n$ near $x$, we have our assertion about $g_{2}(n)$. This completes the proof of Lemma 3.1 and accordingly Theorem 3.1 is established.
4. The mean value of $A(n)$.

Let $g(s)$ be the sum of the Dirichlet Series $\sum c(n) n^{-s}$ whose Euler product is

$$
\frac{\pi}{p}\left(\left\{1-p^{-s}\right\}^{1 / 2}\left\{1+\frac{1}{2}\left(1+\frac{1}{p}\right) p^{-s}+\frac{1}{3}\left(1+\frac{1}{p}+\frac{1}{p^{2}}\right) p^{-2 s}+\ldots\right\}\right),
$$

the square-root being the principal branch. Clearly both series and product converge absolutely for Re $s>\frac{1}{2}$, since the general term of the product has the form

$$
1+\frac{1}{2 p} p^{-s}+\sum_{k=2}^{\infty} \varepsilon_{k}(p) p^{-k s}
$$

where $\left|\varepsilon_{k}(p)\right|<1$. The following result indicates that $A(n)$ behaves like $g(1) n(\pi \log n)^{-1 / 2}$ on average.

THEOREM 4.1. As $x \rightarrow \infty$ we have

$$
\begin{equation*}
\sum_{n \leq x} A(n) \sim \frac{g(1)}{2 \pi^{1 / 2}} \frac{x^{2}}{(\log x)^{1 / 2}} \tag{4.1}
\end{equation*}
$$

PROOF. For Re $s>1$ we have

$$
\begin{align*}
\sum_{n=1}^{\infty} n^{-1} A(n) n^{-s} & =\prod_{p}\left\{1+\frac{1}{2}\left(1+\frac{1}{p}\right) p^{-s}+\frac{1}{3}\left(1+\frac{1}{p}+\frac{1}{p^{2}}\right) p^{-2 s}+\ldots\right\}  \tag{4.2}\\
& =\zeta(s)^{1 / 2} g(s),
\end{align*}
$$

where $\zeta(s)^{1 / 2}$ is real for positive real $s$ greater than 1 . From (4.2) it is possible to deduce in various ways that

$$
\begin{equation*}
B(x)=\sum_{n \leq x} n^{n^{-1}} A(n) \sim \frac{g(1)}{\pi / 2} \frac{x}{(\log x)^{1 / 2}} ; \tag{4.3}
\end{equation*}
$$

in fact we shall sketch below several different methods of going from (4.2) to (4.3). The partial summation fomula

$$
\sum_{n \leq x} A(n)=\left\{_{-}^{x} t d B(t)=x B(x)-\int_{1}^{x} B(t) d t\right.
$$

then readfly enables one to deduce (4.1) from (4.3).
Here are five methods I-V for deducing (4.3) from (4.2).
I One can use the classical method of contour integration (cf. Landau [13], Landau [14], Wilson [24], Stanley [22], and Hardy [8].)

II One can appeal to general theorems established by Kienast [11] and Dixon [4] by the method of contour integration.

III One can start with the result

$$
D_{1 / 2}(x)=\sum_{n \leq x} d_{1 / 2}(n)=\frac{x}{(\pi \log x)^{1 / 2}}+0\left(\frac{x}{(\log x)^{3 / 2}}\right)
$$

of Selberg [22] or Diamond [3], where

$$
\sum_{n=1}^{\infty} d_{1 / 2}(n) n^{-s}=\zeta(s)^{1 / 2} \quad(\operatorname{Re} s>1)
$$

and then use the identity

$$
n \underset{\leq}{\sum_{x}} n^{-1} A(n)=\sum_{n \leq x} c(n) D_{1 / 2}(x / n)
$$

where as above $\left[c(n) n^{-s}=g(s)\right.$.
IV One can use the following Tauberian theorem of Delange:

LEMMA 4.1. Suppose $1 \leq h_{1}<h_{2}<\cdots$ and $h_{n}++\infty$. Suppose the Dirichlet series $\left[a_{n} h_{n}^{-s}\right.$ has non-negative coefficients and converges for $\operatorname{Re} s>1$ to a sum $f(s)$. Suppose there is a real number $r$ which is not a negative integer such that

$$
f(s)=(s-1)^{-r-1} h(s)+k(s)
$$

where $h$ and $k$ are holomorphic functions on some domain containing the closed half-plane $\operatorname{Re} s \geq 1$ and $h(1) \neq 0$. Then as $x \rightarrow+\infty$ we have

$$
n_{n} \sum_{\leq x} a_{n} u \frac{h(1)}{\Gamma(r+1)} x(\log x)^{r} .
$$

Lemma 4.1 is a special case of Theorem 3 of Delange [2]. To obtain (4.3) from (4.2) by using Lemma 4.1 we need only take $r=-1 / 2, h_{n}=n, a_{n}=A(n) / n$, $h(s)=\{(s-1) \zeta(s)\}^{1 / 2} g(s)$, and $k(s)=0$.

V One can apply the Tauberian theorem of Delange quoted as Lemma 5.1 in the next section to

$$
\left\{(\log n) \frac{A(n)}{n} \cdot \frac{1}{n^{s}}=-\frac{d}{d s}\left\{\zeta(s)^{1 / 2} g(s)\right\} .\right.
$$

This approach gives

$$
\sum_{n \leq x}(\log n) \frac{A(n)}{n} \sim \frac{g(1)}{1 / 2} \times(\log x)^{1 / 2}
$$

from which (4.3) follows by partial summation.
Any of the first three of the above five methods will in fact give the more precise result

$$
\sum_{n \leq x} A(n)=\frac{x^{2}}{(\log x)^{1 / 2}}\left\{\frac{g(1)}{2 \pi)^{1 / 2}}+\frac{b_{1}}{\log x}+\frac{b_{2}}{\log ^{2} x}+\ldots+\frac{b_{m}}{\log ^{m} x}+0\left(\frac{1}{\log ^{m+1} x}\right)\right\}
$$

for any positive integer $m$ and suitable constants $b_{1}, b_{2}, \ldots, b_{m}$.
Instead of starting with (4.2) it would also be possible to begin with

$$
\sum_{n=1}^{\infty} n^{-1} A(n) n^{-s}=\sum_{n=1}^{\infty} \frac{1}{d(n)} n^{-s} \cdot \sum_{n=1}^{\infty} \ell(n) n^{-s},
$$

where $\left[\ell(n) n^{-s}\right.$ has abscissa of convergence less than 1 , and then use the result of Wilson [24] that

$$
\sum_{n \leq x} \frac{1}{d(n)} \sim \frac{c x}{(\log x)^{1 / 2}}
$$

for a certain positive constant c.
5. The number of $n$ with $A(n) \leq x$.

We begin by quoting (as Lemma 5.1) another Tauberian theorem of Delange which is somewhat more powerful than Lerma 4.1. While it requires the parameter $r$ to be non-negative, it requires a much less stringent condition than the analyticity of the functions $h$ and $k$ which was imposed in Lemma 4.1. If $r \geq 0$, Lemma 4.1 clearly follows from Lemma 5.1 ; if $r$ is negative but not an integer, Lemma 4.1 can be readily deduced from Lemma 5.1 by repeated differentiation and partial summation.

LEMMA 5.1. Suppose $1 \leq h_{1}<h_{2}<\cdots$ and $h_{n} \rightarrow+\infty$. Suppose the Dirichlet series $\sum a_{n} h_{n}^{-s}$ has non-negative coefficients and converges for $\operatorname{Re} s>1$ to a sum $f(s)$. Suppose

$$
\begin{array}{ll}
\lim _{\substack{ \\
\operatorname{Re} s>1+i y}} f(s) \\
&
\end{array}
$$

exists for each non-zero $y$, And suppose there exist real numbers $A, r, \theta$ with

$$
A>0, r \geq 0,0 \leq \theta<1
$$

such that

$$
f(s)-A(s-1)^{-r-1}=0\left(|s-1|^{-r-\theta}\right)
$$

for $\operatorname{Re} s>1$, $|s-1|<i$. Then as $x++\infty$ we have

$$
h_{n} \sum_{\leq x} a_{n} \sim \frac{A}{\Gamma(r+1)} x(\log x)^{r}
$$

Lemma 5.1 is a special case of Theorem 1 of Delange [2].
The following theorem not only provides valuable information about the distribution of the values $A(n)$, but also is interesting in that it is a clearcut instance where Lerma 5.1 appears to be needed rather than the easier Lenma 4.1.

THEOREM 5.1. As $x++\infty$ we have

$$
\#\{n: A(n) \leq x\}=\sum_{A(n) \leq x} 1 \sim \lambda \times \log x,
$$

where

$$
\lambda=\prod_{p}\left\{\left(7-\frac{1}{p}\right)^{2}\left(7+\frac{2}{p+1}+\frac{3}{p^{2}+p+1}+\frac{4}{p^{3}+p^{2}+p+1}+\ldots\right)\right\}
$$

PROOF. For $\sigma=\operatorname{Re} s>1$ we have

$$
\begin{align*}
\sum A(n)^{-s} & =\pi_{p}\left\{1+A(p)^{-s}+A\left(p^{2}\right)^{-s}+A\left(p^{3}\right)^{-s}+\ldots\right\}  \tag{5.1}\\
& =\zeta(s)^{2^{s}} G(s),
\end{align*}
$$

where

$$
\zeta(s)^{2^{s}}=\exp \left\{2^{s} \sum_{p} \sum_{m=1}^{\infty} m^{-1} p^{-m s}\right\}
$$

and
(5.2) $G(s)=\prod_{p}\left(\left\{1-\frac{1}{p^{s}}\right\}^{2^{s}}\left\{1+\frac{2^{s}}{(p+1)^{s}}+\frac{3^{s}}{\left(p^{2}+p+1\right)^{s}}+\frac{4^{s}}{\left(p^{3}+p^{2}+p+1\right)^{s}}+\ldots\right\}\right)$.

For $\sigma=\operatorname{Re} s>1 / 2$ we have the estimates

$$
1+\frac{2^{s}}{(p+1)^{s}}+\frac{3^{s}}{\left(p^{2}+p+1\right)^{s}}+\frac{4^{s}}{\left(p^{3}+p^{2}+p+1\right)^{s}}+\ldots=1+\frac{2^{s}}{(p+1)^{s}}+0\left(\frac{4^{\sigma}}{p^{2 \sigma}}\right)
$$

and

$$
\left(1-\frac{1}{p^{s}}\right)^{2^{s}}=\exp \left\{2^{s} \log \left(1-\frac{1}{p^{s}}\right)\right\}=\exp \left\{-\frac{2^{s}}{p^{s}}+0\left(\frac{2^{\sigma}}{p^{2 \sigma}}\right)\right\}
$$

so that

$$
\begin{gather*}
\left\{1-\frac{1}{p^{s}}\right\}^{2^{s}}\left\{1+\frac{2^{s}}{(p+1)^{s}}+\frac{3^{s}}{\left(p^{2}+p+1\right)^{s}}+\frac{4^{s}}{\left(p^{3}+p^{2}+p+1\right)^{s}}+\ldots\right\}=  \tag{5.3}\\
1-\frac{2^{s}}{p^{s}}+\frac{2^{s}}{(p+1)^{s}}+0\left(\frac{4^{\sigma}}{p^{2 \sigma}}\right)
\end{gather*}
$$

Since

$$
\left|-\frac{2^{s}}{p^{s}}+\frac{2^{s}}{(p+1)^{s}}\right|=\left|-2^{s} \int_{p}^{p+1} \frac{s}{u^{s+1}} d u\right| \leq \frac{2^{\sigma}|s|}{p^{\sigma+1}},
$$

(5.2) and (5.3) show that $G$ is holomorphic for $\sigma=\operatorname{Re} s>1 / 2$.

Put

$$
H(s)=G(s)\{(s-1) \zeta(s)\}^{2^{s}}
$$

in some domain containing $\operatorname{Re} s \geq 1$ in which $\zeta$ has no zeros. Then for $\operatorname{Re} s>1$ we obtain from (5.1)

$$
\begin{align*}
\sum_{n=1}^{\infty} A(n)^{-s} & =(s-1)^{-2^{s}} H(s)  \tag{5.4}\\
& =(s-1)^{-2} H(1)+(s-1)^{-2}\{H(s)-H(1)\} \\
& +H(s)(s-1)^{-2}\left(\exp \left\{\left(2-2^{s}\right) \log (s-1)\right\}-1\right),
\end{align*}
$$

where the logarithm is the principal branch. Hence if $\operatorname{Re} s>1$ and $|s-1|<1$, we have

$$
\begin{aligned}
\left|\exp \left\{\left(2-2^{s}\right) \log (s-1)\right\}-1\right| & \leq \exp \left|\left(2-2^{s}\right) \log (s-1)\right|-1 \\
& \leq C_{1}\left|\left(2-2^{s}\right) \log (s-1)\right| \\
& \leq C_{2}|(s-1) \log (s-1)| \\
& \left.\leq C_{2}|s-1||\log | s-\left.1\right|^{-1}+\pi\right) \\
& \leq C_{3}|s-1|^{1-\varepsilon}
\end{aligned}
$$

for any fixed positive $\varepsilon<1$ and suitable constants $C_{1}, C_{2}, C_{3}$. Thus (5.4) gives

$$
\left|\sum_{n=1}^{\infty} \frac{1}{A(n)^{s}}-\frac{H(1)}{(s-1)^{2}}\right| \leq C_{4}(s-1)^{-1-\varepsilon} \quad(\operatorname{Re} s>1,|s-1|<1)
$$

for a suitable constant $C_{4}$. We may therefore apply Lemma 5.1 with $h_{n}=$ the $n$-th distinct vàlue in the range of $A, a_{n}=$ the number of times the value $h_{n}$ is taken on by $A, r=1, \theta=\varepsilon$, and

Thus the result of Theorem 5.1 follows.
6. The distribution of the numbers $A(n)$.

THEOREM 6.1. There is a positive constant $v$ such that the number of distinct rationals of the form $\sigma(n) / d(n)$ not exceeding $x$ is $O\left(x /(\log x)^{v}\right)$.

We shall use the following result of Erdös and Wagstaff [5]:
There is a positive constant $\mu$ such that the number of $n \leq x$ such that $n$ has a. divisor $p^{+1}$ with $p$ a prime, $p>T$, is $0\left(x /(\log T)^{\mu}\right)$ uniformly for all $x \geq 1$, $T \geq 2$. Actually Erdos-Wagstaff prove this for $p-1$ in place of $p+1$, but the proof is identical.

PROOF OF THE THEOREM. Let $\varepsilon>0$ be arbitrarily small. Let $x$ be large and let $S=(\log x)^{2 \mu}$, where $\mu$ is the constant in the Erdös-Wagstaff theorem. Any integer $n>0$ can be written uniquely in the form $n=s(n) \cdot m(n)=s m$ where $(s, m)=1$, $m$ is odd and square-free, and $4 s$ is square-full. Let

$$
\begin{aligned}
& N_{1}=\#\left\{n>x(\log x)^{4}: \quad \sigma(n) / d(n) \leq x\right\}, \\
& N_{2}=\#\left\{n \leq x(\log x)^{4}: P(n) \leq x^{1 / \log \log x}\right\}, \\
& N_{3}=\#\left\{n \leq x(\log x)^{4}: P(n)>x^{1 / \log \log x}, P(n)^{2} \mid n\right\},
\end{aligned}
$$

$N_{4}=\#\left\{r \leq x: r=\sigma(n) / d(n)\right.$ for some $n$ with $\left.P(n)>x^{1 / \log \log x}, P(n)| | n, s(n) \leq S\right\}$,

$$
N_{5}=\#\{r \leq x: r=\sigma(n) / d(n) \text { for some } n \text { with } s(n)>S\} \text {, }
$$

where $P(n)$ denotes the largest prime factor of $n$. Thus, if $f(x)$ denotes the number of distinct rationals not exceeding $x$ and having the form $\sigma(n) / d(n)$, we clearly have

$$
\begin{equation*}
f(x) \leq N_{1}+N_{2}+N_{3}+N_{4}+N_{5}, \tag{6.1}
\end{equation*}
$$

so that it remains to estimate these 5 quantities. Note that in the definitions of $N_{1}, N_{2}$, and $N_{3}$ we are counting the number of positive integers $n$ satisfying the conditions in question, but that in the definitions of $N_{4}$ and $N_{5}$ we are counting only distinct values of the ratio $\sigma(n) / d(n)$ arising from at least one $r$ : satisfying the conditions mentioned.

We have (see p. 240 of [24])

$$
\begin{array}{r}
N_{1} \leq \sum_{\substack{n>\\
d(n)}}^{1} \geq n / \log ^{4} x \leq \sum_{i=0}^{\infty} \sum_{2^{i} x} \quad \begin{array}{l}
1 \\
\log ^{4} x<n \leq 2^{i+1} \times \log ^{4} x \\
d(n)>2^{i} \log ^{4} x
\end{array} \\
\leq \sum_{i=0}^{\infty} \frac{1}{2^{2 i} \log ^{8} x} \quad \sum_{n \leq 2^{i+1} \times \log ^{4} x}^{d^{2}(n)}
\end{array}
$$

$$
\ll \sum_{i=0}^{\infty} \frac{1}{2^{2 i} \log ^{8} x} \cdot 2^{i+1} x \log ^{4} x \cdot(i+2 \log x)^{3}
$$

$$
\begin{equation*}
\ll \frac{x}{\log ^{4} x} \sum_{i=0}^{\infty} \frac{(i+2 \log x)^{3}}{2^{i}} \ll \frac{x}{\log x} . \tag{6.2}
\end{equation*}
$$

From Rankin [12]

$$
\begin{equation*}
N_{2} \ll x / \log x . \tag{6.3}
\end{equation*}
$$

Clearly

$$
\begin{equation*}
N_{3} \leq \sum_{d>x^{1 / \log \log x}} x(\log x)^{4} / d^{2} \ll x / \log x . \tag{6,4}
\end{equation*}
$$

We use the Erdö-Wagstaff theorent to estimate $\mathbb{N}_{4}$. If $\sigma(n) / d(n)$ is counted by $N_{4}$, then $\sigma(n) / d(n)=(\sigma(n) / d(m) j / d(s)$ where $\sigma(n) / d(m)$ is an integer and for each $\varepsilon>0$ (cf. (2.2))

$$
d(s) \leq Z \stackrel{\text { def }}{=} \max \{d(s): s \leq s\} \leq(\log x)^{E}
$$

for all large $x$ depending on the choice of $\varepsilon$. Thus the integer $2 \sigma(n) / d(m)$ is at most $2 Z x$ and is divisible by a $p+1$ where $p$ is prime, $p>x^{1 / \log \log x}$. By the Erdös-Wagstaff theorem, we thus have

$$
\begin{equation*}
N_{4} \ll x z^{2} /\left(\log \left(x^{1 / \log \log x}\right)\right)^{\mu} \ll x /(\log x)^{\mu-3 \varepsilon} . \tag{6.5}
\end{equation*}
$$

Note that if $\sigma(n) / d(n)$ is counted by $N_{5}$ and $n=s m$, then $\sigma(n) / d(n)=(\sigma(m) / d(m))(O(s) / d(s))$, so that $(\sigma(m) / d(m)) \sigma(s) \leq x d(s) \leq x s^{\varepsilon}$ for large $x$. But $\sigma(m) / d(m)$ is an integer. Thus for each fixed $s>S$ for which $4 s$ is square full, the number of $\sigma(n) / d(n) \leq x$ with $s(n)=s$ is at most the number of multiples of $\sigma(s)$ below $x s^{\varepsilon}$. Thus

$$
\begin{equation*}
N_{5} \leq \sum_{s>s} x s^{\varepsilon / \sigma(s) \leq} \sum_{s>S x / s^{1-\varepsilon} \ll x /(\log x)^{\mu-2 \varepsilon \mu} .} . \tag{6.6}
\end{equation*}
$$

Our theorem now follows from (6.1), (6.2), (6.3), (6.4), (6.5), (6.6).

Note that if $q$ runs through the primes not exceeding $2 x-1$, the $\pi(2 x-1)$ numbers $\sigma(q) / d(q)=(q+1) / 2$ lie in $[1, x]$ and are all distinct. Thus the constant $v$ in Theorem 6.1 cannot be larger than 1 . In fact, by more complicated arguments we can prove that, if $f(x)$ is as above, then for every positive $\varepsilon$ and every positive integer $k$ we have

$$
\frac{x}{\log x}(\log \log x)^{k} \ll f(x) \ll \frac{x}{(\log x)^{1-\varepsilon}} .
$$

7. Other problems.

In this section we shall state some further results, giving either sketchy proofs or no proofs at all.

THEOREM 7.1. There is a constant $c$ so that

$$
\left.\sum_{i=1}^{n} A(i)\right\}^{1 / n}-c n /(\log n)^{\log 2} .
$$

Since there is a known asymptotic formula for $\left\{\prod_{i=1}^{n} d(i)\right\}^{1 / n}$ due to
Ramanujan (cf. Wilson [24]), Theorem 7.1 can be proved by establishing an asymptotic formula for $\left\{\prod_{i=1}^{n} \sigma(i)\right\}^{1 / n}$. It is possible to give the constant $c$ explicitly and also to give arbitrarily many secondary terms.

THEOREM 7.2. The set of integers $n$ with an integral arithmetic mean for the divisors $d$ of $n$ with $1 \leq d<n$ has density 0 .

That is, Theorem 7.2 asserts that the set of $n$ for which $d(n)-1 \mid \sigma(n)-n$ has density 0 . We now sketch a proof for square-free $n$. The non-square-free case is much harder.

Assume $n$ is square-free and $K$ is large. All but a density 0 of $n$ have $2^{p}-1 \mid \sigma(n)$ for every prime $p \leq K$. For each prime $p, 2 \leq p \leq K$, the $n$ with $p \mid \omega(n)$ have relative density $1 / p$. (Here $\omega(n)$ is the number of distinct prime factors of n.) In fact, the relative density of the square-free $n$ for which

$$
\begin{equation*}
p\left|\omega(n), 2^{p}-1 / n, 2^{p}-1\right| \sigma(n) \tag{7.1}
\end{equation*}
$$

is $p^{-1}\left(1-\left(2^{p}-1\right)^{-1}\right)$. But such an $n$ has $d(n)-1 / 0(n)-n$ since $2^{p}-1 \mid d(n)-1$, $2^{p}-1 / \mathcal{O}(n)-n$. The events (7.1) for different primes $p$ are independent. Thus the relative density of the square-free $n$ for which $d(n)-1 \mid \sigma(n)-n$ is at most

$$
2 \leq \stackrel{\pi}{p} \leq K^{\left\{1-\frac{1}{p}\left(1-\frac{1}{2^{p}-1}\right)\right\} .}
$$

Letting $K \rightarrow \infty$, this product goes to 0 and we have our result for square-free $n$.
The above heuristic argument can be made the backbone of a rigorous proof. Roughly the same idea can be used in the more general case when $d(n)$ is a power of 2 .

Recall that $\sigma_{i}(n)=\int_{d \mid n} d^{i}$. Thus $\sigma(n)=\sigma_{1}(n), d(n)=\sigma_{0}(n)$.
THEOREM 7.3. Let $\delta_{i, j}$ denote the asymptotic density of the set of $n$ for which $\sigma_{i}(n) \mid \sigma_{j}(n)$, where $i, j$ are integers and $0 \leq i<j$. Then

$$
\delta_{i, j}=\left\{\begin{array}{l}
1, \text { if } j / i \text { is an odd integer; } \\
1, \text { if } i=0, j \text { is odd; } \\
0, \text { if } i \geq 1, j / i \text { is not an odd integer. }
\end{array}\right.
$$

Moreover if $i=0, j$ is even, then $\delta_{i, j}$ exists and $0<\delta_{i, j}<1$.

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