where

$$\varepsilon = \sum_{j>r+1} 1/q_j.$$

Now, by letting $N \to +\infty$ in

$$\{1-F_N(x)-F_N(-x)\}-\{1-F_{r+1,N}(x)-F_{r+1,N}(-x)\},$$

we get from (4)

$$|\Delta(x)-\Delta_{r+1}(x)| \leqslant 2\varepsilon,$$

that is,

$$|\Delta(x)| \leqslant |\Delta_{r+1}(x)|+2\varepsilon \leqslant e^{-2(K-\varepsilon)}+2\varepsilon.$$

A passage to the limit, as $r \to +\infty$, yields (12).

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On the distribution of the values of Euler's function

by

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1. Introduction. Let $V(x)$ denote the number of distinct values of $\varphi(n)$ not exceeding x, where φ denotes Euler's function. Since every number of the form $p-1$ where p is a prime is a value of $\varphi(n)$, the prime number theorem immediately gives that $V(x)$ is at least of order of magnitude $x/\log x$. On the other hand it is relatively easy to see that $V(x) = o(x)$ since most values of $\varphi(n)$ are divisible by a high power of 2 and most integers are not (see Niven and Zuckerman [8], Th. 11.9).

In 1935, Erdős [2] showed that

$$V(x) = O\left(x/(\log x)^{1-\varepsilon}\right) \tag{1.1}$$

for every $\varepsilon > 0$. This result was improved in 1973 by Erdős and Hall [3] to

$$V(x) = O\left(\frac{x}{\log x}\exp(B\sqrt{\log\log x})\right) \tag{1.2}$$

for every $B > 2\sqrt{2/\log 2}$. In 1976, Erdős and Hall [4] obtained the lower bound

$$V(x) \gg \frac{x}{\log x}\exp\{A(\log\log\log x)^2\} \tag{1.3}$$

for every $A < 1/\log 16$. (For positive $f(x)$, $g(x)$ the notation $f(x) \gg g(x)$ is equivalent to $g(x) = O(f(x))$.) In [4], Erdős and Hall state that they do not know which of (1.2), (1.3) is nearer to the truth about $V(x)$.

In this paper, I prove an upper bound for $V(x)$ of the same shape as the lower bound (1.3).

THEOREM. *For every* $C > (\log 4 - 2\log(2-\log 2))^{-1} = 1.175018095\ldots$,

$$V(x) = O\left(\frac{x}{\log x}\exp\{C(\log\log\log x)^2\}\right). \tag{1.4}$$

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In fact a slightly stronger result can be proved. Namely, by an argument outlined below, (1.4) holds for every

$$C > (2-2\log(e-1))^{-1} = 1.090096128\ldots.$$

Concerning the lower bound (1.3), a very minor alteration in the Erdős–Hall proof immediately gives (1.3) for every $A < 1/\log 9 = 0.455119613\ldots$. In addition, another improvement can be made and this is discussed below. Probably there is an asymptotic formula for $V(x)$, but I do not know what function to suggest as a choice.

2. Proof of the theorem. We begin with some notation. The letter p shall always denote a prime. Let

$$\Omega(n) = \sum_{p^a \mid n, a\geqslant 1} 1, \quad \omega(n) = \sum_{p\mid n} 1.$$

Define $w(x)$ by the equation

$$V(x) = \frac{x}{\log x} w(x)$$

and let

$$W(x) = \sup_{2\leqslant y\leqslant x} w(y).$$

Let β be an arbitrary but fixed number with

$$1-\frac{\log 2}{2} < \beta < 1. \tag{2.1}$$

Corresponding to the choice of β, let $k \geqslant 3$ be a fixed large integer and $\delta > 0$ fixed such that

$$\frac{1-\beta^k}{1-\beta}(1-\delta) > \frac{2}{\log 2}. \tag{2.2}$$

The proof is organized into five lemmas and a final argument. The first four lemmas treat cases that turn out to be negligible. The heart of the proof can be found in the fifth lemma and in its exploitation in the final argument.

Before the first lemma, we record that from [6], Th. 328, there is an absolute constant c such that if $\varphi(n) \leqslant x$ and $x \geqslant 3$, then

$$n \leqslant cx \log\log x. \tag{2.3}$$

LEMMA 1. *The number $V_1(x)$ of distinct values of $\varphi(n) \leqslant x$ such that either (i) $\Omega(n) \leqslant k+1$, (ii) $n \leqslant x/\log x$, or (iii) $d^2 \mid n$ for some $d \geqslant \log x$ is $o(V(x))$.*

Proof. From [6], Th. 437, and (2.3) the number of $\varphi(n)$ in category (i) is $O(x(\log\log x)^{k+1}/\log x)$. The number of $\varphi(n)$ in category (ii) is obviously

$O(x/\log x)$, while the number of $\varphi(n)$ in category (iii) is $O(x\log\log x/\log x)$, from (2.3). Our lemma thus follows from (1.3).

LEMMA 2. *The number $V_2(x)$ of distinct values of $\varphi(n) \leqslant x$ such that n is divisible by a prime $p > \exp\{(\log x)^{\beta^k}\}$ with $\Omega(p-1) \leqslant (1-\delta)\log\log p$ is $o\left(\frac{x}{\log x} W(x)\right)$.*

Proof. Let $\mathscr{S}$ denote the set of primes p with $p > \exp\{(\log x)^{\beta k}\}$ and $\Omega(p-1) \leqslant (1-\delta)\log\log p$. If $\varphi(n) \leqslant x$ and n is divisible by a prime $p \in \mathscr{S}$, we may assume from Lemma 1 that $n = mp$ where $p \nmid m$ and $\varphi(m) > 1$. Therefore $\varphi(n) = (p-1)\varphi(m)$ and $p \leqslant x/2+1$. The number of distinct values of $\varphi(n) \leqslant x$ associated in this way to p is thus at most $V(x/(p-1))$ and so

$$\begin{aligned} V_2(x) &\leqslant \sum_{\substack{p\leqslant x/2+1\\ p\in\mathscr{S}}} V\left(\frac{x}{p-1}\right) + o(V(x)) \\ &= \sum_{\substack{p\leqslant x/2+1\\ p\in\mathscr{S}}} \frac{x}{(p-1)\log(x/(p-1))} w(x/(p-1)) + o(V(x)) \\ &\leqslant xW(x) \sum_{\substack{p\leqslant x/2+1\\ p\in\mathscr{S}}} \frac{1}{(p-1)\log(x/(p-1))} + o(V(x)). \end{aligned} \tag{2.4}$$

From Erdős [2], there is a $\delta' = \delta'(\delta) > 0$ such that the number of members of $\mathscr{S}$ up to z is $O(z/(\log z)^{1+\delta'})$. For notational simplicity, let $\gamma = \beta^k$. We have by partial summation

$$\begin{aligned} \sum_{\substack{p\leqslant x/2+1\\ p\in\mathscr{S}}} \frac{1}{(p-1)\log(x/(p-1))} &\leqslant \int_{\exp\{(\log x)^{\gamma}\}}^{x/2} \frac{dt}{t(\log x - \log t)(\log t)^{1+\delta'}} \\ &\leqslant (\log x)^{-\gamma\delta'} \int_{\exp\{(\log x)^{\gamma}\}}^{x/2} \frac{dt}{t(\log x-\log t)\log t} \\ &= (\log x)^{-1-\gamma\delta'} [\log\log t - \log(\log x - \log t)]_{\exp\{(\log x)^{\gamma}\}}^{x/2} \\ &\ll \frac{\log\log x}{(\log x)^{1+\gamma\delta'}}. \end{aligned}$$

From (2.4), we thus have $V_2(x) = o(xW(x)/\log x)$.

For $i \geqslant 1$, let $P_i(n)$ denote the ith largest prime factor of n if $\Omega(n) \geqslant i$. Otherwise, let $P_i(n) = 1$.

LEMMA 3. *The number $V_3(x)$ of distinct values of $\varphi(n) \leqslant x$ such that*

$$(2.5)\qquad P_i(n) > \exp\{(\log x)^{\beta^{i-1}}/\log\log x\} \quad \text{for} \quad i = 1, \ldots, k$$

is $o(xW(x)/\log x)$.

Proof. Let n be such that $\varphi(n) \leqslant x$ and (2.5) holds. Let $p_i = P_i(n)$. From Lemma 1 we may assume that $p_1, \ldots, p_k$ are distinct. From Lemma 2, we may assume that

$$\Omega(p_i - 1) > (1-\delta)\log\log p_i \quad \text{for} \quad i = 1, \ldots, k.$$

Therefore from (2.2)

$$\Omega(\varphi(n)) \geqslant \sum_{i=1}^{k} \Omega(p_i - 1) > (1-\delta)\frac{1-\beta^k}{1-\beta}\log\log x - k\log\log\log x \geqslant \frac{2}{\log 2}\log\log x$$

for x large. From Nicolas [7], the number of integers $m \leqslant x$ with $\Omega(m) > (2/\log 2)\log\log x$ is $O(x/\log x) = o(V(x))$. The lemma thus follows.

It is to be remarked that Lemma 3 could equally well exploit Lemma 1 in [3] rather than the Nicolas result.

LEMMA 4. *The number $V_4(x)$ of distinct values of $\varphi(n) \leqslant x$ where n is divisible by a number $m \geqslant \exp\{(\log x)^{\beta^k}\}$ with $P_1(m) \leqslant m^{1/\log\log x}$ is $o(V(x))$.*

Proof. The number of $m \leqslant z$ with $P_1(m) \leqslant m^{1/\log\log x}$ is at most $\Psi(z, z^{1/\log\log x})$, where $\Psi(x, y)$ denotes the number of $n \leqslant x$ with $P_1(n) \leqslant y$. From de Bruijn [1], we have

$$\Psi(z, z^{1/\log\log x}) = z\cdot\exp\{-(1+o(1))\log\log x \log\log\log x\}$$

uniformly for $z \geqslant \exp\{(\log\log x)^3\}$. Therefore, by partial summation, the number of $n \leqslant cx\log\log x$ divisible by such an $m \geqslant \exp\{(\log x)^{\beta^k}\}$ is $o(V(x))$.

LEMMA 5. *The number $V_5(x)$ of distinct values of $\varphi(n) \leqslant x$ such that*

$$(2.6)\qquad P_i(n) \leqslant \exp\{(\log x)^{\beta^{i-1}}/\log\log x\} \quad \text{for some } i = 1, \ldots, k$$

satisfies

$$V_5(x) \leqslant \sum_{j=1}^{k-1} \frac{x(\log\log x)^j}{\log x} W(\exp\{(\log x)^{\beta^j}\}) + o(V(x)).$$

Proof. For $i = 1, \ldots, k$, let $V_{5,i}(x)$ denote the number of distinct values of $\varphi(n) \leqslant x$ such that (2.6) holds for i, but fails for $j = 1, \ldots, i-1$. As in the proof of Lemma 4, from de Bruijn [1], $V_{5,1}(x) = o(V(x))$. Say now $2 \leqslant i \leqslant k$, $\varphi(n) \leqslant x$, and (2.6) holds for i, but fails for $j = 1, \ldots, i-1$. Let

$$q = P_1(n) \ldots P_{i-1}(n), \qquad m = n/q.$$

Since $P_i(m) \leqslant \exp\{(\log x)^{\beta^{i-1}}/\log\log x\}$, from Lemma 4 we may assume

$$(2.7)\qquad m \leqslant \exp\{(\log x)^{\beta^{i-1}}\}.$$

From Lemma 1 and (2.7) we may assume $(q, m) = 1$ and

$$(2.8)\qquad q \geqslant \frac{x}{\log x \cdot \exp\{(\log x)^{\beta^{i-1}}\}}, \qquad \varphi(q) \leqslant x/2.$$

For a fixed q, the number of distinct values of $\varphi(n) \leqslant x$ with $n = qm$, $(q, m) = 1$, and m satisfying (2.7) is at most

$$\min\left\{V\left(\frac{x}{\varphi(q)}\right), V(\exp\{(\log x)^{\beta^{i-1}}\})\right\}.$$

Thus if $\sum'$ denotes a sum over q composed of $i-1$ primes exceeding $\exp\{(\log x)^{\beta^{i-2}}/\log\log x\}$ and satisfying (2.8), then

$$\begin{aligned} V_{5,i}(x) &\leqslant \sum{}' \frac{x}{\varphi(q)\log(x/\varphi(q))} W(\exp\{(\log x)^{\beta^{i-1}}\}) + o(V(x)) \\ &\leqslant \left(\frac{\beta^{i-1}}{(i-2)!} + o(1)\right)\frac{x(\log\log x)^{i-1}}{\log x} W(\exp\{(\log x)^{\beta^{i-1}}\}) + o(V(x)) \\ &\leqslant \frac{x(\log\log x)^{i-1}}{\log x} W(\exp\{(\log x)^{\beta^{i-1}}\}) + o(V(x)). \end{aligned}$$

Summing over i, we have the lemma.

Proof of the theorem. From the lemmas, we have

$$(1+o(1))\frac{x}{\log x} W(x) \leqslant \sum_{j=1}^{k-1} \frac{x(\log\log x)^j}{\log x} W(\exp\{(\log x)^{\beta^j}\}).$$

Thus for $x \geqslant x_0$,

$$W(x) \leqslant 2\sum_{j=1}^{k-1} (\log\log x)^j W(\exp\{(\log x)^{\beta^j}\}).$$

Replacing the sum with its largest term, we have for some $1 \leqslant j_1 < k$,

$$(2.9)\qquad W(x) \leqslant 2k(\log\log x)^{j_1} W(\exp\{(\log x)^{\beta^{j_1}}\}) \quad \text{for} \quad x \geqslant x_0.$$

We iterate the inequality (2.9). Thus there is some $1 \leqslant j_2 < k$ with

$$W(x) \leqslant (2k)^2 \beta^{j_1 j_2}(\log\log x)^{j_1 + j_2} W(\exp\{(\log x)^{\beta^{j_1+j_2}}\}).$$

After t iterations we have

$$(2.10)\qquad W(x) \leqslant (2k)^t \beta^{\Sigma_2}(\log\log x)^{\Sigma_1} W(\exp\{(\log x)^{\beta^{\Sigma_1}}\})$$

where Σ_2, Σ_1 are abbreviations for

$$\sum_{1\le\mu<\nu\le t} j_\mu j_\nu, \qquad \sum_{\mu=1}^{t} j_\mu,$$

respectively, and each $j_\mu \in \{1, \ldots, k-1\}$. We continue this iteration until

$$(\log x)^{\beta^{\Sigma_1}} \le x_0.$$

Thus we choose t so that

$$\sum_{\mu=1}^{t} j_\mu = -\frac{\log\log\log x}{\log\beta} + O(1).$$

Since

$$\sum_{1\le\mu<\nu\le t} j_\mu j_\nu = \tfrac{1}{2}\Big(\sum_{\mu=1}^{t} j_\mu\Big)^2 - \tfrac{1}{2}\sum_{\mu=1}^{t} j_\mu^2$$

and each $j_\mu \in \{1, \ldots, k-1\}$, we thus have

$$\sum_{1\le\mu<\nu\le t} j_\mu j_\nu = \frac{(\log\log\log x)^2}{2\log^2\beta} + O(\log\log\log x).$$

Therefore, from (2.10) we have

$$W(x) \le \exp\left\{-\frac{(\log\log\log x)^2}{2\log\beta} + O(\log\log\log x)\right\}. \tag{2.11}$$

Since we may choose β arbitrarily close to $1-(\log 2)/2$, we have the theorem.

3. An improvement. I indicate now how the theorem can be strengthened to the assertion that (1.4) holds for every $C > \big(2-2\log(e-1)\big)^{-1}$. Indeed, the proof follows the same general outline as in Section 2. The first change is that the number $(\log 2)/2$ appearing in (2.1) is replaced with $1/e$ and the number $2/\log 2$ appearing in (2.2) is replaced with e. Thus β can be chosen arbitrarily close to $1-1/e$.

Let $\omega_x(n)$ denote the number of distinct prime factors of n that exceed $\log x$. The next change is in Lemma 2, where we replace $\Omega(p-1)$ with $\omega_x(p-1)$. From the arguments in Erdös [2], there is a $\delta'' = \delta''(\delta) > 0$ such that the number of primes $p \le z$ with

$$\omega_x(p-1) \le (1-\delta)\log\log p$$

is $O\big(z/(\log z)^{1+\delta''}\big)$. Thus we can use the same proof as for Lemma 2.

The last change is in the proof of Lemma 3. We replace Ω with ω_x, but it is not immediately obvious that we can write

$$\omega_x\big(\varphi(n)\big) \ge \sum_{i=1}^{k} \omega_x(p_i-1). \tag{3.1}$$

However, the numbers $\varphi(n) \le x$ divisible by some d^2 with $d > \log x$ are negligible as in the proof of Lemma 1. Thus we may ignore those values $\varphi(n)$ where (3.1) fails. Thus we will have

$$\omega\big(\varphi(n)\big) \ge \omega_x\big(\varphi(n)\big) > e\log\log x.$$

But from the Hardy–Ramanujan inequality [5], the number of integers $m \le x$ with $\omega(m) > e\log\log x$ is $O(x/\log x) = o\big(V(x)\big)$.

The rest of the proof proceeds in exactly the same way. We finally arrive at (2.11) where $\beta > 1-1/e$ is arbitrary, which proves the stronger assertion.

4. An improved lower bound. Let $0 < \beta < 1$ and consider the set of integers M_k of the form $(p_1-1)(p_2-1)\ldots(p_j-1)$ where $j \le k$, the p_i's are primes, and for each $i = 2, \ldots, j$,

$$(p_1-1)\ldots(p_{i-1}-1) < \exp\{(\log p_i)^\beta\}.$$

Further suppose that uniformly for all $k < 100\log\log\log y$,

$$\sum_{\substack{m_1, m_2 \le y\\ m_1, m_2 \in M_k}} \frac{(m_1, m_2)}{m_1 m_2} \le (\log y)^{1/\beta + o(1)} \tag{4.1}$$

holds for all $y \ge 20$. Then from the proof given in [4] it follows that (1.3) holds for all $A < -1/(2\log\beta)$.

It is remarked in [4] that

$$\sum_{m_1, m_2 \le y} \frac{(m_1, m_2)}{m_1 m_2} = O\big((\log y)^3\big)$$

holds where there is no requirement that $m_1, m_2 \in M_k$. Thus (4.1) holds with $\beta = 1/3$ and so the proof in [4] gives (1.3) for all $A < 1/\log 9$.

By using the restriction $m_1, m_2 \in M_k$, we can show (4.1) for a larger choice of β and thus prove (1.3) for larger values of A. We begin by making the harmless requirement that the primes p_i used in the construction of the set M_k be "normal" primes in that

$$\omega(p_i-1) \approx \Omega(p_i-1) \approx \log\log p_i.$$

Thus if $m \in M_k$, then (as in Section 3)

$$\omega(m) \approx \Omega(m) \approx \frac{1-\beta^j}{1-\beta}\log\log m$$

for some $j \le k$, so that (where $d(m)$ counts the number of divisors of m)

$$d(m) \approx (\log m)^{((1-\beta^j)/(1-\beta))\log 2}.$$

We shall therefore assume that if $m \in M_k$ and $m \le y$, then

$$d(m) \le (\log y)^{\log 2/(1-\beta)}. \tag{4.2}$$

Thus,

$$\sum_{\substack{m_1,m_2\leqslant y\\ m_1,m_2\in M_k}} \frac{(m_1, m_2)}{m_1 m_2} \leqslant \sum_{\substack{m_1\leqslant y\\ m_1\in M_k}} \frac{1}{m_1}\sum_{d|m_1} d \sum_{\substack{m_2\leqslant y\\ d|m_2\\ m_2\in M_k}} \frac{1}{m_2}. \tag{4.3}$$

If we majorize the inner sum trivially by $(\log y)/d$, we obtain

$$\sum_{\substack{m_1,m_2\leqslant y\\ m_1,m_2\in M_k}} \frac{(m_1, m_2)}{m_1 m_2} \leqslant \log y \sum_{\substack{m\leqslant y\\ m\in M_k}} \frac{d(m)}{m} \leqslant (\log y)^{1+\log 2/(1-\beta)} \sum_{\substack{m\leqslant y\\ m\in M_k}} \frac{1}{m},$$

using (4.2). But from (1.1) this last sum is $(\log y)^{o(1)}$. Thus if we choose β so that

$$\frac{1}{\beta} = 1+\frac{\log 2}{1-\beta}$$

we have (4.1). This leads to the value

$$\beta = \tfrac{1}{2}\left(2+\log 2-\sqrt{4\log 2+\log^2 2}\right)$$

and establishes (1.3) for every $A < 0.617122930\ldots$.

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The weighted linear sieve and Selberg's λ^2-method

by

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1. Introduction. In the weighted linear sieve we study sequences $\mathscr{A} = \mathscr{A}_X$, depending on a real parameter $X \geqslant 2$, which satisfy certain general conditions of the type described by Halberstam and Richert [5]. The conditions specified in this paper are labelled (Ω_1), (Ω_2), (R), (D) below. As usual these are chosen with due regard to applicability on one hand (cf. the examples provided in [5], for example), and on the other hand to the requirements of a workable proof of a result of the type established in this paper. The object of the exercise is to deduce, for a suitably small integer $R \geqslant 2$, that the sequence $\mathscr{A}$ contains many numbers having no more than R prime factors.

In [2] the present author obtained an improvement on the results previously known on this problem via a study of expressions of the type

$$\sum_{d|a} \mu(d)\chi_y^-(d)\{W(1)-\sum_{p|d} w(p)\}, \tag{1.1}$$

where $w(p)$ was a 'weight' function of the type appearing in earlier approaches to this problem (see Chapter 9 of [5], for example) and $\chi_y^-(d)$ was the 'characteristic' function appearing in Iwaniec's and Rosser's version (see [7], [9]) of Brun's sieve. Thus in the case $w(p) = 0$ the expression (1.1) reduces to the corresponding expression studied in [7], [9]. In this "unweighted" context an essentially equivalent result had been obtained by Jurkat and Richert [10] (see also Chapter 8 of [5]), using a method in which the well-known λ^2-device of Selberg played a significant rôle. In this paper we replace the expression $\chi_y^-(d)$ in (1.1) by the expression implicit in the paper of Jurkat and Richert. We shall see that in the problem of the weighted linear sieve the expressions are not equivalent, in that we shall obtain an improvement, in certain cases, upon the result in the author's earlier paper [2]. At the same time the result obtained falls short of that which seems to be generally conjectured to be true, and which would be best possible.