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SIEVING BY LARGE INTEGERS AND COVERING SYSTEMS OF CONGRUENCES

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1. Introduction

Notice that every integer n satisfies at least one of the congruences

$$n \equiv 0 \pmod{2},\ n \equiv 0 \pmod{3},\ n \equiv 1 \pmod{4},\ n \equiv 1 \pmod{6},\ n \equiv 11 \pmod{12}.$$

A finite set of congruences, where each integer satisfies at least one them, is called a *covering system*. A famous problem of Erdős from 1950 [4] is to determine whether for every N there is a covering system with distinct moduli greater than N. In other words, can the minimum modulus in a covering system with distinct moduli be arbitrarily large? In regards to this problem, Erdős writes in [6], "This is perhaps my favourite problem."

It is easy to see that in a covering system, the reciprocal sum of the moduli is at least 1. Examples with distinct moduli are known with least modulus 2, 3, and 4, where this reciprocal sum can be arbitrarily close to 1; see [10], §F13. Erdős and Selfridge [5] conjectured that this fails for all large enough choices of the least modulus. In fact, they made the following much stronger conjecture.

Conjecture 1. *For any number B, there is a number N_B, such that in a covering system with distinct moduli greater than N_B, the sum of reciprocals of these moduli is greater than B.*

A version of Conjecture 1 also appears in [7].

Whether or not one can cover all of $\mathbb{Z}$, it is interesting to consider how much of $\mathbb{Z}$ one can cover with residue classes $r(n) \pmod{n}$, where the moduli n come from an interval $(N, KN]$ and are distinct. In this regard, Erdős and Graham [7] have formulated the following conjecture.

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Conjecture 2. *For each number $K > 1$ there is a positive number d_K such that if N is sufficiently large, depending on K, and if we choose arbitrary integers $r(n)$ for each $n \in (N, KN]$, then the complement in $\mathbb{Z}$ of the union of the residue classes $r(n) \pmod{n}$ has density at least d_K.*

In [6], Erdős writes with respect to establishing such a lower bound d_K for the density, "I am not sure at all if this is possible and I give \$100 for an answer."

A corollary of either Conjecture 1 or Conjecture 2 is the following conjecture also raised by Erdős and Graham in [7].

Conjecture 3. *For any number $K > 1$ and N sufficiently large, depending on K, there is no covering system using distinct moduli from the interval $(N, KN]$.*

In this paper we prove strong forms of Conjectures 1, 2, and 3.

Despite the age and fame of the minimum modulus problem, there are still many more questions than answers. We mention a few results. Following earlier work of Churchhouse, Krukenberg, Choi, and Morikawa, Gibson [9] has recently constructed a covering system with minimum modulus 25, which stands as the largest known least modulus for a covering system with distinct moduli. As has been mentioned, if $r_i \pmod{n_i}$ for $i = 1, 2, \dots, l$ is a covering system, then $\sum 1/n_i \geq 1$. Assuming that the moduli n_i are distinct and larger than 1, it is possible to show that equality cannot occur, that is, $\sum 1/n_i > 1$. The following proof (of M. Newman) is a gem. Suppose that $\sum 1/n_i = 1$. If the system then covers, a density argument shows that there cannot be any overlap between the residue classes, that is, we have an *exact covering system*. We suppose, as we may, that $n_1 < n_2 < \cdots < n_l$ and each $r_i \in [0, n_i - 1]$. Then

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{i=1}^{l} \left(z^{r_i} + z^{r_i + n_i} + z^{r_i + 2n_i} + \dots \right) = \sum_{i=1}^{l} \frac{z^{r_i}}{1 - z^{n_i}}.$$

The right side of this equation has poles at the primitive n_l-th roots of 1, which is not true of the left side. Thus, there cannot be an exact covering system with distinct moduli greater than 1 (in fact, the largest modulus must be repeated).

Say an integer H is "covering" if there is a covering system with distinct moduli with each modulus a divisor of H exceeding 1. For example, 12 is covering, as one can see from our opening example. From the above result, if H is covering, then $\sigma(H)/H > 2$, where σ is the sum-of-divisors function. Benkoski and Erdős [2] wondered if $\sigma(H)/H$ were large enough, would this condition suffice for H to be covering. In [11], Haight showed that this is not the case. We obtain a strengthening of this result, and by a shorter proof.

If $n_1, n_2, \dots, n_l$ are positive integers and $C = \{(n_i, r_i) : i = 1, 2, \dots, l\}$ is a set of ordered pairs, let $\delta = \delta(C)$ be the density of the integers that are *not* in the union of the residue classes $r_i \pmod{n_i}$. If $n_1, n_2, \dots, n_l$ are pairwise coprime, there is no mystery about δ. Indeed, the Chinese remainder theorem implies that for any choice of residues $r_1, r_2, \dots, r_l$,

$$\delta = \prod_{i=1}^{l} (1 - 1/n_i),$$

which is necessarily positive if each $n_i > 1$.

One central idea in this paper is to determine how to estimate δ when the moduli are not necessarily pairwise coprime. We note that for any $n_1, n_2, \dots, n_l$, there is a choice for $r_1, r_2, \dots, r_l$ such that

$$\delta \le \prod_{i=1}^{l}(1 - 1/n_i). \tag{1.1}$$

Indeed, this is obvious if $l = 1$. Assume it is true for l, and say we have chosen residues $r_1, r_2, \dots, r_l$ such that the residual set R has density δ satisfying (1.1). The residue classes modulo n_{l+1} partition any subset of $\mathbb{Z}$, and in particular partition R, so that at least one of these residue classes, when intersected with R, has density at least δ/n_{l+1}. Removing such a residue class, the residual set for the $l+1$ congruences thus has density at most

$$\delta - \delta/n_{l+1} = (1 - 1/n_{l+1})\delta \le \prod_{i=1}^{l+1}(1 - 1/n_i).$$

Thus, the assertion follows.

Note that

$$\prod_{N<n\le KN}(1 - 1/n) = \lfloor N \rfloor / \lfloor KN \rfloor \to 1/K \text{ as } N \to \infty,$$

so that d_K in Conjecture 2 must be at most $1/K$. We show in Section 4 that any number $d < 1/K$ is a valid choice for d_K.

A key lemma in our paper allows us to almost reverse the inequality (1.1) for δ. Namely we show that for any choice of residues $r_1, r_2, \dots, r_l$,

$$\delta \ge \prod_{i=1}^{l}(1 - 1/n_i) - \sum_{\substack{i<j \\ \gcd(n_i,n_j)>1}} \frac{1}{n_i n_j}. \tag{1.2}$$

We then maneuver to show that under certain conditions the product is larger than the sum, so that no choice of residue classes r_i allows a covering. As kindly pointed out to us by the referee, the inequality (1.2) bears a resemblance to the Lovász Local Lemma but seems to be independent of it. We shall discuss this connection more in the next section.

If S is a finite set of positive integers, let $\delta^-(S)$ be the minimum value of $\delta(C)$ where C runs over all choices of $\{(n, r(n)) : n \in S\}$. That is, we are given the moduli $n \in S$, and we choose the residue classes $r(n) \pmod n$ so as to cover as much as possible from $\mathbb{Z}$; then $\delta^-(S)$ is the density of the integers not covered. Furthermore, let

$$\alpha(S) = \prod_{n\in S}(1 - 1/n),$$

so that (1.1) implies we have $\delta^-(S) \le \alpha(S)$. With this notation we now state our principal results.

Theorem A. *Let $0 < c < 1/3$ and let N be sufficiently large (depending on c). If S is a finite set of integers $n > N$ such that*

$$\sum_{n\in S}\frac{1}{n} \le c\frac{\log N \log\log\log N}{\log\log N},$$

then $\delta^-(S) > 0$.

Theorem B. *For any numbers c with $0 < c < 1/2$, $N \geq 20$, and K with*

$$1 < K \leq \exp(c \log N \log\log\log N / \log\log N),$$

if S is a set of integers contained in $(N, KN]$, then

$$\delta^-(S) = (1 + o(1))\alpha(S)$$

as $N \to \infty$, where the function "$o(1)$" depends only on the choice of c.

Theorems A and B are proved in Section 4. Using Lemma 3.4 below, we can make the $o(1)$ term in Theorem B explicit in terms of N and c. Both Theorems A and B, as well as several other of our results, are proved in a more general context of multisets S or, equivalently, where multiple residue classes are allowed for each modulus. Note that Theorems A and B prove Conjectures 1 and 2, respectively, and so Conjecture 3 as well.

In the context of Theorem B, if we relax the upper bound on the largest modulus, we are able to construct examples of sets of integers S with least member arbitrarily large and where $\delta^-(S)$ is much smaller than $\alpha(S)$. Proved in Section 5, this result might be interpreted as lending weight towards the existence of covering systems with the least modulus being arbitrarily large.

Similar to the definition of $\delta^-(S)$, let $\delta^+(S)$ be the *largest* possible density for a residual set with S being a set of (distinct) moduli. It was shown by Rogers (see [13], pp. 242–244) that for any finite set of positive integers S, the density $\delta^+(S)$ is attained when we choose the residue class 0 (mod n) for each $n \in S$. That is, $\delta^+(S)$ is the density of integers not divisible by any member of S. There is an extensive literature on estimating $\delta^+(S)$ when S consists of all integers in an interval (see e.g. [8] and Chapter 2 of [14]). In particular, it is known from early work of Erdős [3] that for each $\varepsilon > 0$ there is some $\eta > 0$, such that if S is the set of integers in $(N, N^{1+\eta}]$, then $\delta^+(S) \geq 1 - \varepsilon$ for all large N. In fact, we almost have an asymptotic estimate for $1 - \delta^+(S)$ for such a set S: Among other results, it is shown in Theorem 1 of [8] that for $0 < \eta < 1/2$ and $N \geq 2^{1/\eta}$, $\delta^+(S)$ is between $1 - c_1\eta^\theta(\log 1/\eta)^{-3/2}$ and $1 - c_2\eta^\theta(\log 1/\eta)^{-3/2}$, where c_1, c_2 are positive absolute constants and where $\theta = 1 - (1 + \log\log 2)/\log 2 = 0.08607\ldots$.

In the above example with $\eta > 0$ fixed, we have $\delta^-(S) \leq \alpha(S) = (1+o(1))N^{-\eta} = o(1)$, while for large N, $\delta^+(S)$ is bounded away from 0. If the residue classes are chosen randomly, should we expect the density of the residual set to be closer to $\delta^-(S)$, $\alpha(S)$, or $\delta^+(S)$? We show in Sections 5 and 6 that for any finite integer set S, the average (and typical) case has residual density close to $\alpha(S)$.

Finally we mention a problem we have not been able to settle. Is it true that for each positive number B, there are positive numbers Δ_B, N_B, such that if S is a finite set of positive integers greater than N_B with reciprocal sum at most B, then $\delta^-(S) \geq \Delta_B$? If this holds it would imply each of Conjectures 1, 2, and 3. For more problems and results concerning covering systems, the reader is directed to [16] and [18].

2. A basic lemma and Haight's theorem

To set some notation, we shall always have n a positive integer, with $P(n) = P^+(n)$ the largest prime factor of n for $n > 1$ and $P(1) = 0$. We shall also let $P^-(n)$ denote the least prime factor of n when $n > 1$, and $P^-(1) = +\infty$. The letter p will always represent a prime variable. We use N, K, Q to represent real numbers, usually large. We use the Vinogradov notation $\ll$ from analytic number theory, so that $A \ll B$ is the same as $A = O(B)$, but it is cleaner to use in a chain of inequalities. In addition, $A \gg B$ is the same as $B \ll A$. All constants implied by this notation are absolute and bounds for them are computable in principle. If S is a multiset and we have some product or sum with $n \in S$, it is expected that n is repeated as many times in the product or sum as it appears in S.

Let C be a finite set of ordered pairs of positive integers (n, r), which we interpret as a set of residue classes $r \pmod n$. We say such a set is a *residue system*. Let $S = S(C)$ be the multiset of the moduli n appearing in C. We call the number of times an integer n appears in S the *multiplicity* of n. By $R(C)$ we denote the set of integers *not* congruent to $r \pmod n$ for any $(n, r) \in C$. Since $R(C)$ is a union of residue classes modulo the least common multiple of the members of $S(C)$, it follows that $R(C)$ possesses a (rational) asymptotic density, which we denote by $\delta(C)$. If $C = \{(n_1, r_1), \ldots, (n_l, r_l)\}$, then we set

$$\alpha(C) = \prod_{n \in S(C)} \left(1 - \frac{1}{n}\right) = \prod_{j=1}^{l} \left(1 - \frac{1}{n_j}\right), \quad \beta(C) = \sum_{\substack{i<j \\ \gcd(n_i, n_j) > 1}} \frac{1}{n_i n_j}.$$

Note that $\alpha(C)$ depends only on $S(C)$, so it is notationally consistent with $\alpha(S)$ from Section 1.

Lemma 2.1. *For any residue system C, we have $\delta(C) \geq \alpha(C) - \beta(C)$.*

Proof. Let $\alpha = \alpha(C)$ and $\beta = \beta(C)$. We use induction on l. If $l = 1$, then $\beta = 0$ and the statement is trivial. Let $l > 1$; we will describe an induction step from $l-1$ to l. We denote $C' = \{(n_1, r_1), \ldots, (n_{l-1}, r_{l-1})\}$,

$$\alpha' = \alpha(C') = \prod_{j=1}^{l-1} \left(1 - \frac{1}{n_j}\right), \quad \text{and} \quad \beta' = \beta(C') = \sum_{\substack{i<j\leq l-1 \\ \gcd(n_i, n_j) > 1}} \frac{1}{n_i n_j}.$$

By the induction supposition,

$$\delta(C') \geq \alpha' - \beta'. \tag{2.1}$$

Let $C'' = \{(n_j, r_j) : j < l, \ \gcd(n_j, n_l) = 1\}$, so that

$$\delta(C'') \leq \delta(C') + \sum_{n_j \in S(C' \setminus C'')} \frac{1}{n_j} = \delta(C') + \sum_{\substack{j<l \\ \gcd(n_j, n_l) > 1}} \frac{1}{n_j}. \tag{2.2}$$

The density of integers covered by the residue class $r_l \pmod{n_l}$ but not covered by $r_j \pmod{n_j}$ for every $n_j \in S(C'')$ is equal to $\delta(C'')/n_l$. Therefore,

$$\begin{aligned} \delta(C') - \delta(C) &= \text{density}\{n \equiv r_l \pmod{n_l} : n \in R(C')\} \\ &\leq \text{density}\{n \equiv r_l \pmod{n_l} : n \in R(C'')\} = \delta(C'')/n_l, \end{aligned}$$

so that, by (2.1) and (2.2),

$$\begin{aligned}\delta(C) &\geq \delta(C') - \Bigg(\delta(C') + \sum_{\substack{j<l \\ \gcd(n_j,n_l)>1}} \frac{1}{n_j}\Bigg)\frac{1}{n_l} \\ &= \left(1-\frac{1}{n_l}\right)\delta(C') - \sum_{\substack{j<l \\ \gcd(n_j,n_l)>1}} \frac{1}{n_j n_l} \\ &\geq \left(1-\frac{1}{n_l}\right)(\alpha'-\beta') - (\beta-\beta') \geq \left(1-\frac{1}{n_l}\right)\alpha' - \beta = \alpha-\beta.\end{aligned}$$

This completes the proof of the lemma. □

Remark 1. The proof of Lemma 2.1 actually gives the better bound

$$\delta(C) \geq \alpha(C) - \sum_{\substack{i<j \\ \gcd(n_i,n_j)>1}} \frac{1}{n_i n_j} \prod_{u>j}\left(1-\frac{1}{n_u}\right).$$

Remark 2. The referee has pointed out to us that Lemma 2.1 can be formulated in a more general way involving a finite number of events in a probability space. In particular suppose that $E_1, E_2, \ldots, E_l$ are events in a probability space with the property that if E_i is independent individually of the events $E_{j_1}, E_{j_2}, \ldots E_{j_k}$, then it is independent of every event in the sigma algebra generated by $E_{j_1}, E_{j_2}, \ldots, E_{j_k}$. Then

$$\mathrm{P}\left(\textstyle\bigcap_{i=1}^{l} \overline{E}_i\right) \geq \prod_{i=1}^{l} \mathrm{P}(\overline{E_i}) - \sum_{\substack{1\leq i<j\leq l \\ E_i, E_j \text{ dependent}}} \mathrm{P}(E_i)\mathrm{P}(E_j). \tag{2.3}$$

We can retrieve Lemma 2.1 from this statement if we let E_i be the event that an integer n is in the residue class $r_i \pmod{n_i}$. Indeed, E_i is independent of E_j if and only if n_i and n_j are coprime. The extra condition involving the sigma algebra is easily seen to hold (and was used strongly in our proof). The proof of (2.3) is the same as that of Lemma 2.1, namely an induction on l. This result bears a resemblance to the Lovász Local Lemma (for example, see [1]) and in some situations may be stronger.

There is a very interesting negative result of Haight [11]. As in the introduction, we say an integer H is covering if there is a covering system with the moduli being the (distinct) divisors of H that are larger than 1. It is shown in [11] that there exist integers H that are *not* covering, yet $\sum_{d|H} 1/d = \sigma(H)/H$ is arbitrarily large. Although Haight's theorem follows directly from Theorem A (by taking K fixed, N large and H the product of the primes in $(N, N^K]$), Lemma 2.1 by itself leads to a new (and short) proof of a stronger version of Haight's result:

Theorem 1. *There is an infinite set of positive integers H with*

$$\sigma(H)/H = (\log\log H)^{1/2} + O(\log\log\log H),$$

such that for any residue system C with $S(C) = \{d : d > 1,\ d \mid H\}$, we have

$$\delta(C) \geq (1+o(1))\alpha(C).$$

In particular, for large H in this set, no such C can have $\delta(C) = 0$.

Proof. Let N be a large parameter, and let

$$H = \prod_{e^{\sqrt{\log N}} \log N < p \le N} p.$$

Then

$$\log \sum_{d|H} \frac{1}{d} = \sum_{e^{\sqrt{\log N}} \log N < p \le N} \left(\frac{1}{p} + O\left(\frac{1}{p^2}\right)\right) = \frac{1}{2} \log\log N - \frac{\log\log N + O(1)}{\sqrt{\log N}},$$

by Mertens' theorem. Thus, as $\log H = (1+o(1))N$ by the prime number theorem, we have

$$\frac{\sigma(H)}{H} = \sum_{d|H} \frac{1}{d} = (\log N)^{1/2} - \log\log N + O(1) = (\log\log H)^{1/2} + O(\log\log\log H).$$

Let C be a residue system with $S(C) = \{d : d > 1,\ d \mid H\}$. We have

$$\log \alpha(C) = \sum_{d \in S(C)} \log(1 - 1/d) = - \sum_{d \in S(C)} 1/d + O\left(\exp(-\sqrt{\log N})\right),$$

so that

$$\alpha(C) = \exp\left(-\sqrt{\log N} + O(1)\right) \log N.$$

Also,

$$\beta(C) \le \sum_{d>1} \sum_{\substack{d_1, d_2 \in S(C) \\ d|d_1,\ d|d_2}} \frac{1}{d_1 d_2} \le \sum_{d|H,\ d>1} \frac{1}{d^2} \sum_{\substack{d_1|H \\ d_2|H}} \frac{1}{d_1 d_2} \ll \log N \sum_{d|H,\ d>1} \frac{1}{d^2}.$$

Furthermore,

$$\sum_{d|H,\ d>1} \frac{1}{d^2} \le \sum_{d > e^{\sqrt{\log N}} \log N} \frac{1}{d^2} \ll \exp\left(-\sqrt{\log N}\right) (\log N)^{-1}.$$

Thus,

$$\beta(C) \ll \exp\left(-\sqrt{\log N}\right) = o(\alpha(C))$$

and the theorem follows from Lemma 2.1. □

Remark 3. An examination of our proof shows that we have a more general result. Let $\mathcal{H}$ be the set of integers H which have no prime factors below

$$\exp(\sqrt{\log\log H}) \log\log H.$$

As $H \to \infty$ in $\mathcal{H}$, we have for any residue system C with $S(C) = \{d : d > 1,\ d \mid H\}$ that $\delta(C) \ge (1 + o(1))\alpha(C)$. In particular, at most finitely many integers $H \in \mathcal{H}$ are covering.

We also remark that the proof gives the following result. Say that a positive integer H is s-covering, if for each $d \mid H$ with $d > 1$ there are s integers $r_{d,1}, \dots, r_{d,s}$ such that the union of the residue classes $r_{d,i} \pmod d$ for $i = 1, \dots, s$ and $d \mid H$ with $d > 1$ is $\mathbb{Z}$. Then for each fixed $\varepsilon > 0$ there are values of H where $\sigma(H)/H$ is arbitrarily large, yet H is not s-covering with $s = [(\log\log H)^{1-\varepsilon}]$. Indeed, take H to be the product of the primes in $\left(\exp\left((\log N)^{1-\varepsilon/3}\right), N\right]$ and follow the same proof. This too strengthens a result in [11].

3. The smooth-number decomposition

The relative ease of using Lemma 2.1 in the proof of Haight's theorem is due to the fact that the moduli that we produce for the proof have no small prime factors, so that it is easy to bound the sum for $\beta(C)$. In going over to more general cases, it is clear we have to introduce other tools. For example, if $S(C)$ is the set of all integers in the interval $(N, KN]$, then the sum for $\beta(C)$ tends to infinity with K, while the expression for α is always less than 1. Thus, the lemma would say that the residual set of integers not covered has density bounded below by a negative quantity tending to $-\infty$. This is clearly not useful! To rectify this situation, we choose a paramter Q and factor each modulus n as $n_{\underline{Q}} n_{\overline{Q}}$, where $n_{\underline{Q}}$ is the largest divisor of n composed solely of primes in $[1, Q]$ and $n_{\overline{Q}} = n/n_{\underline{Q}}$. We then find a way to decompose our system C based on these factorizations and use Lemma 2.1 on the parts corresponding to the numbers $n_{\overline{Q}}$ which have no small prime factors.

To set some terminology, for a number $Q \geq 1$, we say a positive integer n is Q-smooth if $P(n) \leq Q$. Thus, $n_{\underline{Q}}$ is the largest Q-smooth divisor of n.

Lemma 3.1. *Let C be an arbitrary residue system. Let $Q \geq 2$ be arbitrary, and set*

$$M = \operatorname{lcm}\{n_{\underline{Q}} : n \in S(C)\}.$$

For $0 \leq h \leq M-1$, let C_h be the set

$$C_h = \left\{ \left(n_{\overline{Q}}, r\right) : (n, r) \in C,\ r \equiv h \,(\text{mod } n_{\underline{Q}}) \right\}.$$

Then

$$\delta(C) = \frac{1}{M} \sum_{h=0}^{M-1} \delta(C_h).$$

Proof. Fix h so that $0 \leq h \leq M-1$. For $(n, r) \in C$, the simultaneous congruences

$$x \equiv r \pmod{n}, \qquad x \equiv h \pmod{M}$$

have a solution if and only if $r \equiv h \pmod{n_{\underline{Q}}}$, since $n_{\underline{Q}} = \gcd(n, M)$, in which case the system is equivalent to the system

$$x \equiv r \,(\text{mod } n_{\overline{Q}}), \qquad x \equiv h \,(\text{mod } M).$$

Thus, $R(C_h) \cap (h \bmod M) = R(C) \cap (h \bmod M)$. Observe that all elements $n_{\overline{Q}}$ of $S(C_h)$ are coprime to M. Thus, the proportion of the numbers in $R(C_h)$ in the class h modulo M is equal to $\delta(C_h)$. Hence, the density of $R(C) \cap (h \bmod M)$ is $\delta(C_h)/M$ and the result follows. $\square$

We now take advantage of the fact that the prime factors of a number $n_{\overline{Q}}$ are all larger than Q to allow us to get a reasonable upper bound for the quantities $\beta(C_h)$. The proof is similar to that in Theorem 1.

Lemma 3.2. *Let $K > 1$, and suppose C is a residue system with $S(C)$ consisting of integers in the interval $(N, KN]$, each with multiplicity at most s. Suppose $Q \geq 2$, and define M and C_h as in Lemma 3.1. Then*

$$\frac{1}{M} \sum_{h=0}^{M-1} \beta(C_h) \ll \frac{s^2 \log^2(QK)}{Q}. \tag{3.1}$$

Proof. For $m|M$, let S_m be the set of distinct numbers $n_{\overline{Q}} = n/\gcd(n,M)$, where $n \in S(C)$ and $n_{\underline{Q}} = \gcd(n,M) = m$. For $m, m' \mid M$, let

$$F(r,m,r',m') = \#\{0 \le h \le M-1 : h \equiv r \pmod m,\ h \equiv r' \pmod{m'}\}.$$

Then

$$\frac{1}{M}\sum_{h=0}^{M-1}\beta(C_h) \le \frac{1}{M}\sum_{\substack{m|M\\ m'|M}}\ \sum_{\substack{n\in S_m\\ n'\in S_{m'}\\ \gcd(n,n')>1}} \frac{1}{nn'} \sum_{\substack{(nm,r)\in C\\ (n'm',r')\in C}} F(r,m,r',m').$$

Since $F(r,m,r',m')$ is either 0 or $M/\mathrm{lcm}[m,m']$, the inner sum is at most

$$s^2 \frac{M}{\mathrm{lcm}[m,m']}.$$

Next,

$$\begin{aligned}\sum_{\substack{n\in S_m\\ n'\in S_{m'}\\ \gcd(n,n')>1}} \frac{1}{nn'} &\le \sum_{p>Q}\ \sum_{\substack{n\in S_m\\ n'\in S_{m'}\\ p|n,\ p|n'}} \frac{1}{nn'}\\ &= \sum_{p>Q}\Bigg(\sum_{\substack{N/m<n\le KN/m\\ p|n,\ P^-(n)>Q}}\frac{1}{n}\Bigg)\Bigg(\sum_{\substack{N/m'<n'\le KN/m'\\ p|n',\ P^-(n')>Q}}\frac{1}{n'}\Bigg).\end{aligned}$$

By standard sieve methods (e.g., Theorem 3.3 of [12]), uniformly in $x \ge 2$, $z \ge 2$, the number of integers $\le x$ which have no prime factor $\le z$ is $\ll x/\log z + 1$. By partial summation,

$$\sum_{\substack{N/m<n\le KN/m\\ p|n,\ P^-(n)>Q}}\frac{1}{n} = \frac{1}{p}\sum_{\substack{\frac{N}{pm}<t\le\frac{KN}{pm}\\ P^-(t)>Q}}\frac{1}{t} \ll \frac{1}{p}\left(\frac{\log K}{\log Q}+1\right) = \frac{\log(QK)}{p\log Q}$$

and similarly with m', n' replacing m, n. We have the estimate $\sum_{p>Q} p^{-2} \ll 1/(Q\log Q)$, which follows from the prime number theorem and partial summation. Thus,

$$\sum_{\substack{n\in S_m\\ n'\in S_{m'}\\ \gcd(n,n')>1}} \frac{1}{nn'} \ll \frac{\log^2(QK)}{Q\log^3 Q},$$

so that

$$\frac{1}{M}\sum_{h=0}^{M-1}\beta(C_h) \ll \frac{s^2\log^2(QK)}{Q\log^3 Q}\sum_{\substack{m|M\\ m'|M}}\frac{1}{\mathrm{lcm}[m,m']} = \frac{s^2\log^2(QK)}{Q\log^3 Q}\sum_{u|M}\ \sum_{\substack{m|M,\ m'|M\\ \mathrm{lcm}[m,m']=u}} u^{-1}.$$

With $\tau(n)$ denoting the number of natural divisors of n, the double sum is equal to

$$\sum_{u|M} u^{-1}\tau(u^2) \le \prod_{p|M}\left(1+\frac{3}{p}+\frac{5}{p^2}+\cdots\right)$$
$$= \prod_{p|M}\frac{1+1/p}{(1-1/p)^2} \le \prod_{p\le Q}\frac{1+1/p}{(1-1/p)^2} \ll \log^3 Q,$$

and this completes the proof. □

To complement Lemma 3.2, we would like a lower bound for the sum of the $\alpha(C_h)$. Key to this estimate will be those moduli in $S(C)$ which are Q-smooth. If the residue classes corresponding to these moduli do not cover everything, we are able to get a respectable lower bound for the sum of the $\alpha(C_h)$.

Lemma 3.3. *Suppose that C is a residue system, $Q \ge 2$, and define M and C_h as in Lemma 3.1. Also let $C' = \{(n,r) \in C : n|M\} = \{(n,r) \in C : P(n) \le Q\}$ and suppose $\delta(C') > 0$. Then*

$$\frac{1}{M}\sum_{h=0}^{M-1}\alpha(C_h) \ge (\alpha(C))^{(1+1/Q)/\delta(C')}.$$

Proof. Note that $1 \in S(C_h)$ if and only if there is a pair $(n,r) \in C'$ with $h \equiv r \pmod n$. Let

$$\mathcal{M}' = \{0 \le h \le M-1 : 1 \notin S(C_h)\}, \quad M' = |\mathcal{M}'|.$$

Then

$$\frac{M'}{M} = \delta(C'). \tag{3.2}$$

The hypothesis $\delta(C') > 0$ thus implies that $M' > 0$. Observe that $1 \in S(C_h)$ implies $\alpha(C_h) = 0$. By the inequality of the arithmetic and geometric means,

$$\frac{1}{M}\sum_{h=0}^{M-1}\alpha(C_h) = \frac{1}{M}\sum_{h\in\mathcal{M}'}\alpha(C_h) \ge \frac{M'}{M}\Big(\prod_{h\in\mathcal{M}'}\alpha(C_h)\Big)^{1/M'}$$
$$= \frac{M'}{M}\Big(\prod_{h\in\mathcal{M}'}\prod_{n'\in S(C_h)}\Big(1-\frac{1}{n'}\Big)\Big)^{1/M'}.$$

Since $\log(1-1/k) > -\frac{1}{k}(1+\frac{1}{k})$ for $k \ge 2$ and since each $n' > Q$, we have

$$1-\frac{1}{n'} > \exp\big(-\lambda/n'\big), \qquad \text{where } \lambda = 1+1/Q.$$

Thus,

$$\frac{1}{M}\sum_{h=0}^{M-1}\alpha(C_h) \ge \frac{M'}{M}\exp\left(-\frac{\lambda}{M'}\sum_{h\in\mathcal{M}'}\sum_{n'\in S(C_h)}\frac{1}{n'}\right)$$
$$\ge \frac{M'}{M}\exp\left(\frac{\lambda(M-M')}{M'} - \frac{\lambda}{M'}\sum_{h=0}^{M-1}\sum_{n'\in S(C_h)}\frac{1}{n'}\right),$$

where the last inequality uses that $1 \in S(C_h)$ for $h \notin \mathcal{M}'$.

Each pair $(n, r) \in C$ maps to those C_h with $h \equiv r \pmod{n_{\underline{Q}}}$, so it produces the pair $(n_{\overline{Q}}, r)$ in exactly $M/n_{\underline{Q}}$ sets C_h for $h \in [0, M-1]$. We thus have

$$\sum_{h=0}^{M-1} \sum_{n' \in S(C_h)} \frac{1}{n'} \leq \sum_{n \in S(C)} \frac{M}{n_{\underline{Q}}} \cdot \frac{1}{n_{\overline{Q}}} = M \sum_{n \in S(C)} \frac{1}{n}.$$

(Note that the inequality holds since several pairs in C may map to the same pair in some C_h, where they would be counted just once.) Thus,

$$\frac{1}{M} \sum_{h=0}^{M-1} \alpha(C_h) \geq \frac{M'}{M} \exp\left(\frac{\lambda(M - M')}{M'} - \frac{\lambda M}{M'} \sum_{n \in S(C)} \frac{1}{n} \right).$$

Also, $(M'/M) \exp\bigl((M - M')/M'\bigr) \geq 1$. Thus,

$$\frac{1}{M} \sum_{h=0}^{M-1} \alpha(C_h) \geq \exp\left(-\frac{\lambda M}{M'} \sum_{n \in S(C)} \frac{1}{n} \right) \geq \bigl(\alpha(C)\bigr)^{\lambda M/M'}.$$

The lemma follows by (3.2). □

We now combine our lemmas into one easily applied statement.

Lemma 3.4. *Suppose $K > 1$, N is a positive integer, and C is a residue system with $S(C)$ consisting of integers in $(N, KN]$, each with multiplicity at most s. Let $Q \geq 2$, and as in Lemma 3.3, let $C' = \{(n, r) \in C : P(n) \leq Q\}$. If $\delta(C') > 0$, then*

$$\delta(C) \geq \alpha(C)^{(1+1/Q)/\delta(C')} + O\left(\frac{s^2 \log^2(QK)}{Q} \right),$$

where the implied constant is uniform in all parameters.

Proof. Define M and C_h as in Lemma 3.1. By Lemmas 2.1, 3.1, 3.2, and 3.3, we have

$$\begin{aligned} \delta(C) = \frac{1}{M} \sum_{h=0}^{M-1} \delta(C_h) &\geq \frac{1}{M} \sum_{h=0}^{M-1} \alpha(C_h) - \frac{1}{M} \sum_{h=0}^{M-1} \beta(C_h) \\ &\geq \alpha(C)^{(1+1/Q)/\delta(C')} + O\left(\frac{s^2 \log^2(QK)}{Q} \right). \end{aligned}$$

Thus, we have the lemma. □

4. Lower bounds on $\delta(C)$

In this section we prove stronger versions of Theorems A and B. We begin with a useful lemma about smooth numbers.

Lemma 4.1. *Suppose $Q \geq 2$ and $Q < N \leq \exp(\sqrt{Q})$. Then*

$$\sum_{\substack{n > N \\ P(n) \leq Q}} \frac{1}{n} \ll (\log Q) e^{-u \log u}, \quad \textit{where } u = \frac{\log N}{\log Q}.$$

Proof. We use standard upper-bound estimates for the distribution of smooth numbers: The number of Q-smooth numbers at most t is $\ll t/u_t^{u_t}$, where $u_t = \log t/\log Q$, provided $Q \le t \le \exp\left(Q^{1-\varepsilon}\right)$ ([15], Theorem 1.2 and Corollary 2.3). Furthermore, for $t > \exp\left(6\sqrt{Q}\right)$, the Q-smooth numbers are distributed more sparsely than the squares. We thus have

$$\begin{aligned}\sum_{\substack{n>N\\P(n)\le Q}} \frac{1}{n} &= \int_N^\infty \frac{1}{t^2} \sum_{\substack{N<n\le t\\P(n)\le Q}} 1\,dt \\ &\le \sum_{0\le i\le 10\sqrt{Q}} \int_{NQ^i}^{NQ^{i+1}} \frac{1}{t^2} \sum_{\substack{n\le t\\P(n)\le Q}} 1\,dt + \int_{\exp(6\sqrt{Q})}^\infty \frac{1}{t^2} \sum_{\substack{n\le t\\P(n)\le Q}} 1\,dt \\ &\ll \sum_{i\ge 0} \frac{\log Q}{(u+i)^{u+i}} + \int_{\exp(6\sqrt{Q})}^\infty \frac{1}{t^{3/2}}\,dt \ll \frac{\log Q}{u^u},\end{aligned}$$

implying the lemma. □

Let

$$L(N,s) = \exp\left(\log N \frac{\log\log(s\log N)}{\log(s\log N)}\right).$$

Theorem 2. *Suppose $0 < b < \frac{1}{2}$, $0 < c < \frac{1}{3}(1-4b^2)$, and let N be sufficiently large, depending on the choice of b and c. Suppose C is a residue system with $S(C)$ consisting of integers $n > N$, each having multiplicity at most s, where $s \le \exp\left(b\sqrt{\log N \log\log N}\right)$, and such that*

$$\sum_{n\in S(C)} \frac{1}{n} \le c \log L(N,s). \tag{4.1}$$

Then $\delta(C) > 0$.

Proof. Throughout we assume that N is sufficiently large, depending only on b and c. Let $\lambda = \frac{1}{3}(1-4b^2)$ and put $\varepsilon = \frac{1}{20}(\lambda - c)$. First, we have

$$-\log\alpha(C) \le \sum_{n\in S(C)} \left(\frac{1}{n} + \frac{1}{n^2}\right) \le \left(1 + \frac{1}{N}\right) \sum_{n\in S(C)} \frac{1}{n} \le (c+\varepsilon)\log L(N,s) = G,$$

say. Define

$$Q_0 = L(N,s)^{1-\varepsilon}, \quad Q_j = \exp\left(Q_{j-1}^{\lambda+\varepsilon}\right) \quad (j \ge 1) \tag{4.2}$$

and

$$K_j = \exp\left(Q_{j-1}^{\lambda+2\varepsilon}\right) \qquad (j \ge 1).$$

Let

$$C_j = \{(n,r) \in C : P(n) \le Q_j\}.$$

Also, define

$$\delta_0 = 1-\varepsilon, \quad \delta_j = e^{-1-G(1+1/Q_0)/\delta_{j-1}} \qquad (j \ge 1), \tag{4.3}$$

where G is defined above. Since C is finite and Q_j tends to infinity with j, it follows that $C = C_j$ for large j. Thus, the theorem will follow if we show that

$$\delta(C_j) \ge \delta_j \qquad (j \ge 0). \tag{4.4}$$

First, by Lemma 4.1,

$$1 - \delta(C_0) \le s \sum_{\substack{n>N \\ P(n)\le Q_0}} \frac{1}{n} \ll s(\log Q_0)e^{-u\log u}, \quad \text{where } u = \frac{\log N}{\log Q_0}.$$

By the definition of Q_0, we have

$$u = \frac{\log(s\log N)}{(1-\varepsilon)\log\log(s\log N)}$$

so that $\log u \ge (1-\varepsilon)\log\log(s\log N)$ and $u\log u \ge \log(s\log N)$. Hence, $\delta(C_0) \ge \delta_0$.

Next, suppose $j \ge 1$ and $\delta(C_{j-1}) \ge \delta_{j-1}$. Let $s_0 = \exp\big(b\sqrt{\log N \log\log N}\big)$ and observe that for N large and $s \le s_0$, we have

$$\frac{\log\log(s\log N)}{\log(s\log N)} \ge \frac{\log\log s_0}{\log(s_0\log N)} \ge \frac{\log\log N}{2\log(s_0\log N)} \ge \frac{(1-\varepsilon)\log\log N}{2b\sqrt{\log N\log\log N}}.$$

Therefore,

$$\begin{aligned} s^2 &\le \exp\big(2b\sqrt{\log N\log\log N}\big) \le L(N,s)^{4b^2/(1-\varepsilon)} \\ &= Q_0^{4b^2/(1-\varepsilon)^2} \le Q_0^{4b^2(1+3\varepsilon)} \le Q_{j-1}^{4b^2(1+3\varepsilon)}. \end{aligned}$$

Let

$$C_j' = \{(n,r)\in C_j : n \le K_j\}, \quad C_j'' = \{(n,r)\in C_j : n > K_j\}.$$

Observe that

$$\delta\big(\{(n,r)\in C_j' : P(n)\le Q_{j-1}\}\big) \ge \delta(C_{j-1}) \ge \delta_{j-1} \quad \text{and} \quad \alpha(C_j') \ge \alpha(C) \ge e^{-G}.$$

By Lemma 3.4 with $Q = Q_{j-1}$ and $K = K_j/N$, there is an absolute constant D such that

$$\begin{aligned} \delta(C_j') &\ge \alpha(C_j')^{(1+1/Q_{j-1})/\delta_{j-1}} - D\frac{s^2\log^2(Q_{j-1}K_j/N)}{Q_{j-1}} \\ &\ge e^{-G(1+1/Q_0)/\delta_{j-1}} - Q_{j-1}^{-1+4b^2(1+3\varepsilon)+2\lambda+5\varepsilon} \ge 2\delta_j - Q_{j-1}^{-\lambda+8\varepsilon}. \end{aligned} \tag{4.5}$$

Also, by Lemma 4.1,

$$1 - \delta(C_j'') \le s \sum_{\substack{n>K_j \\ P(n)\le Q_j}} \frac{1}{n} \ll s(\log Q_j)e^{-u_j\log u_j}, \tag{4.6}$$

where

$$u_j = \frac{\log K_j}{\log Q_j} = Q_{j-1}^{\varepsilon}.$$

Thus, $1 - \delta(C_j'') \le Q_{j-1}^{-1}$. Together with (4.5), this implies

$$\delta(C_j) \ge \delta(C_j') - (1 - \delta(C_j'')) \ge 2\delta_j - Q_{j-1}^{-\lambda+9\varepsilon}. \tag{4.7}$$

To complete the proof of (4.4) and the theorem, it suffices to prove that

$$Q_{j-1}^{-\lambda+9\varepsilon} \le \delta_j \qquad (j \ge 1). \tag{4.8}$$

First,

$$Q_0^{-\lambda+9\varepsilon} = L(N,s)^{(-\lambda+9\varepsilon)(1-\varepsilon)} \le L(N,s)^{-\lambda+10\varepsilon} = L(N,s)^{-c-10\varepsilon},$$

while

$$\delta_1 \ge e^{-1-G(1+1/Q_0)(1+1.1\varepsilon)} \ge L(N,s)^{-c-2\varepsilon}.$$

This proves (4.8) when $j = 1$. Suppose (4.8) holds for some $j \geq 1$. Since $G \leq \log Q_0$, we have

$$-\log \delta_{j+1} = 1 + G(1 + 1/Q_0)/\delta_j \leq 2GQ_{j-1}^{\lambda-9\varepsilon} < Q_{j-1}^{\lambda-8\varepsilon}.$$

Also, by (4.2), we have $-\log(Q_j^{-\lambda+9\varepsilon}) \gg Q_{j-1}^{\lambda+\varepsilon}$. Thus, $Q_j^{-\lambda+9\varepsilon} \leq \delta_{j+1}$ and by induction (4.8) holds for all j. This completes our proof. $\square$

Theorem 2 implies Theorem A of the introduction by setting $s = 1$. Observe that the bound on the sum in Theorem 2 given in (4.1) decreases as s increases. If one is interested in a result similar to Theorem A but with an emphasis on allowing the multiplicity of the moduli to be large, one may take b arbitrarily close to $1/2$ in Theorem 2.

Theorem 2 should be compared with Theorem 5 of the next section, which shows that coverings, even exact coverings with squarefree moduli, exist when we allow the multiplicity of the moduli to be of size $\exp\left(\sqrt{\log N \log\log N}\right)$.

We can also consider the case that $S(C)$ consists of integers from $(N, KN]$ with multiplicities at most $s \leq \exp\left(b\sqrt{\log N \log\log N}\right)$, where $b < \sqrt{3\varepsilon/4}$. If $0 < \varepsilon < 1/3$, N is large, and $K = L(N,s)^{(1/3-\varepsilon)/s}$, then Theorem 2 implies that $\delta(C) > 0$. By a different argument, we can extend the range of K a bit.

Theorem 3. *Suppose* $0 < \varepsilon < (1-\log 2)^{-1}$, $b < \frac{1}{2}\sqrt{(1-\log 2)\varepsilon}$, *and* N *is sufficiently large, depending on the choice of* ε *and* b. *Suppose that* C *is a residue system with* $S(C)$ *consisting of integers from* $(N, KN]$ *with multiplicity at most* s, *where* $s \leq \exp\left(b\sqrt{\log N \log\log N}\right)$ *and* $K = L(N,s)^{((1-\log 2)^{-1}-\varepsilon)/s}$. *Then* $\delta(C) > 0$.

Note that for $s \geq \log N$, $K = 1 + o(1)$. Before proving Theorem 3, we present a lemma.

Lemma 4.2. *Suppose* s *is a positive integer and* C *is a residue system with* $S(C)$ *consisting of integers from* $(1, B]$ *with multiplicity at most* s. *Let*

$$C_0 = \{(n,r) \in C : P(n) \leq \sqrt{sB}\}.$$

If $\delta(C_0) > 0$, *then* $\delta(C) > 0$.

Proof. Suppose that $\delta(C_0) > 0$. Denote by P the product of all primes in $(\sqrt{sB}, B]$, and let L be the least common multiple of the elements of $S(C_0)$.

Let p be a prime divisor of P. Since $p > \sqrt{sB}$ implies $sB/p < p$, there are at most $p-1$ multiples of p in the multiset $S(C)$. Call them $m_1, \dots, m_t$, and let $r_1, \dots, r_t$ be the corresponding residue classes. Then there is a choice for $b = b(p) \in \{0, 1, \dots, p-1\}$ such that each integer satisfying $x \equiv b \pmod{p}$ is not covered by (i.e., does not satisfy) any of the congruences $x \equiv r_j \pmod{m_j}$ with $1 \leq j \leq t$.

By assumption, there is a residue class a mod L contained in $R(C_0)$. Let A be a solution to the Chinese remainder system $A \equiv a \pmod{L}$ and $A \equiv b(p) \pmod{p}$ for each prime p dividing P. Then not only do we have $A \not\equiv r \pmod{n}$ for each $(n,r) \in C_0$, we also have $A \not\equiv r \pmod{n}$ for each $(n,r) \in C$ with $p \mid n$ for some prime $p \mid P$. Since this exhausts the pairs $(n,r) \in C$, we have $A \in R(C)$, so we have the lemma. $\square$

Proof of Theorem 3. We may suppose that $\varepsilon > 0$ is sufficiently small and $K \geq 2$. Let C_0 be as in Lemma 4.2 where we take $B = KN$. Then

$$\sum_{n\in S(C_0)} \frac{1}{n} \leq s \sum_{\substack{N<n\leq KN \\ P(n)\leq\sqrt{sKN}}} \frac{1}{n} = s \sum_{N<n\leq KN} \frac{1}{n} - s \sum_{\substack{N<n\leq KN \\ P(n)>\sqrt{sKN}}} \frac{1}{n}$$

$$= s\log K + O(s/N) - s \sum_{\sqrt{sKN}<p\leq KN} \frac{1}{p} \sum_{N/p<m\leq KN/p} \frac{1}{m}.$$

Now,

$$\sum_{N/p<m\leq KN/p} \frac{1}{m} = \begin{cases} \log K + O(p/N), & p \leq N, \\ \log(KN/p) + O(1), & N < p \leq KN. \end{cases}$$

Thus,

$$\sum_{\sqrt{sKN}<p\leq KN} \frac{1}{p} \sum_{N/p<m\leq KN/p} \frac{1}{m}$$

$$= \sum_{\sqrt{sKN}<p\leq N} \left(\frac{\log K}{p} + O(1/N)\right) + \sum_{N<p\leq KN} \left(\frac{\log K}{p} + \frac{\log N - \log p + O(1)}{p}\right)$$

$$= \sum_{\sqrt{sKN}<p\leq KN} \frac{\log K}{p} + \sum_{N<p\leq KN} \frac{\log N - \log p}{p} + O(\log K/\log N)$$

$$= \log 2\log K + O\left(\frac{\log K\log(sK)}{\log N}\right) = (\log 2 + o(1))\log K.$$

Hence, since $-\log\alpha(C_0) \leq \sum_{n\in S(C_0)} 1/n + O(s/N)$, we have

$$-\log\alpha(C_0) \leq s\big(1 - \log 2 + o(1)\big)\log K \leq \big(1 - (1-\log 2)\varepsilon + o(1)\big)\log L(N,s).$$

Let $Q = L(N,s)^{1-\lambda}$, where $\lambda = \frac{1}{4}((1-\log 2)\varepsilon - 4b^2)$. Also let $C' = \{(n,r) \in C_0 : P(n) \leq Q\}$. As before, using Lemma 4.1 yields

$$\delta(C') = 1 + O\Bigg(s \sum_{\substack{n>N \\ P(n)\leq Q}} \frac{1}{n}\Bigg) = 1 + o(1) \qquad (N \to \infty).$$

Hence,

$$\alpha(C_0)^{(1+1/Q)/\delta(C')} \gg L(N,s)^{-1+(1-\log 2)\varepsilon-\lambda}.$$

On the other hand,

$$\frac{s^2\log^2(QK)}{Q} \ll L(N,s)^{-1+4b^2+2\lambda}.$$

By Lemma 3.4, we have $\delta(C_0) > 0$ for N sufficiently large. Thus, $\delta(C) > 0$ by Lemma 4.2. $\square$

We now show that if K is a bit smaller than in Theorem 3, then in fact

$$\delta(C) \geq (1 + o(1))\alpha(C).$$

The following result generalizes Theorem B from the introduction.

Theorem 4. *Suppose $0 < \varepsilon < 1/2$, $0 < b < \frac{1}{2}\sqrt{\varepsilon}$, and $N \geq 100$. Suppose that C is a residue system with $S(C)$ consisting of integers from $(N, KN]$ with multiplicity at most s, where $s \leq \exp\left(b\sqrt{\log N \log\log N}\right)$ and $K = L(N,s)^{(1/2-\varepsilon)/s}$. Then*

$$\delta(C) \geq \left(1 + O\left(\frac{1}{(\log N)^{\lambda}}\right)\right)\alpha(C),$$

where λ is a positive constant depending only on ε and b.

Proof. We follow the same general plan as in the proof of Theorem 2. Since the sum of $1/n$ for all $n \in (N, KN]$ is $\log K + O(1/N)$, we have

$$\alpha(C) \gg L(N,s)^{-1/2+\varepsilon}.$$

Let $Q = L(N,s)^{1/2-\lambda}$, where $\lambda = \frac{1}{3}(\varepsilon - 4b^2)$. In particular $Q \geq \log^2 N$. Let $u = \log N/\log Q$, and let C' be as in Lemma 3.4. By Lemma 4.1, we have

$$1 - \delta(C') \ll \frac{s\log Q}{u^u} \ll \frac{s\log N}{(s\log N)^{2+\lambda}},$$

so that $1/\delta(C') = 1 + O\left((s\log N)^{-1-\lambda}\right)$. Since $|\log\alpha(C)| \leq \log N$, we have

$$\alpha(C)^{(1+1/Q)/\delta(C')} = (1 + O(1/(\log N)^{\lambda}))\alpha(C).$$

So, by Lemma 3.4 it suffices to show that $s^2(\log QK)^2/Q = O(\alpha(C)(\log N)^{-\lambda})$. But, for large N we have $s^2 \leq L(N,s)^{4b^2+\lambda}$. Thus,

$$\begin{aligned}\frac{s^2(\log QK)^2}{Q} &\ll \frac{s^2\log^2 L(N,s)}{L(N,s)^{1/2-\lambda}}\\ &\ll \frac{1}{L(N,s)^{1/2-2\lambda-4b^2}} \ll \frac{1}{L(N,s)^{1/2-\varepsilon+\lambda}} \ll \frac{\alpha(C)}{L(N,s)^{\lambda}}.\end{aligned}$$

This completes the proof. □

5. Coverings and near-coverings of the integers

In this section, we address two items. The first shows that there are coverings of the integers with the moduli bounded below by N and the multiplicity of the moduli *near* the upper bound on the multiplicity of the moduli given by Theorem 2. The second shows that, when we allow K to be large, the density of the integers which are not covered by a covering system using distinct moduli from $(N, KN]$ can be considerably smaller than what is suggested by Theorem 4.

Theorem 5. *For sufficiently large N and $s = \exp(\sqrt{\log N \log\log N})$, there exists an exact covering system with squarefree moduli greater than N such that the multiplicity of each modulus does not exceed s.*

Proof. Let p denote a prime and let $X_j = (j+1)^{j+1}$ for $j = 0, 1, \ldots$. We first show that

$$\sum_{X_{j-1} < p \leq X_j} [X_j/p] \geq X_{j-1} \quad (j \geq 1). \tag{5.1}$$

Here $[x]$ denotes the largest integer which is $\leq x$. Note that (5.1) holds for $j \leq 5$. Suppose then that $j \geq 6$. Using the estimates (3.4), (3.17), and (3.18) in Rosser and Schoenfeld [17], we have that

$$\sum_{X_{j-1}<p\le X_j} [X_j/p] \ge X_j \sum_{X_{j-1}<p\le X_j} 1/p - \pi(X_j)$$
$$\ge X_j \left(\log \frac{\log X_j}{\log X_{j-1}} - \frac{1}{\log^2 X_{j-1}} - \frac{1}{\log X_j - \frac{3}{2}}\right).$$

The expression in the parentheses is

$$\log \frac{(j+1)\log(j+1)}{j\log j} - \frac{1}{j^2\log^2 j} - \frac{1}{(j+1)\log(j+1) - \frac{3}{2}}$$
$$> \frac{1}{j+1}\left(\frac{j+1}{j} - \frac{j+1}{2j^2} - \frac{j+1}{j^2\log^2 j} - \frac{1}{\log(j+1) - 3/(2j+2)}\right) > \frac{0.43}{j+1}.$$

Also, $X_j = (j+1)j^j(1+1/j)^j > 2.5(j+1)j^j$. Thus,

$$\sum_{X_{j-1}<p\le X_j} [X_j/p] > \frac{2.5(j+1)0.43}{j+1} j^j > j^j,$$

which proves (5.1).

We describe now an explicit construction of a covering system, which we will then show satisfies the conditions of the theorem. For $J \ge 1$ and $s = X_J$, we establish that there exists an exact covering system C_J with squarefree moduli greater than

$$N_J = \prod_{j=0}^{J-1} X_j$$

such that the multiplicity of each modulus does not exceed s. Set

$$P_j = \{p : X_{j-1} < p \le X_j\}.$$

We construct C_J, through induction on J, by choosing moduli of the form $p_1 \cdots p_J$ where each $p_j \in P_j$. Observe that such a product $p_1 \cdots p_J$ is necessarily $> N_J$. One checks that $C_1 = \{(2,0),(2,1)\}$ satisfies the conditions for C_J with $J = 1$. Now, suppose that we have C_J as above for some $J \ge 1$. Thus, we have an exact covering system C_J with moduli of the form $p_1 \cdots p_J$ where each $p_j \in P_j$. Fix such a modulus $n = p_1 \cdots p_J$, and let $(n,r_1),\dots,(n,r_t)$, with $t \le X_J$, be the pairs of the form (n,r) in C_J. Let $q_1 < q_2 < \cdots$ be the complete list of primes from P_{J+1}. To construct C_{J+1}, we replace each pair (n,r_i), $i \le [X_{J+1}/q_1]$, with the q_1 pairs $(nq_1, r_i + n\mu)$, where $\mu = 0,\dots,q_1-1$. Notice that the multiplicity of the modulus nq_1 is at most $[X_{J+1}/q_1]q_1 \le X_{J+1}$. Next, we replace each pair (n,r_i), $[X_{J+1}/q_1] < i \le [X_{J+1}/q_1] + [X_{J+1}/q_2]$, with the q_2 pairs $(nq_2, r_i+n\mu)$, where $\mu = 0,\dots,q_2-1$. We proceed with this construction until all the pairs $(n,r_1),\dots,(n,r_t)$ are replaced with new pairs. As $t \le X_J$, this will happen at some point by (5.1). This completes the inductive construction of our exact covering systems C_J.

To complete the proof of the theorem, it suffices to show that $\log N_J \log\log N_J \ge \log^2 s$ for large J. Now

$$\log N_J = \sum_{j=1}^{J} j\log j \ge \int_1^J t\log t\,dt > \frac{1}{2}J^2\log J - \frac{1}{4}J^2,$$

so that

$$\log\log N_J > 2\log J + \log\log J - \log 2 + \log(1 - 1/(2\log J)) > 2\log J + \log\log J - 1,$$

for $J \geq 7$. Thus,

$$\begin{aligned}\log N_J \log\log N_J &> J^2\log^2 J + \frac{1}{2}J^2\log J\left(\log\log J - 1.5 - \frac{\log\log J}{2\log J}\right)\\ &> J^2\log^2 J + 3J\log^2 J,\end{aligned}$$

for $J \geq 350$. But $\log^2 s = (J+1)^2\log^2(J+1) < J^2\log^2 J + 3J\log^2 J$ in the same range. This completes the proof of the theorem. □

Remark 4. A more elementary proof, one that does not use the estimates from [17], is possible by defining the sequence X_j inductively as the minimal numbers for which (5.1) holds.

Suppose $s = 1$ and N, KN are integers in Theorem 4. Then $S(C)$ consists of distinct integers chosen from $(N, KN]$ so that

$$\alpha(C) \geq \prod_{j=N+1}^{KN}\left(1 - \frac{1}{j}\right) = \frac{1}{K}.$$

Thus, Theorem 4 implies a lower bound of approximately $1/K$ for any $\delta(C)$ with $S(C) \subseteq (N, KN]$, provided K is not too large. It is clear that the expression $1/K$ is not far from the truth, since the argument of the introduction gives a residue system C with $\delta(C) \leq 1/K$. However, we might ask about the situation when K is large compared to N. The following result shows that $\delta(C)$ can in fact be considerably smaller than $1/K$ when K is much larger than N.

Theorem 6. *Suppose N and K are integers with $N \geq 1$ and K sufficiently large. Then there is some residue system C consisting of distinct moduli from $(N, KN]$ such that*

$$\delta(C) \leq \frac{1}{K}\exp\left(-\frac{\log K}{3N}\right).$$

Before giving a proof of the above theorem, we give a lemma that will also play a role in the next section. For a set T of positive integers, we let $\mathcal{C}(T)$ be the set of residue systems C with $S(C) = T$ and where $(n, r) \in C$ implies $1 \leq r \leq n$. Also, define

$$W(T) = \#\mathcal{C}(T) = \prod_{n\in T} n.$$

Lemma 5.1. *Let T be a set of positive integers. Then the expected value of $\delta(C)$ over $C \in \mathcal{C}(T)$, denoted $\mathbf{E}\delta(C)$, is $\prod_{n\in T}(1 - 1/n)$.*

Proof. Put $W = W(T)$ and say $1 \leq m \leq W$. The number of systems $C \in \mathcal{C}(T)$ with $m \in R(C)$ is $\prod_{n\in T}(n-1)$, since for each $n \in T$, there are $n - 1$ choices for r

with $1 \le r \le n$ and $r \not\equiv m \pmod n$. Thus,

$$\sum_{C \in \mathcal{C}(T)} \delta(C) = \sum_{C \in \mathcal{C}(T)} \frac{1}{W} \sum_{\substack{1 \le m \le W \\ m \in R(C)}} 1$$
$$= \frac{1}{W} \sum_{m=1}^{W} \sum_{\substack{C \in \mathcal{C}(T) \\ m \in R(C)}} 1 = \frac{1}{W} \sum_{m=1}^{W} \prod_{n \in T} (n-1) = \prod_{n \in T} (n-1).$$

The result follows by dividing this equation by W. □

Remark 5. It is not hard to prove a version of Lemma 5.1 that allows for taking moduli from T with multiplicity greater than 1.

Proof of Theorem 6. There is a covering system with distinct moduli and smallest modulus 25 (a result of Gibson [9]), so Theorem 6 follows for $N \le 24$. Henceforth we may assume that $N \ge 25$; however our argument holds for $N \ge 4$. We shall construct a residue system $C = \{(n, r(n)) : N < n \le KN\}$ as follows. We will randomly choose the values of $r(n) \in [1, n]$ for $N < n \le 2N$ so that each residue class modulo n is taken with the same probability $1/n$ and the variables $r(n)$ are independent. Based on the random choice of such $r(n)$ for $N < n \le 2N$, we then select the remaining values of $r(n)$ with $2N < n \le KN$ via a greedy algorithm. In fact, we show that, under our construction, the expected value of $\delta(C)$ over all randomly chosen values of $r(n) \in [1, n]$ for $N < n \le 2N$ is

$$\le \frac{1}{K} \exp\left(-\frac{\log K}{3N}\right).$$

The result thus follows.

Let $C_{2N} = \{(n, r(n)) : N < n \le 2N\}$, where each $r(n)$ is chosen randomly from $[1, n]$. From Lemma 5.1, it follows that $\mathbf{E}\delta(C_{2N}) = 1/2$. Hence, by the arithmetic mean-geometric mean inequality,

$$\mathbf{E} \log \delta(C_{2N}) \le -\log 2. \tag{5.2}$$

We will also make use of Lemma 5.1 in another way. If D is a subset of the integers in $(N, 2N]$ and $\widetilde{C} = \{(d, r(d)) : d \in D\}$, then it is not difficult to see that the expected value of $\delta(\widetilde{C})$ over all randomly chosen values of $r(d) \in [1, d]$ for $d \in D$ is the same as the expected value of $\delta(\widetilde{C})$ over all randomly chosen values of $r(n) \in [1, n]$ for $n \in (N, 2N]$; in other words, the random selection of extra residue classes not associated with $\widetilde{C}$ will not affect the expected value $\delta(\widetilde{C})$. Thus, Lemma 5.1 implies $\mathbf{E}\delta(\widetilde{C}) = \alpha(\widetilde{C})$ where the expected value is over all randomly chosen $r(n) \in [1, n]$ for $N < n \le 2N$.

Suppose then that the values of $r(n) \in [1, n]$ for $N < n \le 2N$ have been chosen randomly. For $2N < j \le KN$, we describe how to select $r(j)$. For this purpose, we set $C_j = \{(n, r(n)) : N < n \le j\}$. We use the greedy algorithm to choose $r(j)$ to be a residue class modulo j containing the largest proportion of $R(C_{j-1})$. As in the introduction, this gives trivially

$$\delta(C_j) \le \left(1 - \frac{1}{j}\right) \delta(C_{j-1}).$$

We can sometimes do better. If j has a divisor d with $N < d \le 2N$, then there are residue classes modulo j not intersecting $R(C_{j-1})$. In particular, the residue class $r(d) \pmod d$ contains $r \pmod j$ when $r \equiv r(d) \pmod d$. Let

$$D(j) = \{d : d|j, N < d \le 2N\}, \quad \widetilde{C}_j = \{(d, r(d)) : d \in D(j)\}.$$

Let $f(j)$ be the number of residue classes $r \pmod j$ for which $r \not\equiv r(d) \pmod d$ for each $d \in D(j)$. If we choose $r(j)$ appropriately from among these $f(j)$ choices for r, we have

$$\delta(C_j) \le \left(1 - \frac{1}{f(j)}\right) \delta(C_{j-1}). \tag{5.3}$$

The last equality is nonsense if $f(j) = 0$, but in that case we have $R(C_{j-1}) = \emptyset$, and the theorem is trivial. Also, there is nothing to prove if $f(j) = 1$ since then $R(C_j) = \emptyset$. Throughout the following we assume that $f(j) > 1$.

We see from (5.3) and linearity of expectation that

$$\mathbf{E} \log \delta(C_j) - \mathbf{E} \log \delta(C_{j-1}) \le \mathbf{E} \log \left(1 - \frac{1}{f(j)}\right) \le -\mathbf{E}\left(\frac{1}{f(j)}\right). \tag{5.4}$$

Using Lemma 5.1 as described above, we have

$$\mathbf{E}\delta\big(\widetilde{C}_j\big) = \prod_{d \in D(j)} \left(1 - \frac{1}{d}\right).$$

Since j is a common multiple of the members of $D(j)$, it follows that $\delta\big(\widetilde{C}_j\big) = f(j)/j$, so that

$$\mathbf{E} f(j) = j \mathbf{E}\delta\big(\widetilde{C}_j\big) = j \prod_{d \in D(j)} \left(1 - \frac{1}{d}\right).$$

By the arithmetic mean-harmonic mean inequality, we thus have

$$\mathbf{E}\left(\frac{1}{f(j)}\right) \ge j^{-1} \prod_{d \in D(j)} \left(1 - \frac{1}{d}\right)^{-1} \ge \frac{1}{j} + \sum_{d \in D(j)} \frac{1}{dj}.$$

After substituting the last inequality into (5.4), we get

$$\mathbf{E} \log \delta(C_j) - \mathbf{E} \log \delta(C_{j-1}) \le -\frac{1}{j} - \sum_{d \in D(j)} \frac{1}{dj}.$$

Thus,

$$\begin{aligned}
\mathbf{E} \log \delta(C) - \mathbf{E} \log \delta(C_{2N}) &\le -\sum_{j=2N+1}^{KN} \frac{1}{j} - \sum_{j=2N+1}^{KN} \sum_{d \in D(j)} \frac{1}{dj} \\
&= -\sum_{j=2N+1}^{KN} \frac{1}{j} - \sum_{d=N+1}^{2N} \sum_{2N/d < l \le KN/d} \frac{1}{d^2 l} \\
&= -\log(K/2) + O(1/N) - \sum_{d=N+1}^{2N} \frac{\log K + O(1)}{d^2}.
\end{aligned}$$

We have for $N \geq 4$ the estimate

$$\sum_{d=N+1}^{2N} \frac{1}{d^2} \geq \int_{N+1}^{2N+1} \frac{dt}{t^2} = \frac{N}{(N+1)(2N+1)} \geq \frac{1}{2.9N}.$$

Therefore, by (5.2),

$$\mathbf{E} \log \delta(C) \leq -\log K - \frac{\log K + O(1)}{2.9N}.$$

The theorem now follows. ∎

6. Normal value of $\delta(C)$

It is reasonable to expect that $\delta(C) \approx \alpha(C)$ for almost all residue systems C with fixed $S(C)$. In this section, we establish such a result when $S(C)$ consists of distinct integers, by considering the variance of $\delta(C)$ over $C \in \mathcal{C}(T)$, where, as before, $\mathcal{C}(T)$ is the set of residue systems C with $S(C) = T$.

Theorem 7. *Let T be a set of distinct positive integers with minimum element $N \geq 3$. Let α be the common value of $\alpha(C)$ for $C \in \mathcal{C}(T)$. Then,*

$$\frac{1}{W(T)} \sum_{C \in \mathcal{C}(T)} |\delta(C) - \alpha|^2 \ll \frac{\alpha^2 \log N}{N^2}.$$

Proof. From Lemma 5.1, we have $\mathbf{E}\delta(C) = \alpha(C) = \alpha$. Writing $W = W(T)$, we deduce then that

$$\frac{1}{W} \sum_{C \in \mathcal{C}(T)} |\delta(C) - \alpha|^2 = \frac{1}{W} \sum_{C \in \mathcal{C}(T)} \big(\delta(C)^2 - \alpha^2\big). \tag{6.1}$$

We have

$$\sum_{C \in \mathcal{C}(T)} \delta(C)^2 = \sum_{C \in \mathcal{C}(T)} \Bigg(\frac{1}{W} \sum_{\substack{1 \leq m \leq W \\ m \in R(C)}} 1\Bigg)^2 = \frac{1}{W^2} \sum_{1 \leq m_1, m_2 \leq W} \sum_{\substack{C \in \mathcal{C}(T) \\ m_1, m_2 \in R(C)}} 1.$$

As in the proof of Lemma 5.1, the inner sum is

$$\prod_{\substack{n \in T \\ m_1 \equiv m_2 \pmod n}} (n-1) \prod_{\substack{n \in T \\ m_1 \not\equiv m_2 \pmod n}} (n-2) = W \prod_{n \in T} \left(1 - \frac{2}{n}\right) \prod_{\substack{n \in T \\ m_1 \equiv m_2 \pmod n}} \frac{n-1}{n-2}$$

$$= \alpha^2 W \prod_{n \in T} \frac{1 - \frac{2}{n}}{\left(1 - \frac{1}{n}\right)^2} \prod_{\substack{n \in T \\ n | (m_1 - m_2)}} \frac{n-1}{n-2} = \alpha^2 W \prod_{n \in T} \left(1 - \frac{1}{(n-1)^2}\right) \prod_{\substack{n \in T \\ n | (m_1 - m_2)}} \frac{n-1}{n-2}.$$

Let $u = \sum_{n \in T} 1/n^2$ and define $f(m_1, m_2) = \prod_{n \in T,\, n | (m_1 - m_2)} (n-1)/(n-2)$. Thus,

$$\sum_{C \in \mathcal{C}(T)} \delta(C)^2 = \frac{\alpha^2}{W} \left(1 - u + O\left(\frac{1}{N^2}\right)\right) \sum_{1 \leq m_1, m_2 \leq W} f(m_1, m_2). \tag{6.2}$$

For S a finite set of integers which are ≥ 3, let $M(S)$ denote $\prod_{n \in S} (n-2)$, and let $L(S)$ denote the least common multiple of the members of S. We have

$$f(m_1, m_2) = \prod_{\substack{n \in T \\ n | (m_1 - m_2)}} \left(1 + \frac{1}{n-2}\right) = \sum_{\substack{S \subseteq T \\ L(S) | (m_1 - m_2)}} \frac{1}{M(S)}.$$

Thus,

$$\sum_{1\le m_1,m_2\le W} f(m_1,m_2) = \sum_{S\subseteq T}\frac{1}{M(S)}\sum_{\substack{1\le m_1,m_2\le W\\ L(S)|(m_1-m_2)}} 1 = W^2\sum_{S\subseteq T}\frac{1}{M(S)L(S)}. \tag{6.3}$$

In this last sum we separately consider the terms with $\#S\le 1$ and $\#S\ge 2$. We have

$$\sum_{\substack{S\subseteq T\\ \#S\le 1}}\frac{1}{M(S)L(S)} = 1+\sum_{n\in T}\frac{1}{(n-2)n} = 1+u+O\left(1/N^2\right). \tag{6.4}$$

If $S\subseteq T$ and $\#S\ge 2$, let $k>h$ be the largest two members of S. Then $L(S)\ge \operatorname{lcm}[h,k]=hk/\gcd(h,k)$, so that

$$E := \sum_{\substack{S\subseteq T\\ \#S\ge 2}}\frac{1}{M(S)L(S)} \le \sum_{k>h\ge N}\frac{\gcd(h,k)}{(h-2)(k-2)hk}\sum_{U\subseteq[N,h-1]}\frac{1}{M(U)}.$$

The inner sum here is identical to $\prod_{N\le n\le h-1}(n-1)/(n-2) = (h-2)/(N-2)$, so that

$$\begin{aligned} E &\ll \frac{1}{N}\sum_{k>h\ge N}\frac{\gcd(h,k)}{hk^2} \le \frac{1}{N}\sum_{d\ge 1}\sum_{\substack{k>h\ge N\\ d|h,\ d|k}}\frac{d}{hk^2}\\ &= \frac{1}{N}\sum_{d\ge 1}\sum_{v>w\ge N/d}\frac{1}{d^2wv^2} \ll \frac{1}{N}\sum_{d\ge 1}\sum_{w\ge N/d}\frac{1}{d^2w^2}. \end{aligned}$$

In this last double sum, if $d\le N$, then the sum on w is $\ll d/N$, so that the contribution to E is $\ll(\log N)/N^2$. Moreover if $d>N$, the sum on w is $\ll 1$, so that the contribution to E is $\ll 1/N^2$. We conclude that $E\ll(\log N)/N^2$. Thus, with (6.3) and (6.4) we have

$$\sum_{1\le m_1,m_2\le W} f(m_1,m_2) = W^2\left(1+u+O((\log N)/N^2)\right),$$

so that from (6.2) and $u\ll 1/N$, we get

$$\sum_{C\in\mathcal{C}(T)}\delta(C)^2 = \alpha^2 W\left(1+O((\log N)/N^2)\right).$$

The result now follows immediately from (6.1). $\square$

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Reference [8] is to appear in Annals of Math.
Reference [18] has appeared: Internat. J. Math. **17** (2006), 1047–1064.

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