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Collinear Subsets of Lattice Point Sequences—An Analog of Szemerédi's Theorem

CARL POMERANCE

Department of Mathematics, University of Georgia, Athens, Georgia 30602

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Szemerédi's theorem states that given any positive number B and natural number k, there is a number $n(k, B)$ such that if $n \geqslant n(k, B)$ and $0 < a_1 < \cdots < a_n$ is a sequence of integers with $a_n \leqslant Bn$, then some k of the a_i form an arithmetic progression. We prove that given any B and k, there is a number $m(k, B)$ such that if $m \geqslant m(k, B)$ and $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_m$ is a sequence of plane lattice points with $\sum_{i=1}^{m} \| \mathbf{u}_i - \mathbf{u}_{i-1} \| \leqslant Bm$, then some k of the $\mathbf{u}_i$ are collinear. Our result, while similar to Szemerédi's theorem, does not appear to imply it, nor does Szemerédi's theorem appear to imply our result.

1. INTRODUCTION

Recently, Szemerédi [17] gave a proof of an old and pretty conjecture of Erdös and Turán [6]. Szemerédi proved that given any number B and any positive integer k, there is a number $n_1(k, B)$ such that if $n \geqslant n_1(k, B)$ and $0 < a_1 < \cdots < a_n$ are integers with $a_n \leqslant Bn$, then k of the a_i form an arithmetic progression. Szemerédi's proof, although elementary, is very complicated. Furstenberg [8] has given a new proof involving ergodic methods.

A well-known and old result of van der Waerden [18] is that if the natural numbers are partitioned into two subsets, then one of the subsets has arbitrarily long arithmetic progressions. It is not very difficult to show (see [2]) that van der Waerden's theorem has the following equivalent formulation. For every number B and positive integer k, there is a number $n_0(k, B)$ such that if $n \geqslant n_0(k, B)$ and $0 < a_1 < \cdots < a_n$ are integers with each $a_{i+1} - a_i \leqslant B$, then k of the a_i form an arithmetic progression. Thus we can say that Szemerédi's theorem improves on van der Waerden's theorem by replacing a uniform upper bound on each difference in the sequence with an upper bound for the average difference.

A possible arena for generalizations of the results of van der Waerden and Szemerédi is $\mathbb{Z}^2$, the set of plane lattice points. We first notice that there are

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two directions we can take. First, we can consider subsets of $\mathbb{Z}^2$ that are "fairly dense" in $\mathbb{Z}^2$. Second, we can consider sequences in $\mathbb{Z}^2$ with "fairly small" gaps. Both concepts are the same if we are considering $\mathbb{Z}$ and not $\mathbb{Z}^2$.

On the first approach, we have the following result of Gallai (= Grünwald) (See Rado [14]): if $\mathbb{Z}^2$ is partitioned into two subsets, then one of the subsets has for each k, a subset $I \times I$ where I is an integer arithmetic progression of length k. Gallai's theorem is thus a van der Waerden analog for $\mathbb{Z}^2$ (actually Gallai proved a more general result which holds in each $\mathbb{Z}^m$). A Szemerédi analogue of Gallai's theorem would read: given any B and k, there is a number $n_2(k, B)$ such that if $n \geqslant n_2(k, B)$ and S is a subset of $\mathbb{Z}^2$ with n points and diameter at most $B(n)^{1/2}$, then S has a subset $I \times I$ where I is an integer arithmetic progression of length k. This statement, generalized to each $\mathbb{Z}^m$, is a conjecture of Erdös. In [17], Szemerédi announced that he and Ajtai have established Erdös's conjecture for the case $k = 2$. Choi [3] has shown Erdös's conjecture but with the weaker conclusion that S has a subset $I \times J$ where I and J are integer arithmetic progressions of length k.

For the second approach of generalizing the van der Waerden and Szemerédi results to $\mathbb{Z}^2$, namely the consideration of sequences with "fairly small" gaps, it is not immediately clear how the conclusion of a result should read. However, it is obvious that a conclusion as in Gallai's theorem is not to be had. It is not so obvious that we cannot even expect to find long arithmetic progressions as subsets of a slowly growing sequence. That this is the case follows as an easy corollary of a theorem of Justin (See Brown [1]). Thus it is possible to construct an infinite sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots,$ in $\mathbb{Z}^2$ such that each $\mathbf{u}_i - \mathbf{u}_{i-1}$ is (1, 0) or (0, 1) and such that no six of the $\mathbf{u}_i$ form an arithmetic progression. From a very recent result of Dekking [4], this last result can be improved so that no five of the $\mathbf{u}_i$ form an arithmetic progression.

One may think geometrically of an aithmetic progression in $\mathbb{Z}^2$ as a set of collinear points that are equally spaced on their line. Thus there are two natural weakenings of the concept of arithmetic progression. First, we can ask for points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ such that

$$\| \mathbf{u}_2 - \mathbf{u}_1 \| = \| \mathbf{u}_3 - \mathbf{u}_2 \| = \cdots = \| \mathbf{u}_k - \mathbf{u}_{k-1} \| .$$

Second, we can ask for points $\mathbf{u}_1, \ldots, \mathbf{u}_k$ that are collinear.

In Ramsey [15], the following result is proved. For each number B and each k, there is a number $m_0(k, B)$ such that if $m \geqslant m_0(k, B)$ and $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_m$ are points in $\mathbb{Z}^2$ with each

$$\| \mathbf{u}_i - \mathbf{u}_{i-1} \| \leqslant B, \; 1 \leqslant i \leqslant m,$$

then k of the $\mathbf{u}_i$ are collinear. (Ramsey's theorem for the case $B = 1$ is accomplished in [12], but with a stronger hypothesis: each $\mathbf{u}_i - \mathbf{u}_{i-1} = (1, 0)$

or (0, 1).) Thus Ramsey's theorem, like Gallai's theorem, is a van der Waerden analog for $\mathbb{Z}^2$.

It is the purpose of this paper to prove a Szemerédi analog for $\mathbb{Z}^2$ in the direction opened up by Ramsey. Thus we prove that for every B, k there is a number $m_1(k, B)$ such that if $m \geqslant m_1(k, B)$ and $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_m$ are points in $\mathbb{Z}^2$ with

$$\sum_{i=1}^{m} \| \mathbf{u}_i - \mathbf{u}_{i-1} \| \leqslant Bm,$$

then k of the $\mathbf{u}_i$ are collinear. Our proof is elementary.

Erdös's conjecture and Choi's theorem mentioned above are true generalizations of Szemerédi's theorem in that both statements clearly imply Szemerédi's theorem. However the result in this paper does not appear to imply Szemerédi's theorem (nor does Szemerédi's theorem appear to imply our result). Thus properly speaking our result is an analog of Szemerédi's theorem, not a generalization.

An important defect in the proof of our theorem is that it is very indirect. It would be nice to have an effective upper bound for our function $m_1(k, B)$, but we are not very close to finding such a bound. In Gerver [9] and Ramsey and Gerver [16] it is shown that there exist positive constants c_1, c_2 with

$$B^2 \exp(c_1(\log k)^2) < m_0(k, B) < \exp(c_2 B^4 K^4),$$

where $m_0(k, B)$ is the function in Ramsey's theorem. Since $m_0(k, B) \leqslant m_1(k, B)$, the above lower bound also gives a result for $m_1(k, B)$.

In Ramsey and Gerver [16], an infinite sequence $\mathbf{u}_0, \mathbf{u}_1, \ldots,$ in $\mathbb{Z}^3$ is constructed such that each $\mathbf{u}_i - \mathbf{u}_{i-1}$ is (1, 0, 0), (0, 1, 0), or (0, 0, 1) and such that no $5^{11} + 1$ of the $\mathbf{u}_i$ are collinear. Thus our result (and even Ramsey's result) has no direct generalization to higher dimensions. However probably our result does generalize directly to each $\mathbb{Z}^m$ if we replace the conclusion of points being collinear with being co-hyperplanar. Ramsey's theorem [15] is so generalizable. If collinearity is insisted on in $\mathbb{Z}^3$, perhaps the following is true. If in addition to assumptions about the sequence in $\mathbb{Z}^3$ having small (or average small) gaps, the additional hypothesis is made that the number of points in the sequence is $>CR$ where C is some large constant that depends on k and B and where R is the diameter of the sequence, then for every k there are k collinear points in the sequence.

In a letter, P. Erdös suggested the following problem. Find a weak condition on the rate of growth of the infinite integer sequence $a_1 < a_2 < \cdots$ that assures for every k that there are k points (n, a_n) collinear. It was work on this problem that led me to the theorem in this paper. Our theorem has the corollary that if $\{a_n\}$ has positive upper density, then for every k there are k

points (n, a_n) that are collinear. A more general corollary is that if for increasing integer sequences $\{a_n\}$ and $\{b_n\}$ we have positive upper density for $\{a_n\}$ and positive lower density for $\{b_n\}$, then for every k there are k points (a_n, b_n) which are collinear. It is not hard to construct an example of sequences $\{a_n\}$, $\{b_n\}$ both with positive upper density such that no three points (a_n, b_n) are collinear. In [13] we proved that for every k there are k points (n, p_n) collinear where p_n denotes the n-th prime.

We leave as an open question the problem mentioned above of finding hypotheses on plane lattice point sequences that would permit a conclusion that there are k equally spaced points in a subset.

We take this opportunity to thank P. Erdös for suggesting his problem we mentioned above, R. L. Graham for informing me of Brown [1], and E. R. Canfield for critically listening to the details of the proof.

2. The Proof

If U denotes the sequence of plane lattice points $(\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_m)$, let

$$d(U) = \frac{1}{m} \sum_{i=1}^{m} \| \mathbf{u}_i - \mathbf{u}_{i-1} \|.$$

We shall prove the following

THEOREM. *Given any positive integer k and any positive real B, there is a number $m(k, B)$ such that if U is a plane lattice point sequence with more than $m(k, B)$ terms and if $d(U) \leqslant B$, then U has at least k collinear terms.*

Let k be an arbitrary but fixed positive integer. For each positive integer m, let

$$C(m) = \{U \in (\mathbb{Z}^2)^{m+1}\text{: no } k \text{ points of } U \text{ are collinear}\},$$

$$d(m) = \min\{d(U)\colon U \in C(m)\},$$

$$d = \liminf_{m \to \infty} d(m).$$

It is easy to see that the theorem for k is equivalent to the assertion that $d = +\infty$. Indeed, if the theorem is true, then for each B and $m > m(k, B)$, we have $d(m) > B$. Hence $d = +\infty$. Conversely, if the theorem is false, there is a number B and arbitrarily long plane lattice point sequences U with no k points collinear and $d(U) \leqslant B$. Thus there are infinitely many m with $d(m) \leqslant B$, so $d \leqslant B$.

In what follows, we shall assume the theorem is false for k. Thus $d < \infty$. Our method of attack will be to look at members of $C(m)$ for which $d(U)$ is close to d. Then we shall consider a subset consisting of sequences which are as "stretched out" as possible, length being measured as the distance from the first to the last term. Then among these "stretched out" sequences we shall pass to a subset consisting of sequences which are as "steep" as possible, slope being computed from just the first and last terms of a sequence. We then show that at least half the points of one of our "stretched out", "steep" sequences are near the line joining the first and last terms. We then use a pigeon hole argument to show that these sequences have many collinear points, thus providing a contradiction and establishing the theorem for k.

For each $t > 0$, let

$$D(m, t) = \{U \in C(m): |\, d(U) - d \,| < t\}.$$

If $U = (\mathbf{u}_0, \ldots, \mathbf{u}_m)$ and $0 \leqslant j \leqslant m$, let

$$U_j = (\mathbf{u}_0, \ldots, \mathbf{u}_j),$$
$${}_jU = (\mathbf{u}_j, \ldots, \mathbf{u}_m).$$

LEMMA 1. *For each $\epsilon > 0$, there is an $m_0(\epsilon)$ such that for all $m \geqslant m_0(\epsilon)$, if $U \in D(m, \epsilon/7)$ and $m/4 \leqslant j \leqslant 3m/4$, then $U_j \in D(j, \epsilon)$ and ${}_jU \in D(m-j, \epsilon)$.*

Proof. There is an $m_1(\epsilon)$ such that for all $m \geqslant m_1(\epsilon)$, $d(m) > d - \epsilon/7$. Let $m_0(\epsilon) = 4m_1(\epsilon)$ and let $m \geqslant m_0(\epsilon)$. Suppose $U \in D(m, \epsilon/7)$ and $m/4 \leqslant j \leqslant 3m/4$. Since $U_j \in C(j)$ and ${}_jU \in C(m-j)$, we have

$$d(U_j) > d - \epsilon/7, \; d({}_jU) > d - \epsilon/7.$$

Say $d(U_j) \geqslant d + \epsilon$. Then

$$\begin{aligned} d + \epsilon/7 > d(U) &= \frac{j}{m}\, d(U_j) + \left(1 - \frac{j}{m}\right) d({}_jU) \\ &> \frac{j}{m}(d + \epsilon) + \left(1 - \frac{j}{m}\right)(d - \epsilon/7) \\ &\geqslant (d + \epsilon)/4 + 3(d - \epsilon/7)/4 = d + \epsilon/7, \end{aligned}$$

a contradiction. We similarly have ${}_jU \in D(m-j, \epsilon)$. ∎

If U is a finite sequence of plane lattice points, let $\lambda(U)$ denote the line segment joining the first and last points of U if these points are not the same, and let $\lambda(U)$ denote this point if they are the same. If U has $m + 1$ points, let $l(U)$ denote $1/m$ times the length of $\lambda(U)$. Let

$$l(m, t) = \begin{cases} \max\{l(U)\colon U \in D(m, t)\}, & \text{if } D(m, t) \neq \varnothing \\ 0, & \text{if } D(m, t) = \varnothing, \end{cases}$$

$$l(t) = \limsup_{m\to\infty} l(m, t),$$

$$l = \lim_{t\to 0^+} l(t).$$

We note that for any fixed $t > 0$, there are infinitely many m for which $D(m, t) \neq \varnothing$. Moreover, if $0 < t' < t$, then $D(m, t') \subset D(m, t)$, $l(m, t') \leqslant l(m, t)$, and $l(t') \leqslant l(t)$.

LEMMA 2. $1/3k \leqslant l \leqslant d$.

Proof. If $U \in D(m, t)$, by the triangle inequality we have

$$l(U) \leqslant d(U) < d + t.$$

Hence each $l(m, t) < d + t$, so that $l(t) \leqslant d + t$, and $l \leqslant d$.

Let $\epsilon > 0$ be arbitrary and let $m_0(\epsilon)$ be as in Lemma 1. Let $m \geqslant m_0(\epsilon)$ be such that $D(m, \epsilon/7) \neq \varnothing$. Let $U \in D(m, \epsilon/7)$, $U = (\mathbf{u}_0, \ldots, \mathbf{u}_m)$. Then the points $\mathbf{u}_j$, $m/4 \leqslant j \leqslant 3m/4$, cover a horizontal breadth of at least $m/2k$. For if not, at least k of these points $\mathbf{u}_j$ lie on the same vertical line, contradicting $U \in C(m)$. Thus $\mathbf{u}_0$ is at least distance $m/4k$ from one of these $\mathbf{u}_j$. That is, for some j, $m/4 \leqslant j \leqslant 3m/4$,

$$l(U_j) \geqslant m/4kj \geqslant 1/3k.$$

But $U_j \in D(j, \epsilon)$ by Lemma 1. Hence $l(j, \epsilon) \geqslant 1/3k$. There are infinitely many m for which the above argument holds. Hence there are infinitely many values of j for which $l(j, \epsilon) \geqslant 1/3k$. Hence $l(\epsilon) \geqslant 1/3k$. Since $\epsilon > 0$ is arbitrary, we have $l \geqslant 1/3k$. ∎

Now let

$$L(m, t) = \{U \in D(m, t) : | l(U) - l | < t\}.$$

It is easy to see that for each $t > 0$, there are infinitely many m such that $L(m, t) \neq \varnothing$. Indeed there is an ϵ with $t \geqslant \epsilon > 0$ such that $0 \leqslant l(\epsilon) - l < t/2$. There are infinitely many m such that $| l(m, \epsilon) - l(\epsilon)| < t/2$. For these m we have $| l(m, \epsilon) - l | < t$. If we choose $t \leqslant 1/3k$, by Lemma 2 we have $l(m, \epsilon) > 0$ for these m. Thus for each such m, there is a $U \in D(m, \epsilon)$ with

$l(U) = l(m, \epsilon)$. Since $D(m, \epsilon) \subset D(m, t)$ we have $U \in L(m, t)$. If $t > 1/3k$, we note that $L(m, 1/3k) \subset L(m, t)$.

If $U = (\mathbf{u}_0, ..., \mathbf{u}_m)$ is a sequence of plane lattice points, let $x(U)$ denote $1/m$ times the maximal distance of a $\mathbf{u}_j$, $m/4 \leqslant j \leqslant 3m/4$, from $\lambda(U)$.

LEMMA 3. *For each $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ and an $m_2(\epsilon)$ such that for all $m \geqslant m_2(\epsilon)$, if $U \in L(m, \delta(\epsilon))$, then $x(U) < \epsilon$.*

Proof. Assume the lemma is false. Then there is an $\epsilon > 0$ and an infinite sequence $t(1) > t(2) > \cdots$ of positive numbers converging to 0 such that for each $t(i)$ there are infinitely many m for which there is a $U \in L(m, t(i))$ with $x(U) \geqslant \epsilon$. Let $m(1) > m_0(t(1))$ (cf. Lemma 1) such that there is a $U^1 \in L(m(1), t(1))$ with $x(U^1) \geqslant \epsilon$. If $m(1), ..., m(i-1)$, $U^1, ..., U^{i-1}$ have already been defined, let $m(i) > \max\{m_0(t(i)), m(i-1)\}$ such that there is a $U^i \in L(m(i), t(i))$ with $x(U^i) \geqslant \epsilon$. Say $U^i = (\mathbf{u}_0^i, ..., \mathbf{u}_{m(i)}^i)$. Let $j(i) \in [m(i)/4, 3m(i)/4]$ be such that the distance of $\mathbf{u}_{j(i)}^i$ from $\lambda(U^i)$ is at least $\epsilon \cdot m(i)$.

From Lemma 1, we have each $U_{j(i)}^i \in D(j(i), 7t(i))$ and each ${}_{j(i)}U^i \in D(m(i) - j(i), 7t(i))$. Since $t(i)$ is monotone decreasing to 0 and $m(i) \to \infty$, we have

$$\limsup_{i\to\infty} l(U_{j(i)}^i) \leqslant l, \qquad \limsup_{i\to\infty} l({}_{j(i)}U^i) \leqslant l. \tag{1}$$

Since each $U^i \in L(m(i), t(i))$, we have

$$\lim_{i\to\infty} l(U^i) = l. \tag{2}$$

Consider for each i the triangle with vertices

$$A_i = \mathbf{u}_0^i/m(i), \qquad B_i = \mathbf{u}_{j(i)}^i/m(i), \qquad C_i = \mathbf{u}_{m(i)}^i/m(i).$$

By (1) and (2), for each $\mu > 0$, there is an $i_0(\mu)$ such that for $i > i_0(\mu)$ we have

$$A_iB_i = \frac{j(i)}{m(i)}\, l(U_{j(i)}^i) < \frac{j(i)}{m(i)}\,(l + \mu), \tag{3}$$

$$B_iC_i = \left(1 - \frac{j(i)}{m(i)}\right) l({}_{j(i)}U^i) < \left(1 - \frac{j(i)}{m(i)}\right)(l + \mu), \tag{4}$$

$$l - \mu < l(U^i) = A_iC_i < l + \mu. \tag{5}$$

Now the altitude of triangle $A_iB_iC_i$ from B_i is, by assumption, at least ϵ. Then if we take $\mu = \epsilon^2/l$ and use (3), (4), (5), we have for $i > i_0(\mu)$

$$\begin{aligned} l + \mu &> A_iB_i + B_iC_i \geqslant ((A_iC_i)^2 + 4\epsilon^2)^{1/2} \\ &> ((l-\mu)^2 + 4\epsilon^2)^{1/2} = l + \mu, \end{aligned}$$

a contradiction. ∎

Let

$$X(m, t) = \{U \in L(m, t): x(U) < t\}.$$

It follows from Lemma 3 that for each $t > 0$, there are infinitely many m such that $X(m, t) \neq \varnothing$. Indeed, there are infinitely many m such that $m > m_2(t)$ and $L(m, \delta'(t)) \neq \varnothing$, where $\delta'(t) = \min\{t, \delta(t)\}$. But if $U \in L(m, \delta'(t))$, by Lemma 3, $x(U) < t$. Since $L(m, \delta'(t)) \subset L(m, t)$, it follows that $U \in X(m, t)$.

Let $s(U)$ denote the slope of $\lambda(U)$ (if $\lambda(U)$ is vertical, let $s(U) = +\infty$; if $\lambda(U)$ is one point, let $s(U) = -\infty$). For each $t > 0$ and each m, let

$$s(m, t) = \begin{cases} \max\{s(U): U \in X(m, t)\}, & \text{if } X(m, t) \neq \varnothing, \\ 0, & \text{if } X(m, t) = \varnothing, \end{cases}$$

$$s(t) = \limsup_{m \to \infty} s(m, t),$$

$$s = \lim_{t \to 0^+} s(t).$$

Note that if $0 < t' < t$, then $X(m, t') \subset X(m, t)$, $s(m, t') \leqslant s(m, t)$, and $s(t') \leqslant s(t)$.

LEMMA 4. $1 \leqslant s \leqslant 2kl$.

Proof. Note that if $U \in X(m, t)$ and $0 < t < l$, then $\lambda(U)$ is not a single point. Also, by reflecting U in the line $y = 0$ and/or in the line $y = x$ we get a congruent copy U' of U that is also in $X(m, t)$. Thus $s(m, t) \geqslant 1$. Since there are infinitely many m with $X(m, t) \neq \varnothing$, we thus have $s(t) \geqslant 1$. Since $0 < t < l$ is arbitrary, we have $s \geqslant 1$.

Let $t < 1/4k$ and let $U = (\mathbf{u}_0, \ldots, \mathbf{u}_m)$, $U \in X(m, t)$, $s(U) \geqslant 1$. Now as in the proof of Lemma 2, the horizontal breadth of the points $\mathbf{u}_j$ for $m/4 \leqslant j \leqslant 3m/4$ is at least $m/2k$. But each of these points has distance from $\lambda(U)$ less than $mt < m/4k$. Hence the segment $\lambda(U)$ has horizontal breadth at least

$$m/2k - 2mt > 0.$$

Thus

$$s(U) \leqslant \frac{m \cdot l(U)}{m/2k - 2mt} = \frac{2k \cdot l(U)}{1 - 4kt} < \frac{2k(l + t)}{1 - 4kt}.$$

Thus $s(t) \leqslant 2k(l + t)/(1 - 4kt)$, so that $s \leqslant 2kl$. ∎

Let now $1 < b_1 < b_2 < \cdots$ be integers for which there exist integers $a_1, a_2, \ldots,$ such that

$$| s - a_i/b_i | < 1/b_i^2.$$

(Note that if s is irrational, the existence of the a_i/b_i is a well-known result, while if s is rational, we can let each $a_i/b_i = s$, since we do not insist that $(a_i, b_i) = 1$.) Then there are integers $m_1 < m_2 < \cdots$ and sequences $U^1, U^2, \ldots$, such that each

$$U^i \in X(m_i, 1/b_i^2), \mid s(U^i) - s \mid < 1/b_i^2.$$

(The proof that such U^i exist is similar to the proof above that for each $t > 0$ there are infinitely many m for which $L(m, t) \neq \varnothing$.)

For each i, consider the set of lines with slope a_i/b_i which pass through lattice points and whose distance from $\lambda(U^i)$ is less than m_i/b_i^2. Since $x(U^i) < 1/b_i^2$, it follows that these lines touch at least $m_i/2$ points of U^i. We now count the number of lines we are considering.

Note that

$$\mid s(U^i) - a_i/b_i \mid < 2/b_i^2$$

and that

$$s(U^i) > s - 1/b_i^2 \geqslant 1 - 1/b_i^2 \geqslant 3/4.$$

Since $l(U^i) < l + 1/b_i^2$, an elementary calculation shows that each of our lines is within horizontal distance less than

$$(4 + 4l)\, m_i/b_i^2$$

of $\mathbf{u}_0{}^i$, the first point of U^i. Indeed, the exact distance is

$$(1 + s(U^i)^2)^{-1/2}\, [(s(U^i) + b_i/a_i)\, M + \mid 1 - s(U^i)\, b_i/a_i \mid l(U^i)],$$

where M is the maximal distance of one of our lines from $\lambda(U^i)$. Hence there are at most

$$8a_i(1 + l)\, m_i/b_i^2$$

such lines.

From the above considerations it follows that at least one of these lines has at least

$$\frac{m_i/2}{8a_i(1 + l)\, m_i/b_i^2} = \frac{b_i^2}{16a_i(1 + l)} > \frac{b_i}{16(s + 1)(1 + l)}$$

points of U^i on it. Since $b_i \to \infty$ as $i \to \infty$, this last expression is eventually bigger than k, contradicting $U^i \in C(m_i)$. This contradiction establishes our theorem for k.

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