

**The covering congruences of
Paul Erdős**

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Conjecture (Erdős, 1950): *For each number B , one can cover \mathbb{Z} with finitely many congruences to distinct moduli all $> B$.*

Erdős (1995):

“Perhaps this is my favorite problem.”

Early origins

Are there infinitely many primes of the form $2^n - 1$?

Euclid: *n must be prime, but this is not sufficient.*

For example,

$2^2 - 1$, $2^3 - 1$, $2^5 - 1$, $2^7 - 1$ are prime, but $2^{11} - 1 = 23 \times 89$.

Euclid: *If $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is **perfect**. (That is, it is equal to the sum of its proper divisors.)*

Euler: *All even perfect numbers are in Euclid's form.*

Primes of the form $2^n - 1$ are called **Mersenne** primes. There are 44 of them known, the largest being

$$2^{32\,582\,657} - 1.$$

See www.mersenne.org .

Early origins, cont'd

Are there infinitely many primes of the form $2^n + 1$?

Fermat: *A necessary condition is that n is a power of 2.* He conjectured this is also sufficient.

For example, $2^1 + 1$, $2^2 + 1$, $2^4 + 1$, $2^8 + 1$, $2^{16} + 1$ are all prime.

Euler:

$$2^{32} + 1 = 641 \times 6700417.$$

No other Fermat primes are known; $2^{2^k} + 1$ is composite for $k = 5, 6, \dots, 32$ and for many higher, sporadic values of k .

Gauss, Wantzel: *A regular n -gon is constructible with straight-edge and compass if and only if n is a power of 2 times a product of distinct Fermat primes.*

A mathematician's credo:

**If you can't solve it,
generalize!**

For each odd number k , are there infinitely many primes of the form $2^n + k$?

OK, way too hard! Lets try:

For each odd number k , there is at least one prime of the form $2^n + k$.

(conjectured by de Polignac in 1849)

$$61 + 2 = 63, \quad \{3, 7\}.$$

Mod 3, the powers of 2 are
2, 1, 2, 1, ...
(period 2).

So,

$$n \equiv 1 \pmod{2} \Rightarrow 61 + 2^n \equiv 0 \pmod{3}.$$

Mod 7, the powers of 2 are
2, 4, 1, 2, 4, 1, ...
(period 3).

So,

$$n \equiv 1 \pmod{3} \Rightarrow 61 + 2^n \equiv 0 \pmod{7}.$$

Also

$$61 + 2^2 = 65, \quad \{5, 13\}.$$

Mod 5, powers of 2 are

2, 4, 3, 1, ...

(period 4).

So,

$$n \equiv 2 \pmod{4} \Rightarrow 61 + 2^n \equiv 0 \pmod{5}.$$

Conclude:

$61 + 2^n$ is composite for

$$n \equiv 1 \pmod{2},$$

$$n \equiv 1 \pmod{3},$$

$$n \equiv 2 \pmod{4}.$$

$n \equiv 1 \pmod{2}$:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...

$n \equiv 1 \pmod{3}$:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...

$n \equiv 2 \pmod{4}$:

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...

$$n \equiv 1 \pmod{2}, n \equiv 1 \pmod{3},$$

or

$$n \equiv 2 \pmod{4}:$$

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...

And, $61 + 2^8 = 317$, a prime.

So [de Polignac](#) is still safe, but not for long.

Lets automate the idea:

p period of powers of 2

3 2

5 4

7 3

13 12

17 8

241 24

We can use the moduli 2, 4, 3, 12, 8, 24 to **cover** \mathbb{Z} :

Every $n \in \mathbb{Z}$ is either

1 (mod 2), 2 (mod 4),
1 (mod 3), 8 (mod 12),
4 (mod 8), 0 (mod 24).

So, if k **simultaneously** is

$$\begin{array}{ll} -2^1 \pmod{3}, & -2^2 \pmod{5}, \\ -2^1 \pmod{7}, & -2^8 \pmod{13}, \\ -2^4 \pmod{17}, & -2^0 \pmod{241}, \end{array}$$

then

$$\gcd(2^n + k, 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241) > 1$$

for all n .

We also ask for k to be odd.
By the magic of the Chinese
Remainder Theorem, we can
find an infinite arithmetic
progression of such numbers k :

$$k \equiv 9\,262\,111 \pmod{11\,184\,810}.$$

In particular, $2^n + 9\,262\,111$ is
composite for all n .

Erdős (1950): *de Polignac's conjecture is false.*

Note, the same calculations show that $k \cdot 2^n + 1$ is composite for all n for the same values of k . Sierpiński had a short paper about such k in 1960.

An odd number k with $k \cdot 2^n + 1$ composite for all n is now known as a **Sierpiński number**. They are useful in finding factors of large Fermat numbers.

Conjecture (Selfridge): *The least Sierpiński number is $k = 78\,557$.*

In 2002, for all but 17 values of $k < 78\,557$, a prime had been found of the form $k \cdot 2^n + 1$.

Thus began the website

www.seventeenorbust.com ([Helm](#) and [Norris](#)). Now there are just 6 remaining values of k for which no prime is known:

10223, 21181, 22699,
24737, 55459, 67607.

They've only been looking for primes $k \cdot 2^n + 1$ with $n > 0$, so my contribution:

k	n	k	n
10223	– 19	21181	– 28
22699	– 26	24737	– 17
55459	– 14	67607	– 16389

Seventeen or bust? **Busted!**

Unsolved:

Erdős: If k is a Sierpiński number, must the sequence of least prime factors of $k \cdot 2^n + 1$ be bounded?

Filaseta, Finch, Kozek: Is the sequence of least prime factors of $5 \cdot 2^n + 1$ unbounded?

Erdős: Lets forget about powers of 2 and just look for congruences that cover \mathbb{Z} .

For example: $0 \pmod{1}$

Another example: $0 \pmod{2}$,
 $1 \pmod{2}$

Too easy!

Insist that the moduli be distinct and > 1 .

Example: $0 \pmod{2}$,
 $0 \pmod{3}$, $1 \pmod{4}$,
 $1 \pmod{6}$, $11 \pmod{12}$

What about least modulus
 $> 2?$, $> 3?$, ...

Conjecture (Erdős, 1950): *For each number B , one can cover \mathbb{Z} with finitely many congruences to distinct moduli all $> B$.*

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Records:

min modulus	discovered by
9	Churchhouse (1968)
18	Krukenberg (1971)
20	Choi (1971)
24	Morikawa (1981)
25	Gibson (2006)
36	Nielsen (2007)

Erdős, Selfridge: Is there a covering of \mathbb{Z} with distinct odd moduli > 1 ?

Erdős: Yes.

Selfridge: No.

Note: $0 \pmod{2}$, $1 \pmod{2}$
exactly covers \mathbb{Z} in that each n
satisfies exactly one
congruence.

Erdős: Can one exactly cover \mathbb{Z}
with distinct moduli > 1 ?

Mirsky, Newman, Znam: *No.*

Say $\{a_i \pmod{b_i}\}$, $i \leq k$, exactly covers \mathbb{Z} . Numbers $\equiv a \pmod{b}$ are represented by

$$z^a + z^{a+b} + z^{a+2b} + \dots = \frac{z^a}{1 - z^b}.$$

So,

$$\sum_{i=1}^k \frac{z^{a_i}}{1 - z^{b_i}} = \frac{1}{1 - z},$$

and the largest b_i is 1 or is repeated.

Note: A covering $\{a_i \pmod{b_i}\}$ is exact iff $\sum 1/b_i = 1$.

Can one have a covering with distinct moduli $b_i > 1$ and $\sum 1/b_i$ arbitrarily close to 1?

Yes, take progressions 2^{i-1}
(mod 2^i) for $i = 1, 2, \dots, 1000$,
say. This covers everything
except $0 \pmod{2^{1000}}$.

Cover this with

$0 \pmod{2 \cdot 2^{1000}}$,
 $0 \pmod{3 \cdot 2^{1000}}$, etc., where
we are copying over the
 $0 \pmod{2}$, $0 \pmod{3}$, etc.
covering from before.

Similarly, one can find coverings with distinct moduli with least modulus 3, and with least modulus 4, with the moduli reciprocal sum arbitrarily close to 1.

What about least modulus 5, or larger?

Conjecture (Erdős, Selfridge).

For each N there is a B :

*if $\{a_i \pmod{b_i}\}$ is a covering
with distinct moduli $> B$, then*

$$\sum 1/b_i > N.$$

Theorem. *Yes.*

(Filaseta, Ford, Konyagin, P,
Yu 2007).

Corollary. *For each $K > 1$, there is some B_0 so that for $B \geq B_0$ there is no covering with distinct moduli from $[B, KB]$.*

Conjecture (Erdős, Graham).

For each $K > 1$, there are $d_K > 0, B_0$ such that for $B \geq B_0$ and for any congruences with distinct moduli from $[B, KB]$, at least density d_K of \mathbb{Z} remains uncovered.

Theorem. *Yes.*

(Filaseta, Ford, Konyagin, P,
Yu 2007).

In fact, any d_K with
 $0 < d_K < 1/K$ works.

For example, if B is large, at
most $1/2 + \epsilon$ of \mathbb{Z} can be
covered with congruences with
distinct moduli from $[B, 2B]$.

In analogy to the Lovász local lemma:

Suppose we have moduli b_1, \dots, b_t . Let

$$\alpha = \prod \left(1 - \frac{1}{b_i}\right), \quad \beta = \sum_{\substack{i < j \\ \gcd(b_i, b_j) > 1}} \frac{1}{b_i b_j}.$$

Then no matter the choice of residues, at least $\alpha - \beta$ of \mathbb{Z} remains uncovered.

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