# On the proportion of numbers coprime to a given integer

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**Abstract**

For a positive integer $n$ and its Euler function $\phi(n)$ we write $\phi(n)/n = a/b$, where $a = a(n)$ and $b = b(n)$ are coprime. For a fixed integer $a$, we consider the number of integers $b$ for which the above relation holds for some $n$, and we also fix $b$ and count corresponding $a$'s. We discuss the greatest common divisor of $n$ and $\phi(n)$, applying it to the relation $\phi(n) \mid f(n)$ for $f$ a polynomial.

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# 1 Introduction

For a positive integer $n$ we write $\phi(n)$ for the Euler function of $n$, namely the number of integers in $[1, n]$ coprime to $n$. The fraction $\phi(n)/n$ is thus the asymptotic density of the set of the positive integers relatively prime to $n$. This proportion has been extensively studied. For example, it was known to Euler that $\{\phi(n)/n\}_{n\geq 1}$ is dense in $[0, 1]$, and one of the earliest results concerning the distribution of values of arithmetic functions is Schoenberg's 1928 theorem (see [15]) to the effect that $\phi(n)/n$ possesses a continuous distribution in $[0, 1]$. That is, $D(u)$, defined as the asymptotic density of the set of those positive integers $n$ such that $\phi(n)/n \leq u$, exists for every real number $u$. In addition, $D(u)$ is continuous and strictly increasing on $[0, 1]$ (and clearly $D(0) = 0$ and $D(1) = 1$). This result might be argued to mark the dawn of probabilistic number theory.

This paper discusses several arithmetic functions that are directly related to the ratio $\phi(n)/n$. First, it is natural to reduce this fraction to its lowest terms: $\phi(n)/n = a/b$. We write $a = a(n)$ and $b = b(n)$, so that

$$\phi(n)/n = a(n)/b(n), \quad \gcd(a(n), b(n)) = 1.$$

Let $\mathrm{rad}(n)$ denote the largest squarefree number that divides $n$. Since

$$\phi(n)/n = \prod_{p|n,\ p \text{ prime}} (1 - 1/p) = \phi(\mathrm{rad}(n))/\mathrm{rad}(n),$$

we have

$$a(n) = a(\mathrm{rad}(n)), \quad b(n) = b(\mathrm{rad}(n))$$

for every positive integer $n$. For positive integers $a, b$, we put

$$f(a) = \#\{n \text{ squarefree} : a(n) = a\},$$

$$g(b) = \#\{n \text{ squarefree} : b(n) = b\}.$$

For example, $f(1)$ is the number of squarefree $n$ with $\phi(n) \mid n$. It is easy to see that $n = 1, 2, 6$ are the only such numbers, so that $f(1) = 3$, a fact recorded in [16], p. 232.

As we shall see in the next section, the function $n \mapsto \phi(n)/n$ is one-to-one when restricted to squarefree numbers, so we have the alternate definitions

$$f(a) = \#\{b \ : \ \gcd(a, b) = 1 \text{ and } a/b = \phi(n)/n \text{ for some } n\},$$

$$g(b) = \#\{a \ : \ \gcd(a,b) = 1 \text{ and } a/b = \phi(n)/n \text{ for some } n\}.$$

For any given numbers $a, b$, we present simple finite procedures for computing the values $f(a), g(b)$. We discuss the maximal orders of the arithmetic functions $f(a), g(b)$, and we show these functions are normally 0. We also discuss $\gcd(n, \phi(n)) = n/b(n) = \phi(n)/a(n)$. We show that normally it is the largest divisor of $n$ composed of primes at most $\log\log n$, and on average it is bounded by $n^{o(1)}$. We also consider its maximal order when restricted to squarefree values of $n$. Our result on the average order of $\gcd(n, \phi(n))$ has an application in the counting of solutions of certain polynomial congruences.

Throughout this paper, we use the order symbols $\gg$, $\ll$, $\asymp$, $O$, and $o$ with their ususal meanings in analytic number theory. They are all absolute except where specified differently as in Theorem 11 and in Section 6. For a positive real number $x$, we use $\log x$ for the natural logarithm of $x$, $\log_1 x = \max\{1, \log x\}$, and if $k \geq 2$ we use $\log_k x$ for the $k$-fold iterated composition of the function $\log_1$ evaluated at $x$. We use $p$ and $q$ with or without subscripts for prime numbers. We use $\pi(x)$ and $\pi(x; b, a)$ for the number of primes $p \leq x$ and the number of primes $p \leq x$ in the arithmetic progression $p \equiv a \pmod b$, respectively. We use the notation $v_p(n)$ for the exponent on the prime $p$ in the prime factorization of the natural number $n$. (In particular, if $p \mid n$, then $p^{v_p(n)} \| n$, and if $p \nmid n$, then $v_p(n) = 0$.) We let $P(n)$ denote the greatest prime factor of $n$ if $n > 1$, and we let $P(1) = 1$. We let $\tau(n)$ denote the number of divisors of $n$, and $\omega(n)$ denotes the number of these divisors that are prime. We let $p_i$ denote the $i$-th prime. We also use $c_0, \ c_1, \ \ldots$ for positive computable constants.

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## 2 An algorithm

In this section we give simple procedures for the evaluation of the functions $f(a)$ and $g(b)$. There is no emphasis on efficiency, only on the existence of a deterministic procedure for the evaluations. We begin with a simple lemma.

**Lemma 1.** *Let $a, b$ be coprime natural numbers. If there is a natural number $m$ with $\phi(m)/m = a/b$, then*

*(i)* $0 < a/b \leq 1$;

*(ii)* $b$ *is squarefree;*

*(iii) there is a unique squarefree number $n$ with $\phi(n) = a/b$;*

*(iv)* $P(b) = P(n)$;

*(v)* $\gcd(b, \phi(b)) = 1$;

*(vi)* $\omega(n) \leq v_2(a) + 2$.

*Proof.* The first assertion follows immediately from $0 < \phi(m) \leq m$ for all $m$. As we saw in the Introduction, if $\phi(m)/m = a/b$, then $n = \mathrm{rad}(m)$ is squarefree and $\phi(n)/n = a/b$. Thus, $b \mid n$, so that $b$ is squarefree. Suppose $n_1, n_2$ are squarefree numbers with $\phi(n_i)/n_i = a/b$ for $i = 1, 2$. Since $\phi(n) < n$ for $n > 1$, it follows that $n_1 = n_2 = 1$ if $a/b = 1$. Suppose that $a/b < 1$, so that $n_1, n_2 > 1$. As $P(n_i) \nmid \phi(n_i)$, we have $P(n_i) \mid b$. But $b \mid n_i$ implies $P(b) \leq P(n_i)$ for $i = 1, 2$, so that $P(n_1) = P(n_2) = P(b)$. That is, if $n_1, n_2$ are squarefree and $\phi(n_1)/n_1 = \phi(n_2)/n_2$, then $P(n_1) = P(n_2)$. We thus may replace $n_i$ with $n_i/P(n_i)$ and iterate, coming to the conclusion that $n_1 = n_2$. This proves the uniqueness assertion in (iii), and in the course of the proof, we saw (iv).

To prove (v), assume not and let $p$ be a common prime factor of $b$ and $\phi(b)$. Let $n$ be squarefree with $a/b = \phi(n)/n$, so that $b \mid n$, and $\phi(b) \mid \phi(n)$. It follows that $v_p(\phi(n)) \geq 1 = v_p(n)$, so that in the reduction $\phi(n)/n$ to $a/b$, the denominator $b$ is not divisible by $p$ after all. Thus it must be that $\gcd(b, \phi(b)) = 1$.

To see the last assertion, let $k = \omega(n)$. Then $n$ is divisible by at least $k - 1$ odd prime factors, so that $v_2(\phi(n)) \geq k - 1$. But $n$ is squarefree, so that $v_2(n) \leq 1$ and $v_2(a) = v_2(\phi(n)) - v_2(n) \geq k - 2$. □

So, given coprime natural numbers $a, b$ we might ask how we might determine if $a/b$ is in the range of $\phi(n)/n$, and if it is, how we might find the unique squarefree pre-image $n$. The following algorithm gives such a procedure.

**Algorithm A**. Let $a, b$ be coprime natural numbers. If there is a squarefree number $n$ with $\phi(n)/n = a/b$, this algorithm finds $n$. If there is no such number $n$, this algorithm reports NONE.

1. Let $n = 1$;
2. If $a = b$, report $n$;
3. If $a > b$, report NONE;
4. If $b$ is not squarefree, report NONE;
5. Let $p = P(b)$, $d = \gcd(a, p-1)$, $a = a/d$, $b = (p-1)b/pd$, $n = pn$, and go to step 2;

It is clear from Lemma 1 that the algorithm correctly reports NONE when it does so. The iteration in the last step is based on the fact that if a squarefree number $n$ exists with $\phi(n)/n = a/b$, then Lemma 1 implies that $P(n) = P(b)$, so that if $p = P(n)$ and $N = n/p$, we have

$$\frac{\phi(N)}{N} = \frac{a/(p-1)}{b/p} = \frac{a/\gcd(a, p-1)}{(p-1)/\gcd(a,(p-1)) \cdot b/p} = \frac{a'}{b'},$$

say. We thus may reduce the problem to the new pair $a', b'$, justifying the last step of the algorithm.

We may use Algorithm A to compute $f(a)$, $g(b)$ as follows. Given a natural number $b$, run Algorithm A for each pair $a, b$ with $1 \le a \le b$ and $\gcd(a, b) = 1$. Count 1 for each time the algorithm reports a value of $n$ with $\phi(n)/n = a/b$, the total count being $g(b)$. For the $f(a)$ computation, note that if $\phi(n)/n = a/b$ for some $n$ and $b$ with $b$ coprime to $a$, then from Lemma 1, we have $k := v_2(a) + 2 \ge \omega(n)$. If such a number $n$ exists, then $\phi(n)/n \ge \prod_{i=1}^{k}(1 - 1/p_i)$. Let this product be denoted $z = z(a)$. Then $b = an/\phi(n) \le a/z$. So, to compute $f(a)$, for each integer $b$ coprime to $a$ with $a \le b \le a/z$, run Algorithm A and count 1 for each such $b$ where the algorithm reports a value for $n$ with $\phi(n)/n = a/b$. The total count is $g(b)$.

We remark that the algorithms in this section have not been optimized, our point merely being that there exists a deterministic procedure. The issue of computing the solutions $n$ to $\phi(n) = m$ has been discussed from the point of view of complexity in [4].

# 3 The function $f(a)$

We start with the maximal order of $f(a)$.

**Theorem 2.** *The inequality $f(a) \le (1+o(1))a \log_2 a/\log_3 a$ holds as $a \to \infty$. On the other hand, there exists a positive constant $c_0$ such that $f(a) > a^{c_0}$ holds for infinitely many $a$.*

*Proof.* Assume that $n$ is a positive integer such that $\phi(n)/n = a/b$ for some integer $b$ coprime to $a$. As we saw at the end of the last section, we have

$$b \le B := a \prod_{i=1}^{k} (1 - 1/p_i)^{-1},$$

where $k = v_2(a) + 2$. But $v_2(a) \le \log a/\log 2$, so that by the prime number theorem, $p_k \le (1 + o(1)) \log a \log\log a/\log 2$. Thus, by Mertens's theorem,

$$b \le B \le (e^{\gamma} + o(1))a \log\log a, \tag{1}$$

where $\gamma$ is the Euler–Mascheroni constant. But, by Lemma 1,

$$f(a) \le \#\{b \le B \ : \ \gcd(b, \phi(b)) = 1\}.$$

By a result of Erdős (see [6]), the cardinality of the set

$$\{n \le y \ : \ \gcd(n, \phi(n)) = 1\}$$

is $(e^{-\gamma} + o(1))y/\log_3 y$ as $y \to \infty$. Applying this with $y = B$ and using our upper bound for $B$, we get the first claim of the theorem.

For the second claim, it is known that there exists a positive constant $c_1$ such that for infinitely many positive integers $m$, the inequality

$$\mathcal{A}_m = \#\{n \ : \ \mu^2(n) = 1 \text{ and } \phi(n) = m\} \ge m^{c_1}$$

holds. Here, $\mu(n)$ is the Möbius function of $n$. The above result is due to Erdős and appears in [5]. Let $m$ be one of these integers, and let $n \in \mathcal{A}_m$. Then $\phi(n)/n = m/n = a/b$ for some divisor $a$ of $m$. Thus, there are at least $\#\mathcal{A}_m/\tau(m) \ge m^{c_1}/\tau(m)$ values of $n$ corresponding to the same value of $a$. Since for each such $n$ we have that $b = na/m$, we get that the values of $b$ are distinct. Since $\tau(m) = m^{o(1)}$ as $m$ tends to infinity, the second claim of the theorem follows with any constant $c_0$ smaller than $c_1$. □

The best (largest) known constant $c_1$ above is $0.7067\ldots$ and is due to Baker and Harman [2]. We conjecture that $f(a) \geq a^{1-(1+o(1))\log_3 a/\log_2 a}$ holds for infinitely many positive integers $a$. In fact, we conjecture that this function of $a$ is also an upper bound for $f(a)$ as $a \to \infty$, but we cannot even prove that the inequality $f(a) < a$ holds for all sufficiently large values of $a$.

**Theorem 3.** *We have $f(a) = 0$ for almost all positive integers $a$.*

*Proof.* We prove more. Namely, we show that if we put

$$\mathcal{A}(x) = \{a \leq x \ : \ f(a) \neq 0\},$$

then

$$\#\mathcal{A}(x) \leq x \exp\big(-(\log 2 + o(1))\log_2 x/\log_3 x\big) \qquad \text{as } x \to \infty. \tag{2}$$

We let $x$ be large, let $k < \log_2 x$ be a positive integer tending to infinity with $x$ in a way which will be specified later, and let $K = \lfloor 2\log_2 x \rfloor$. Let $\mathcal{A}_1$ be the set of $a \leq x$ such that $2^{k-1} \mid a$ and $\mathcal{A}_2$ be the set of $a \leq x$ such that $\omega(a) > K$. It is obvious that

$$\#\mathcal{A}_1 \leq \frac{x}{2^{k-1}}. \tag{3}$$

Further, it follows by a result of Hardy and Ramanujan [10] that

$$\#\mathcal{A}_2 \ll \frac{x}{(\log x)^{2\log 2 - 1}}. \tag{4}$$

Let $\mathcal{A}_3$ be the set of all $a \leq x$ not in $\mathcal{A}_1 \cup \mathcal{A}_2$ for which $f(a) \neq 0$. Let $a \in \mathcal{A}_3$ and assume that $a/b = \phi(n)/n$ for some squarefree integers $b, n$, where $a$ and $b$ are coprime. Write $n = uv$ where $u = \gcd(n, \phi(n))$, so that $a = \phi(v)\phi(u)/u$. Since $\omega(u) \leq \omega(n) \leq k < \log_2 x$, we have

$$\phi(v) \leq xu/\phi(u) \ll x\log_3 x. \tag{5}$$

We fix a value $w = \phi(v)$ of the Euler function that satisfies (5) and ask how many values of $u$ with $w\phi(u)/u \in \mathcal{A}_3$ can correspond to it. We will show that for a given $w$ there are at most $O((K+k)^k)$ choices for $u$ that can work, and it will remain to multiply this quantity by the number of choices for $w$.

Write $u = q_1 \ldots q_l$ where $q_1 < \cdots < q_l$. Since $v_2(a) \leq k-2$ and $a = w\phi(u)/u$, it follows that $v_2(\phi(u)) \leq k-1$ and $l = \omega(u) \leq k$. Note too that

for any valid choice for $u$, we have $\omega(w\phi(u)) = \omega(au) \le \omega(a)+\omega(u) \le K+k$. Since $q_l \nmid \phi(u)$, we must have $q_l \mid w$, so there are at most $K+k$ choices for $q_l$. Once $q_l$ is chosen, note that $q_{l-1} \nmid \phi(u/q_l)$, so $q_{l-1} \mid w\phi(q_l)$ and there are at most $K+k$ choices for $q_{l-1}$. In general, $q_{l-i} \mid w\phi(q_{l-i+1}\dots q_l)$, so that once $q_{l-i+1}, \dots, q_l$ are chosen, there are at most $K+k$ choices for $q_{l-i}$. Thus the number of choices for $u$ is at most $\sum_{l=0}^{k}(K+k)^l \ll (K+k)^k$.

For any positive real number $y$ put

$$V(y) = \{\phi(m) \ : \ m \text{ is a positive integer, } \phi(m) \le y\}.$$

In 1988, Maier and Pomerance [14] showed that that there is a positive constant $c_2$ with

$$\#V(y) = \frac{y}{\log y}\exp\left((c_2+o(1))(\log_3 y)^2\right) \qquad \text{as } y \to \infty.$$

(The exact order of magnitude of $V(y)$ has been determined by Ford in [8].) Therefore, from (5),

$$\#\mathcal{A}_3 \le \frac{x}{\log x}\exp\left(O((\log_3 x)^2)\right)(K+k)^k.$$

We equate this bound for $\#\mathcal{A}_3$ with the bound for $\#\mathcal{A}_1$ in (3), leading us to choose

$$k = \left\lfloor \frac{\log_2 x}{\log_3 x} - 2\log 2\frac{\log_2 x}{(\log_3 x)^2}\right\rfloor.$$

With this value of $k$ we have (2) and the theorem. □

We remark that it may be the case that there is some positive constant $c$ such that $\#\mathcal{A}(x) \ll x/(\log x)^c$, but we have not been able to prove this. Since $p-1 \in \mathcal{A}(x)$ for every prime $p \le x+1$, we have $\#\mathcal{A}(x) \gg x/\log x$.

# 4 The function $g(b)$

The first result here addresses the maximal order of the function $g(b)$.

**Theorem 4.** *We have* $g(b) \le b^{(1+o(1))\log_3 b/\log_2 b}$ *as* $b \to \infty$.

*Proof.* Let $a/b = \phi(n)/n$ for some squarefree integer $n$. Then $an = b\phi(n)$, therefore $n \mid b\phi(n)$. Let $r(n) = \mathrm{rad}(\phi(n))$ and let $r_k$ be the $k$-fold iteration of $r$, with $r_1 = r$ and $r_0$ the identity. Note that if $u$ is squarefree and $u \mid vw$, then $r(u) \mid r(v)r(w)$. Since $n \mid br(n)$, it follows by induction on $k \geq 0$, that $r_k(n) \mid r_k(b)r_{k+1}(n)$. Let $k(b)$ be the smallest positive integer $k$ such that $r_k(b) = 1$. The above divisibility relation shows that $r_{k(b)}(n) \mid r_{k(b)+1}(n)$, which easily leads to the conclusion that $r_{k(b)}(n) = 1$. Hence, writing

$$F(b) = \prod_{0 \leq k \leq k(b)} r_k(b), \tag{6}$$

we get

$$n \;\Big|\; br(n) \;\Big|\; br(b)r_2(n) \;\Big|\; \ldots \;\Big|\; F(b).$$

Thus, $a \mid \phi(n) \mid \phi(F(b))$ and also $a \leq b$. By a generalization of a result of Pratt, see Theorem 4.6 in [7], we have that $\omega(F(b)) < \log b/\log 2 + 1$. Put $t = \lfloor \log b/\log 2 \rfloor + 1$ and assume that $b$ is large. Let $\mathcal{P}_b$ be the set of all prime factors of $F(b)$ and $\Psi_{\mathcal{P}}(x)$ denote the number of positive integers $a \leq x$ all whose prime factors are in $\mathcal{P}$. Then $g(b) \leq \Psi_{\mathcal{P}_b}(b)$. The prime number theorem (or estimates of Chebyshev) imply that $p_k \leq 2k \log k$ holds for all sufficiently large $k$. Put $P(n)$ for the largest prime factor of $n$. The above argument shows that if $b$ is large enough, then

$$g(b) \leq \Psi_{\mathcal{P}_b}(b) \leq \Psi(b, p_t) \leq \Psi(b, 2t \log t), \tag{7}$$

where we use

$$\Psi(x, y) = \#\{n \leq x \;:\; P(n) \leq y\}.$$

By the de Bruijn estimates for the function $\Psi(x, y)$ (see, for example, Theorem 2 on p. 359 in [17]), we have

$$\Psi(b, 2t \log t) \leq \exp\left((1 + o(1))\frac{\log b}{\log t} \log\left(1 + \frac{2t \log t}{\log b}\right)\right). \tag{8}$$

Comparing estimates (7) and (8) and recalling the definition of $t$, we get the conclusion of the theorem as $b \to \infty$. □

A natural question is the average order of $g(b)$, but we have not been able to substantially improve on the estimate afforded by Theorem 4. Using this theorem, we can get the following result for the average order of $f(a)$.

**Corollary 5.** *As $x \to \infty$, we have $\sum_{a \le x} f(a) \le x^{1+(1+o(1))\log_3 x/\log_2 x}$.*

*Proof.* If $\gcd(a,b) = 1$ and $a/b = \phi(n)/n$ for some integer $n$, then for $a$ large, (1) implies that $b \le 2a\log_2 a$. Thus, for $x$ large,

$$\sum_{a \le x} f(a) \le \sum_{b \le 2x\log_2 x} g(b),$$

and so the result follows from Theoerem 4. □

**Theorem 6.** *We have $g(b) = 0$ for almost all positive integers $b$. In fact the number of integers $b \le x$ with $g(b) > 0$ is $\sim e^{-\gamma} x/\log_3 x$ as $x \to \infty$.*

*Proof.* From Lemma 1, if $g(b) > 0$, then $\gcd(b, \phi(b)) = 1$. Conversely, if $\gcd(b, \phi(b)) = 1$, then $\phi(b)/b$ is already reduced, so $g(b) > 0$. Thus the theorem follows immediately from the result of Erdős [6] quoted in the proof of Theorem 2. □

# 5 The greatest common divisor of $n$ and $\phi(n)$

Our first result in this direction addresses the maximal order of the greatest common divisor of $n$ and $\phi(n)$.

**Theorem 7.** *The inequality*

$$\gcd(n, \phi(n)) \le 2n \exp\left(-\sqrt{\log 2 \log n}\right)$$

*holds for all squarefree $n \ge 1$. On the other hand, there is an infinite set $\mathcal{S}$ of squarefree numbers $n$ such that*

$$\gcd(n, \phi(n)) \ge n^{1-(1+o(1))\log_3 n/\log_2 n}$$

*as $n \to \infty$, $n \in \mathcal{S}$.*

*Proof.* Assume that $n \ge 3$ is squarefree. Write $d = \gcd(n, \phi(n))$. Since $n$ is squarefree, we have that $P(n) \nmid \phi(n)$, which shows that $d \le n/P(n)$. Thus, the first inequality follows immediately if $P(n) > \exp(\sqrt{\log 2 \log n})$ even without the factor of 2 on the right hand side. On the other hand, if $P(n) \le \exp(\sqrt{\log 2 \log n})$, then, using that $n$ is squarefree, we get

$$\omega(n) \ge \frac{\log n}{\log P(n)} \ge \sqrt{\log n/\log 2}. \tag{9}$$

Let $\beta = v_2(\phi(n))$. If $n$ is odd we have $\beta \geq \omega(n)$ and $d \mid \phi(n)/2^\beta$, so that $d \leq n/2^{\omega(n)}$. Thus (9) gives the first inequality, again without the factor of 2. Finally, if $n$ is even, we have $d \mid \phi(n)/2^{\beta-1}$, since $n$ is squarefree. But in this case $\beta \geq \omega(n)-1$ and $\phi(n) \leq n/2$, so that $d \leq n/2^\beta \leq n/2^{\omega(n)-1}$. Using (9) gives the first inequality for $d$.

For the lower bound, note that from Theorem 3 in [13], there is a set of numbers $\mathcal{T}$ having asymptotic density 1, such that for $t \to \infty$, $t \in \mathcal{T}$, we have $\mathrm{rad}(F(t)/t) > t^{(1+o(1))\log_2 t/\log_3 t}$, where $F(t)$ is defined in (6). Note too that for any squarefree number $t$ we have $\gcd(\mathrm{rad}(F(t)), \phi(\mathrm{rad}(F(t))))$ equal to $\mathrm{rad}(F(t)/t)$. Let $\mathcal{S}$ be the set of numbers $n = \mathrm{rad}(F(t))$ for $t \in \mathcal{T}$ with $t$ squarefree. The second inequality of the theorem follows. □

Our next result addresses the normal order of the greatest common divisor of $n$ and $\phi(n)$.

**Theorem 8.** *For almost all $n$, $\gcd(n, \phi(n))$ is the largest divisor of $n$ supported on the prime divisors of $n$ in the interval $[1, \log\log n]$.*

*Proof.* Let $x$ be a large positive real number. We first note that most numbers $n \leq x$ are "nearly" squarefree, in that $v_p(n) \leq 1$ for all primes $p \geq \log_3 x$. Indeed if $\mathcal{E}_0(x)$ is the set of integers $n$ which violate this property, then

$$\#\mathcal{E}_0(x) \leq \sum_{p \geq \log_3 x} \frac{x}{p^2} \sim \frac{x}{\log_3 x \log_4 x} = o(x).$$

Assume $n \leq x$ and $n \notin \mathcal{E}_0(x)$. Let $\mathcal{E}_1(x)$ be the set of such $n$ with $p \mid \gcd(n, \phi(n))$ for some prime $p \geq \log_2 x$. To bound $\#\mathcal{E}_1(x)$, we fix a prime $p \geq \log_2 x$ and look at the number of $n \leq x$ such that $p \mid \gcd(n, \phi(n))$. It is clear that either $p^2 \mid n$ or $pq \mid n$ for some prime $q \equiv 1 \pmod p$. Since $n \notin \mathcal{E}_0(x)$ the first choice does not occur. Thus,

$$\begin{aligned} \#\mathcal{E}_1(x) &\leq \sum_{p \geq \log_2 x} \sum_{\substack{q \leq x/p \\ q \equiv 1 \pmod p}} \left\lfloor \frac{x}{pq} \right\rfloor \leq x \sum_{p \geq \log_2 x} \frac{1}{p} \sum_{\substack{q \leq x \\ q \equiv 1 \pmod p}} \frac{1}{q} \\ &\ll x \sum_{p \geq \log_2 x} \frac{\log_2 x}{p(p-1)} \ll \frac{x}{\log_2 x \log_3 x} = o(x), \end{aligned} \tag{10}$$

where we used the estimate

$$\sum_{\substack{q \leq x \\ q \equiv 1 \pmod b}} \frac{1}{q} \ll \frac{\log_2 x}{\phi(b)}, \tag{11}$$

which is uniform for $2 \le b \le x$ (see, for example, inequality (3.1) in [7]). Thus we may assume that $n \notin \mathcal{E}_1(x)$.

In Lemma 2 in [12] it is shown that there exists a positive constant $c_3$ such that if we put $M(x)$ for the least common multiple of all numbers $m \le B := c_3 \log_2 x / \log_3 x$, then all numbers $n \le x$ have the property that $M(x) \mid \phi(\mathrm{rad}(n)) \mid \phi(n)$ except for a set $\mathcal{E}_2(x)$ of them of cardinality $o(x)$.

We now consider the set of all $n \le x$ which do not belong to $\mathcal{E}_0(x) \cup \mathcal{E}_1(x) \cup \mathcal{E}_2(x)$. If $\gcd(n, \phi(n))$ is not the divisor of $n$ concentrated on the prime factors of $n$ from $[1, \log_2 n]$, then one of the following must hold:

(i) there is a prime factor $p$ of $n$ in the interval $I := [B, \log_2 x]$.

(ii) there is a prime $p \le B$ and a positive integer $a$ such that $p^a > B$ and $p^a \mid n$.

Let $\mathcal{E}_3(x)$ and $\mathcal{E}_4(x)$ be the subsets of such $n \le x$ satisfying (i) and (ii), respectively.

A simple computation with Mertens's theorem shows that $\sum_{p \in I} 1/p = o(1)$, so $\#\mathcal{E}_3(x) \le \sum_{p \in I} x/p = o(x)$.

We now bound $\#\mathcal{E}_4(x)$. Note that $a \ge 2$, so that $n \notin \mathcal{E}_0(x)$ implies that $p < \log_3 x$. Then $B < p^a < (\log_3 x)^a$, which implies that $a \ge K := \lceil \log B / \log_4 x \rceil$ if $x$ is sufficiently large. Thus, $n$ is a multiple of $p^K$ for some prime $p$, so the number of such $n \le x$ does not exceed

$$\sum_{p \ge 2} \frac{x}{p^K} \le x(\zeta(K) - 1) \ll \frac{x}{2^K} = o(x).$$

Thus $\#\mathcal{E}_4(x) = o(x)$, which, together with our estimates for the other counts $\#\mathcal{E}_j(x)$ for $j = 0, 1, 2, 3$, completes the proof of the theorem. □

Theorem 8 has the following consequences.

**Corollary 9.** *The normal order of* $\omega(\gcd(n, \phi(n)))$ *is* $\log_4 n$.

*Proof.* This result follows from Theorem 8 and the fact that the normal order for the number of prime factors of $n$ below $\log_2 n$ is $\log_4 n$, see Theorem 8 on p. 312 of [17]. □

Note that Corollary 9 was stated without proof in [6].

**Corollary 10.** *For each real number $u > 0$, the asymptotic density of the set of natural numbers $n$ with*

$$\gcd(n, \phi(n)) > (\log\log n)^u$$

*is $e^{-\gamma} \int_u^\infty \rho(t)\, dt$, where $\rho$ is the Dickman–de Bruijn function.*

*Proof.* In light of Theorem 8 we may replace the function $\gcd(n, \phi(n))$ in the corollary with the function $D(n)$, defined as the largest divisor of $n$ supported on the prime factors of $n$ in the interval $[1, \log\log n]$. Let $y = y(x) = \log\log x$ and let $D_y(n)$ be the largest $y$-smooth divisor of $n$. That is, $D_y(n)$ is the largest divisor of $n$ supported on the prime factors of $n$ in $[1, y]$. Since the function $\log\log x$ grows so slowly, it suffices to show that the number of $n$ in $[1, x]$ with $D_y(x) > y^u$ is $\sim \delta_u x$, where $\delta_u = e^{-\gamma} \int_u^\infty \rho(t)\, dt$.

First note that the number of $n \le x$ with $D_y(n) \ge x^{1/2}$ is $o(x)$. Fix a positive real number $u$. Then the number of integers $n \le x$ with $D_y(n) > y^u$ is

$$\sum_{m > y^u,\, P(m) \le y} \; \sum_{n \le x,\, D_y(n) = m} 1 = \sum_{x^{1/2} > m > y^u,\, P(m) \le y} \; \sum_{n \le x,\, D_y(n) = m} 1 \; + \; o(x).$$

Let $L$ denote the product of the primes in $[1, y]$. The inner sum above is

$$\sum_{k \le x/m,\, \gcd(k, L) = 1} 1 = \left( \frac{\phi(L)}{L} + o(1) \right) \frac{x}{m}$$

uniformly for all $m$ in consideration, where we use a complete inclusion-exclusion over the primes in $L$. By the theorem of Mertens, we have $\phi(L)/L \sim e^{-\gamma}/\log y$, so that our count is

$$(1 + o(1)) \frac{x}{e^\gamma \log y} \sum_{x^{1/2} > m > y^u,\, P(m) \le y} \frac{1}{m}.$$

Our corollary then follows by partial summation and standard results on the distribution of $y$-smooth integers $m$. □

Our last result in this section addresses the average value of $\gcd(n, \phi(n))$.

**Theorem 11.** *Let*

$$A(x) = \frac{1}{x}\sum_{n\le x}\gcd(n,\phi(n)).$$

*Then, for any $k>0$ we have*

$$(\log x)^k \le A(x) \le x^{(1+o(1))\log_3 x/\log_2 x}$$

*as $x\to\infty$.*

*Proof.* We start with the upper bound since it is easier. We have

$$\sum_{n\le x}\gcd(n,\phi(n)) \;=\; \sum_{\substack{r\le x\\ r \text{ squarefree}}}\;\sum_{\substack{m\le x/r\\ \mathrm{rad}(m)|r}}\gcd(rm,\phi(rm))$$

$$= \sum_{\substack{r\le x\\ r \text{ squarefree}}}\gcd(r,\phi(r))\sum_{\substack{m\le x/r\\ \mathrm{rad}(m)|r}} m \;\le\; x\sum_{\substack{r\le x\\ r \text{ squarefree}}}\frac{\gcd(r,\phi(r))}{r}\sum_{\substack{m\le x\\ \mathrm{rad}(m)|r}} 1.$$

We estimate the inner sum, call it $S(r)$. It is majorized by replacing $r$ with $P_j = p_1\dots p_j$ where $j=\omega(r)$, so we have $S(r)\le S(P_j)=\Psi(x,p_j)$. Since $p_j\le (1+o(1))\log x$, it follows from Theorem 2 (of de Bruijn) on p. 359 of [17] that $S(r)\le x^{(\log 4+o(1))/\log_2 x}$ as $x\to\infty$, uniformly in $r$. Using this, we have for large $x$,

$$\sum_{n\le x}\gcd(n,\phi(n)) \;\le\; x^{1+2/\log_2 x}\sum_{\substack{r\le x\\ r \text{ squarefree}}}\frac{\gcd(r,\phi(r))}{r}$$

$$= \; x^{1+2/\log_2 x}\sum_{b\le x}\frac{g(b)}{b} \;\le\; x^{1+(1+o(1))\log_3 x/\log_2 x},$$

where for the last estimate we used Theorem 4. Dividing by $x$, we have the asserted upper bound for $A(x)$.

We now deal with the lower bound. Let $\mathcal{D}$ be the set of positive integers $d$ such that

$$\sum_{\substack{p\le d^{10}\\ p\equiv 1 \pmod d}}\frac{1}{p}\ge\frac{1}{2d}.$$

We state the following lemma for future use.

**Lemma 12.** *The set $\mathcal{D}$ contains all positive integers except for at most $O(x/\log x)$ of them.*

*Proof.* Indeed, let $x$ be large and let $d \in (x/\log x, x)$. Theorem 2.1 in [1] shows that there exists a constant $c_4$ such that the inequality

$$\pi(y; 1, d) \geq \frac{\pi(y)}{2\phi(d)}$$

holds for large $x$ and for all positive integers $d \in (x/\log x, x)$ uniformly in the range $y \in (x^{2.6}, x^{10})$ except for a set $\mathcal{D}'(x)$ of such $d$, where $\mathcal{D}'(x)$ consists of all positive integers $d \in (x/\log x, x)$ which are divisible by one of at most $c_4$ positive integers $d_i = d_i(x)$, all of which exceed $\log x$. Certainly,

$$\#\mathcal{D}'(x) \leq \sum_{i \leq c_4} \left\lfloor \frac{x}{d_i} \right\rfloor \ll \frac{x}{\log x}.$$

Assume now that $d \notin \mathcal{D}'(x)$. Then $d^3 \geq (x/\log x)^3 \geq x^{2.6}$ when $x$ is large, therefore

$$\begin{aligned} \sum_{\substack{p \leq d^{10} \\ p \equiv 1 \pmod d}} \frac{1}{p} &\geq \int_{d^3}^{d^{10}} \frac{d\pi(y; 1, d)}{y} \geq \frac{\pi(y)}{2\phi(d)y}\Big|_{y=d^3}^{y=d^{10}} + \int_{d^3}^{d^{10}} \frac{\pi(y)dy}{2\phi(d)y^2} \\ &= \frac{1}{2\phi(d)} \sum_{d^3 < p \leq d^{10}} \frac{1}{p} = \frac{\log(10/3) + o(1)}{2\phi(d)} \geq \frac{1}{2d}, \end{aligned}$$

which shows that $(x/\log x, x) \backslash \mathcal{D}'(x) \subset \mathcal{D} \cap [1, x]$. This completes the proof. □

Assume that $k$ is a positive integer. We put $\alpha_i = 2^{-1}11^{-2(i+2)}$ for $i = 1, \ldots, k+1$. Set $y = x^{\alpha_{k+1}}$. For each $i = 1, \ldots, k$, we put $\mathcal{I}_i = [x^{\alpha_i}, x^{10\alpha_i}]$. We consider $k$-tuples of positive integers $(d_1, \ldots, d_k)$ such that

(i) $d_i \in \mathcal{D} \cap \mathcal{I}_i$ for all $i = 1, \ldots, k$;

(ii) $P(d_i) \leq y$ for all $i = 1, \ldots, k$;

(iii) $\omega(d_i) \leq 2\log_2 x$ for all $i = 1, \ldots, k$;

(iv) $\gcd(d_i, d_j) = 1$ for all $i \neq j$;

(v) the smallest prime factor of $d_i$ exceeds $(\log_2 x)^4$ for all $i = 1, \dots, k$.

We now let $\mathcal{E}$ be the set of such $k$-tuples $(d_1, \dots, d_k)$. For each $k$-tuple $\mathbf{d} = (d_1, \dots, d_k)$ in $\mathcal{E}$ we consider the set $\mathcal{F}_{\mathbf{d}}$ of $k$-tuples of primes $\mathbf{p} = (p_1, \dots, p_k)$ with $p_i \in \mathcal{P}_{\mathbf{d},i}$, where $\mathcal{P}_{\mathbf{d},i}$ consists of primes with the following properties:

(i) $p_i \le d_i^{10}$;

(ii) $p_i \equiv 1 \pmod{d_i}$;

(iii) $\omega(p_i - 1) \le 10 \log_2 x$;

(iv) $p_i - 1$ is coprime to $\prod_{j \ne i} d_j$,

for all $i = 1, \dots, k$. Finally, for a fixed $k$-tuple $\mathbf{d}$ in $\mathcal{E}$ and a fixed $k$-tuple of primes $\mathbf{p}$ in $\mathcal{F}_{\mathbf{d}}$ consider the set $\mathcal{G}_{\mathbf{d},\mathbf{p}}$ of all positive integers

$$m \in \left[ \frac{x}{2 \prod_{i=1}^k d_i p_i}, \frac{x}{\prod_{i=1}^k d_i p_i} \right].$$

which are coprime to $(p_1 - 1) \dots (p_k - 1)$ and have $P(m) \le y$.

We shall first prove that

**Lemma 13.** *The estimate*

$$\#\mathcal{G}_{\mathbf{d},\mathbf{p}} \gg_k \frac{x}{\log_2 x \prod_{i=1}^k d_i p_i}.$$

*holds for all* $\mathbf{d} \in \mathcal{E}$ *and* $\mathbf{p} \in \mathcal{F}_{\mathbf{d}}$.

*Proof.* For a positive integer $Q$ let

$$\Psi_Q(x, y) = \sum_{\substack{n \le x \\ P(n) \le y \\ \gcd(n, Q) = 1}} 1$$

denote the number of $y$-smooths in $[1, x]$ coprime to $Q$. We fix $\mathbf{d}$ and $\mathbf{p}$ and put $P = \prod_{i=1}^k d_i p_i$ and $Q = \prod_{i=1}^k (p_i - 1)$. Then

$$\#\mathcal{G}_{\mathbf{d},\mathbf{p}} = \Psi_Q(x/P, y) - \Psi_Q(x/2P, y).$$

Note that

$$Q \le P \le (d_1 \dots d_k)^{11} \le x^{11(\alpha_1 + \dots + \alpha_k)} = x^{(1 + 11^{-2} + \dots + 11^{-2(k-1)})/242} < x^{1/240}.$$

It thus follows from Theorem 1 in [9] that

$$\#\mathcal{G}_{\mathbf{d},\mathbf{p}} = \frac{\phi(Q)}{Q}(\Psi(x/P,y) - \Psi(x/2P,y))\left(1 + O_k\left((\log_2 x)^2/\log x\right)\right).$$

We have $\phi(Q)/Q \gg 1/\log_2 x$ and $\Psi(x/P,y) - \Psi(x/2P,y) \gg_k x/P$, so the result follows. □

We now look at numbers of the form $n = d_1 \dots d_k p_1 \dots p_k m$, where $\mathbf{d} = (d_1, \dots, d_k) \in \mathcal{E}$, $\mathbf{p} \in \mathcal{F}_{\mathbf{d}}$ and $m \in \mathcal{G}_{\mathbf{d},\mathbf{p}}$. Clearly, $n \in (x/2, x)$. We show that each such $n$ arises from a unique triple $(\mathbf{d}, \mathbf{p}, m)$. Note first that since

$$p_i \le d_i^{10} < x^{100\alpha_i} < x^{\alpha_{i-1}} \le d_{i-1} < p_{i-1}$$

holds for all $i \ge 2$, while $y = x^{1/2(11)^{2(k+3)}} < d_k$, it follows that $p_1 > p_2 > \dots > p_k > y \ge P(d_1 \dots d_k m)$. Now assume that the same $n$ arises also from $\mathbf{d}' = (d_1', \dots, d_k')$, $\mathbf{p}' = (p_1', \dots, p_k')$ and $m'$. Then $d_1 \dots d_k p_1 \dots p_k m = d_1' \dots d_k' p_1' \dots p_k' m'$. By identifying the $k$ largest prime factors of $n$, we get that $p_i = p_i'$ for $i = 1, \dots, k$. Once $p_1, \dots, p_k$ are known, then $d_1 \dots d_k = \gcd(n, (p_1 - 1) \dots (p_k - 1))$, so $m = m'$. Thus, $d_1 \dots d_k = d_1' \dots d_k'$ and the unicity of the $d_i$'s follows from the fact that $d_i = \gcd(d_1, \dots, d_k, p_i - 1)$. Hence, each such $n$ arises from a unique triple $(\mathbf{d}, \mathbf{p}, m)$.

Note now that if $n$ arises from $(\mathbf{d}, \mathbf{p}, m)$, then $d_1 \dots d_k \mid \gcd(n, \phi(n))$. Thus, we get

$$\begin{aligned} A(x) &\ge x^{-1} \sum_{\mathbf{d}\in\mathcal{E}} d_1 \dots d_k \sum_{\mathbf{p}\in\mathcal{F}_{\mathbf{d}}} \sum_{m\in\mathcal{G}_{\mathbf{d},\mathbf{p}}} 1 = x^{-1} \sum_{\mathbf{d}\in\mathcal{E}} d_1 \dots d_k \sum_{\mathbf{p}\in\mathcal{F}_{\mathbf{d}}} \#\mathcal{G}_{\mathbf{d},\mathbf{p}} \\ &\gg_k \frac{1}{\log_2 x} \sum_{\mathbf{d}\in\mathcal{E}} \sum_{\mathbf{p}\in\mathcal{F}_{\mathbf{d}}} \frac{1}{p_1 \dots p_k} = \frac{1}{\log_2 x} \sum_{\mathbf{d}\in\mathcal{E}} \prod_{i=1}^{k} \left( \sum_{p\in\mathcal{P}_{\mathbf{d},i}} \frac{1}{p} \right), \end{aligned} \tag{12}$$

where in the above chain of inequalities we used Lemma 13.

We shall also need the following two lemmas whose proofs we postpone for the moment.

**Lemma 14.** *There exists $x_k$ such that the inequality*

$$\sum_{p\in\mathcal{P}_{\mathbf{d},i}} \frac{1}{p} \ge \frac{1}{3d_i}$$

*holds for all $i = 1, \dots, k$ and all $\mathbf{d} \in \mathcal{E}$ provided that $x \ge x_k$.*

**Lemma 15.** *There exists $x_k$ and $\beta(k) > 0$ such that the inequality*

$$\sum_{\mathbf{d}\in\mathcal{E}} \frac{1}{d_1 \dots d_k} \geq \frac{1}{2}\left(\beta(k)\frac{\log x}{\log_3 x}\right)^k$$

*holds for all $x \geq x_k$.*

Clearly, estimate (12) and Lemmas 14 and 15 show that if $x$ is sufficiently large (with respect to $k$), then

$$\begin{aligned} A(x) &\gg_k \frac{1}{\log_2 x}\sum_{\mathbf{d}\in\mathcal{E}}\prod_{i=1}^{k}\left(\sum_{p\in\mathcal{P}_{\mathbf{d},i}}\frac{1}{p}\right) && \text{(by (12))} \\ &\gg \frac{1}{3^k \log_2 x}\sum_{\mathbf{d}\in\mathcal{E}}\frac{1}{d_1\dots d_k} && \text{(by Lemma 14)} \\ &\gg_k \frac{1}{\log_2 x}\left(\beta(k)\frac{\log x}{\log_3 x}\right)^k && \text{(by Lemma 15)} \\ &\geq (\log x)^{k/2} \end{aligned}$$

if $x$ is sufficiently large with respect to $k$, which completes the proof of the lower bound asserted by Theorem 11, since $k$ was arbitrary. □

It remains to prove Lemmas 14 and 15.

*Proof of Lemma 14.* Fix $\mathbf{d}$ in $\mathcal{E}$ and $i = 1, \dots, k$. We let $\mathcal{P}_1$ be the set of primes satisfying conditions (i) and (ii) from the definition of the $p_i$'s, $\mathcal{P}_2$ be the subset of $\mathcal{P}_1$ failing (iii) and $\mathcal{P}_3$ be the subset of $\mathcal{P}_1$ failing (iv). From the fact that $d_i \in \mathcal{D}$, we have that

$$S_1 = \sum_{p\in\mathcal{P}_1}\frac{1}{p} \geq \frac{1}{2d_i}.$$

Since

$$\sum_{p\in\mathcal{P}_{\mathbf{d},i}}\frac{1}{p} \geq S_1 - S_2 - S_3,$$

where

$$S_j = \sum_{p\in\mathcal{P}_j}\frac{1}{p}$$

for $j = 2,\ 3$, it suffices to show that both $S_2 = o(d_i^{-1})$ and $S_3 = o(d_i^{-1})$ hold as $x$ goes to infinity. If $p \in \mathcal{P}_2$, it then follows that $p - 1 = d_i m$, where $\omega(m) \geq \omega(p-1) - \omega(d_i) \geq 6 \log_2 x$ because of condition (iii) on the $d_i$. Putting $K = \lfloor 6 \log_2 x \rfloor$ and using the multinomial formula, unique factorization, the Stirling formula and the known fact that

$$\sum_{p^\alpha \leq t} \frac{1}{p^\alpha} = \log_2 t + O(1)$$

uniformly in $t \geq 3$, we get that

$$\begin{aligned} S_2 &= \sum_{p \in \mathcal{P}_2} \frac{1}{p} \leq \sum_{p \in \mathcal{P}_2} \frac{1}{p-1} \leq \frac{1}{d_i} \sum_{\substack{m \leq x \\ \omega(m) \geq K}} \frac{1}{m} \\ &= \frac{1}{d_i} \sum_{\ell \geq K} \sum_{\substack{m \leq x \\ \omega(m) = \ell}} \frac{1}{m} \leq \frac{1}{d_i} \sum_{\ell \geq K} \frac{1}{\ell!} \left( \sum_{p^\alpha \leq x} \frac{1}{p^\alpha} \right)^\ell \\ &= \frac{1}{d_i} \sum_{\ell \geq K} \left( \frac{e \log_2 x + O(1)}{\ell} \right)^\ell \ \leq \ \frac{1}{d_i} \sum_{\ell \geq K} \frac{1}{2^\ell} \ll \frac{1}{2^K d_i} \ = \ o(d_i^{-1}) \end{aligned}$$

as $x \to \infty$, where we used also the fact that $6 > 2e$, therefore $(e \log_2 x + O(1))/K < 1/2$ if $x$ is sufficiently large.

For $S_3$, let $\mathcal{Q}_i$ be the set of prime factors of $\prod_{j \neq i} d_j$. Note that $\#\mathcal{Q}_i \leq 2(k-1) \log_2 x$ (by the property (iii) on the $d_j$'s), and that if $q_1$ is the smallest element in $Q_i$ then $q_1 \geq (\log_2 x)^4$ (by the property (v) on the $d_j$'s). If $p \in \mathcal{P}_3$, there exists $q \in \mathcal{Q}_i$ such that $p \equiv 1 \pmod q$. Since $q$ divides $d_j$ for some $j \neq i$, it does not divide $d_i$ (by the property (iv) of the $d_i$'s), so $p \equiv 1 \pmod{d_i q}$. Hence,

$$\begin{aligned} S_3 &\leq \sum_{q \in \mathcal{Q}_i} \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{d_i q}}} \frac{1}{p} \ll \sum_{q \in \mathcal{Q}_i} \frac{\log_2 x}{\phi(d_i q)} \ = \ \frac{\log_2 x}{\phi(d_i)} \sum_{q \in \mathcal{Q}_i} \frac{1}{q-1} \\ &\leq \ \frac{(\log_2 x)^2 \#\mathcal{Q}_i}{d_i(q_1 - 1)} \ \ll_k \ \frac{(\log_2 x)^3}{d_i (\log_2 x)^4} \ = \ \frac{1}{d_i \log_2 x} \ = \ o(d_i^{-1}) \end{aligned}$$

as $x \to \infty$, where we used aside from the minimal order $\phi(d_i)/d_i \geq 1/\log_2 x$ of the Euler function for $d_i \in [1, x]$ also the estimate (11). $\square$

*Proof of Lemma 15.* We let $\mathcal{D}_i$ be the set of all positive integers $d_i$ satisfying all the properties (i)–(v) *except* (iv). We shall later show that

$$T_i := \sum_{d\in\mathcal{D}_i} \frac{1}{d} \asymp_k \frac{\log x}{\log_3 x}. \tag{13}$$

Assuming estimate (13), let $\mathcal{E}_1 = \times_{i=1}^k \mathcal{D}_i$ and $\mathcal{E}_2$ be the subset of $\mathcal{E}_1$ consisting in $k$-tuples $\mathbf{d} = (d_1, \dots, d_k)$ such that there exist $i \neq j$ with $\gcd(d_i, d_j) \neq 1$. It is clear that $\mathcal{E} = \mathcal{E}_1 \backslash \mathcal{E}_2$, so

$$\sum_{\mathbf{d}\in\mathcal{E}} \frac{1}{d_1 \dots d_k} = S_1 - S_2, \tag{14}$$

where $S_j = \sum_{\mathbf{d}\in\mathcal{E}_j} (d_1 \dots d_k)^{-1}$ for $j = 1,\ 2$. By estimate (13), we have

$$S_1 = \prod_{i=1}^k T_i \asymp_k \left(\frac{\log x}{\log_3 x}\right)^k.$$

Furthermore, note that

$$\begin{aligned}
S_2 &\leq \sum_{i\neq j} \left( \prod_{\substack{1\leq \ell\leq k \\ \ell\neq i,j}} T_\ell \right) \left( \sum_{p\geq (\log_2 x)^4} \sum_{\substack{d_i\leq x \\ p\mid d_i}} \sum_{\substack{d_j\leq x \\ p\mid d_j}} \frac{1}{d_i d_j} \right) \\
&\ll_k \left(\frac{\log x}{\log_3 x}\right)^{k-2} \left( \sum_{p\geq (\log_2 x)^4} \frac{1}{p^2} \right) \left( \sum_{u\leq x} \frac{1}{u} \right)^2 \\
&\ll \left(\frac{\log x}{\log_3 x}\right)^{k-2} \left( \frac{1}{(\log_2 x)^4 \log_3 x} \right) (\log x)^2 \\
&= \frac{(\log x)^k}{(\log_3 x)^{k-1} (\log_2 x)^4} = O\left( \frac{S_1 \log_3 x}{(\log_2 x)^4} \right) = o(S_1),
\end{aligned}$$

as $x \to \infty$, which together with inequality (14) completes the proof of the lemma. Hence, it suffices to prove estimate (13).

Let $i = 1, \dots, k$ be fixed. Let $R = \prod_{q\leq (\log_2 x)^3} q$. As in the proof of Lemma 13, one checks that if we put $\mathcal{D}_i'(t)$ for the set of positive integers in $[1, t]$ satisfying (ii) and (v), then

$$\#\mathcal{D}_i'(t) = \rho(v_t) \frac{\phi(R)}{R} t \left( 1 + O\left( \frac{(\log_2 x)^3}{\log x} \right) \right)$$

uniformly for $t \in \mathcal{I}_i$, where $v_t = (\log t)/\log y$). Of these numbers, the number of numbers that fail (iii) are $O_k(x/(\log x)^{c_3})$ with $c_3 = 1 - 2\log(2/e) = 0.38629436122\ldots$, while the number of numbers that fail (i) is, by Lemma 12, $O(t/\log t) = O_k(t/\log x)$. Hence,

$$\#\mathcal{D}_i(t) \;=\; \rho(v_t)\frac{\phi(R)}{R}t\left(1 + O_k\left(\frac{(\log_2 x)^3}{\log x}\right)\right) \;\asymp_k\; \frac{t}{\log_3 x},$$

where the last estimate above follows from Mertens's formula. By partial summation, we get that

$$\begin{aligned} T_i \;&=\; \sum_{d\in\mathcal{D}_i}\frac{1}{d} \;=\; \int_{x^{\alpha_i}}^{x^{10\alpha_i}} \frac{d\#\mathcal{D}(t)}{t} \\ &\geq\; -1 + \int_{x^{\alpha_i}}^{x^{10\alpha_i}} \frac{\#\mathcal{D}(t)dt}{t^2} \;\asymp_k\; \frac{\log t}{\log_3 x}\Big|_{t=x^{\alpha_i}}^{t=x^{10\alpha_i}} \;\asymp_k\; \frac{\log x}{\log_3 x}, \end{aligned}$$

which completes the proof of Lemma 15 and so the proof of Theorem 11. □

We remark that an examination of the proof (and the tool we used from [9]) shows that we may take $k$ in the theorem of size $c\log_3 x$ for some small positive constant $c$, and so obtain the lower bound $A(x) > (\log x)^{c\log_3 x}$. We are not sure what to suggest for the true order of $A(x)$.

# 6 An application

For a nonzero polynomial $f(X) \in \mathbb{Z}[X]$, let $\mathcal{L}_f(x)$ denote the set of integers $n \leq x$ with $\phi(n) \mid f(n)$. In the recent paper [3], it was shown that there are certain "canonical" solutions to this relation of the form $pm$ where $m$ is a positive integral root of $f$ and $p$ is a prime in one of a few prescribed residue classes mod $\phi(m)$, and that non-canonical solutions are few in number. In particular, it was shown that

**Theorem 16.** *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree $k$, with $f(0) \neq 0$, whose roots all have multiplicity at most $\nu$. For each positive integral root $m$ of $f(X)$, there exist certain residue classes $\{\alpha_{j,m} \pmod{\phi(m)} : j = 1,\ldots,r_m\}$ for which the following estimate holds:*

$$\begin{aligned} \#\mathcal{L}_f(x) = \sum_{\substack{m\in\mathbb{N}\\ f(m)=0}}\sum_{j=1}^{r_m} \pi(x/m;\phi(m),\alpha_{j,m}) \\ + O\left(x^{1-1/(2\nu+1)+o(1)} + x^{1-1/(k+1)+o(1)}\right), \end{aligned} \tag{15}$$

*where the functions implied by $o(1)$ and the constant implied by $O$ depend only on $f$.*

In the case $f(0) = 0$ a weaker error estimate of the shape $x^{1-o(1)}$ was obtained in [3]. Here, we use the results from Section 5 to show that Theorem 16 holds without the assumption that $f(0) \neq 0$.

**Theorem 17.** *The conclusion of Theorem 16 holds even when $f(0) = 0$.*

*Proof.* We follow the proof of Theorem 5 in [3]. Put

$$\alpha = \frac{\nu}{2\nu + 1}.$$

It is shown in the proof of Theorem 5 in [3] that the number of $n \leq x$ with $P(n) > x^\alpha$ for which $\phi(n) \mid f(n)$ is given by the right hand side of formula (15), and this argument from [3] does not use the fact that $f(0) \neq 0$. Thus, it remains to show that the number $N(x)$ of $n \leq x$ with $P(n) \leq x^\alpha$ for which $\phi(n) \mid f(n)$ satisfies

$$N(x) \leq x^{1-\alpha/\nu+o(1)}. \tag{16}$$

Such a number $n$ clearly has a divisor $m$ with $x^\alpha < m \leq x^{2\alpha}$. We fix a number $m$ in this interval and ask how many integers $r \leq x/m$ there are with $\phi(mr) \mid f(mr)$. Let this count be denoted $N_m(x)$.

We assume that $f(0) = 0$. Let $F(X)$ denote the product of the distinct irreducible factors (in $\mathbb{Z}[X]$) of $f(X)$ and the content of $f$ (the gcd of the coefficients of $f$). Let

$$t(m) = \prod_{p^k \| \phi(m)} p^{\lceil k/\nu \rceil},$$

so that $\phi(m) \mid \phi(mr)$ and $\phi(mr) \mid f(mr)$ imply that $t(m) \mid F(mr)$. Also, let $F(X) = XG(X)$, where $G(X) \in \mathbb{Z}[X]$, and let

$$s(m) = t(m)/\gcd(m, t(m)).$$

Then $t(m) \mid mrG(mr)$ implies that $s(m) \mid rG(mr)$. For $d \mid s(m)$, we consider separately those $r$ with $\gcd(r, s(m)) = d$. For these values of $r$ we have $s(m) \mid rG(mr)$ if and only if $s(m)/d \mid G(mr)$. We let $r = du$, where $u \leq x/md$ and $u$ is coprime to $s(m)/d$. Dropping the coprimality condition, let

$$N_{m,d}(x) \;=\; \#\{u \leq x/md \;:\; s(m)/d \mid G(mdu)\},$$

so that

$$N_m(x) \leq \sum_{d|s(m)} N_{m,d}(x).$$

We use the Nagell–Ore theorem, a strong form of it appearing in [11]. It asserts that for a squarefree polynomial $g(X)$ in $\mathbb{Z}[X]$, the number of solutions to the congruence $g(u) \equiv 0 \pmod{n}$ in a complete residue system mod $n$ is bounded above by $\deg(g)^{\omega(n)} D(g)^2$, where $D(g)$ is the discriminant of $g$. Say $p$ is a prime and $p^a \| s(m)/d$. If $p$ does not divide $md$, then the number of solutions to $G(mdu) \equiv 0 \pmod{p^a}$ in a complete residue system mod $p^a$ is equal to the number of solutions of $G(u) \equiv 0 \pmod{p^a}$, which is $O(1)$, by the Nagell–Ore theorem, the constant depending on the polynomial $G(X)$ (which in turn depends on $f(X)$). So, say $p$ does divide $md$, say $p^b \| md$. If $G(mdu) \equiv 0 \pmod{p^a}$ has any solutions at all, we must have $p^{\min\{a,b\}} \mid G(0)$. Since $G(0) \neq 0$, we have that $p$ is in a finite set depending on $G(X)$, and that $\min\{a, b\}$ is bounded as well. If $a = \min\{a, b\}$, we take $p^a$ as the (trivial) upper bound for the number of solutions to $G(mdu) \equiv 0 \pmod{p^a}$. If $b = \min\{a, b\}$, we consider the polynomial $H(X) = G(p^b X)$ and again use the Nagell–Ore theorem. The discriminant of $H$ is $p^{b(l^2-l)}$ times the discriminant of $G$, where $l = \deg G$, so that the number of solutions to $H(u) \equiv 0 \pmod{p^a}$ is bounded by $O(p^{2bl^2}) = O(1)$, again with the constant depending ultimately on $f$. But $mdp^{-b}$ is coprime to $p$, so the number of solutions to $H(mdp^{-b}u) \equiv 0 \pmod{p^a}$ is exactly the same quantity as with $H(u)$, and it only remains to note that $H(mdp^{-b}X) = G(mdX)$. So, in each case, the number of solutions to $G(mdu) \equiv 0 \pmod{p^a}$ is at most some constant $C$ that depends only on $f$. Thus, by the Chinese remainder theorem,

$$N_{m,d}(x) \ \leq \ C^{\omega(s(m)/d)} \left( \frac{x}{mds(m)/d} + 1 \right) \ = \ C^{\omega(s(m)/d)} \left( \frac{x}{ms(m)} + 1 \right).$$

Thus,

$$N_m(x) \ \leq \ \tau(s(m)) C^{\omega(s(m))} \left( \frac{x}{ms(m)} + 1 \right) \ = \ m^{o(1)} \left( \frac{x}{ms(m)} + 1 \right),$$

where we use the well-known maximal orders for the functions $\tau$ and $\omega$.

Note that

$$s(m) \ \geq \ \frac{\phi(m)^{1/\nu}}{\gcd(m, \phi(m))} \ = \ \frac{m^{1/\nu + o(1)}}{\gcd(m, \phi(m))}.$$

Thus,

$$N(x) \le \sum_{x^\alpha < m \le x^{2\alpha}} N_m(x) \le x^{o(1)} \sum_{x^\alpha < m \le x^{2\alpha}} \left( \frac{x \gcd(m, \phi(m))}{m^{1+1/\nu}} + 1 \right)$$
$$= x^{1+o(1)} \sum_{x^\alpha < m \le x^{2\alpha}} \frac{\gcd(m, \phi(m))}{m^{1+1/\nu}} + x^{2\alpha+o(1)}.$$

Let $S$ denote the last sum above. To compute $S$, we use Theorem 11 and partial summation and get that it is equal to

$$x^{-2\alpha(1+1/\nu)} \sum_{x^\alpha < m \le x^{2\alpha}} \gcd(m, \phi(m)) + \int_{x^\alpha}^{x^{2\alpha}} \frac{1+1/\nu}{t^{2+1/\nu}} \sum_{x^\alpha < m \le t} \gcd(m, \phi(m))\, dt$$

$$\le x^{2\alpha+o(1)-2\alpha(1+1/\nu)} + x^{o(1)} \int_{x^\alpha}^{x^{2\alpha}} \frac{dt}{t^{1+1/\nu}} = x^{-\alpha/\nu+o(1)}.$$

Thus,

$$N(x) \le x^{1-\alpha/\nu+o(1)} + x^{2\alpha+o(1)} = x^{1-\alpha/\nu+o(1)}.$$

This last estimate establishes (16) and so completes the proof of the theorem. □

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