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ON MULTIPLY PERFECT NUMBERS WITH A SPECIAL PROPERTY

CARL POMERANCE

If m is a multiply perfect number and $m = p^a n$ where p is prime and $n \mid \sigma(p^a)$, then $m = 120, 672, 523776$, or m is an even perfect number.

1. Introduction. Suppose p is a prime a, n are natural numbers, and

$$p^a \mid \sigma(n) , \qquad n \mid \sigma(p^a) \tag{1.1}$$

where σ is the sum of the divisors function. Then $1 = (p^a, \sigma(p^a)) = (p^a, n)$, so that $p^a n \mid \sigma(p^a)\sigma(n) = \sigma(p^a n)$; that is $p^a n$ is a multiply perfect number. In this paper we identify all multiply perfect numbers which arise in this fashion.

Let M be the set of Mersenne exponents, that is, $M = \{k: 2^k - 1$ is prime$\}$. We shall prove

THEOREM 1.1. *If p, a, n is a solution of* (1.1) *where p is prime, then either*

$$p^a = 2^k - 1 , \quad n = 2^{k-1} \quad \text{for some} \quad k \in M \tag{1.2}$$

$$p^a = 2^{k-1} , \quad n = 2^k - 1 \quad \text{for some} \quad k \in M \tag{1.3}$$

$$p^a = 2^3 , \quad n = 15 \tag{1.4}$$

$$p^a = 2^5 , \quad n = 21 \tag{1.5}$$

$$p^a = 2^9 , \quad n = 1023 . \tag{1.6}$$

COROLLARY 1.1. *If m is a multiply perfect number and $m = p^a n$ where p is prime and $n \mid \sigma(p^a)$, then $m = 120, 672, 523776$, or m is an even perfect number.*

Note that in [2] all solutions of (1.1) with $p^a = \sigma(n)$ are enumerated: they are (1.2) and (1.5). Hence in the proof of Theorem 1.1, we may assume $p^a < \sigma(n)$.

We recall that a natural number n is said to be super perfect it $\sigma(\sigma(n)) = 2n$. In [2] and Suryanarayana [8] it is shown that if n is super perfect and if either n or $\sigma(n)$ is a prime power, then $n = 2^{k-1}$ for $k \in M$. Here we will say n is *super multiply perfect* if $\sigma(\sigma(n))/n$ is an integer.

COROLLARY 1.2. *If n is super multiply perfect, and if n or $\sigma(n)$ is a prime power, then $n = 8, 21, 512$, or $n = 2^{k-1}$ for some $k \in M$.*

If p is a prime, denote by $\sigma_p(n)$ the sum of all those divisors of n which are powers of p. Then $\sigma_p(n) \mid \sigma(n)$.

COROLLARY 1.3. *If $n > 1$ and $n \mid \sigma_p(\sigma(n))$ for some prime p, then $p = 2$ and $n = 15, 21$, or 1023 or $p = 2^k - 1$ for some $k \in M$ and $n = 2^{k-1}$.*

We remark that in general the super multiply perfect numbers appear to be quite intractable. Partly complicating matters is that for every K, $\sigma(\sigma(n))/n \geqq K$ on a set of density 1. Professor David E. Penney of the University of Georgia, in a computer search, found that there are exactly 37 super multiply perfect numbers $\leqq 150000$. Of these, the only odd ones are 1, 15, 21, 1023, and 29127.

Recently, Guy and Selfridge [4], p. 104, published a proof of a stronger version of Theorem 1.1 for the special case $p = 2$.

2. Preliminaries. If n is a natural number, we let $\omega(n)$ be the number of distinct prime factors of n, and we let $\tau(n)$ be the number of natural divisors of n. If a, b are natural numbers with $(a, b) = 1$, we let $\mathrm{ord}_a(b)$ be the least positive integer k for which $a \mid b^k - 1$. If p is a prime and x is a natural number, then $\sigma(p^x) = (p^{x+1} - 1)/(p - 1) < (p/(p - 1))p^x$.

THEOREM 2.1 (Bang [1]). *If p is a prime, a is a natural number, and $1 < d \mid a + 1$, then there is a prime $q \mid \sigma(p^a)$ with $\mathrm{ord}_q(p) = d$, unless*

(i) *$p = 2$ and $d = 6$, or*
(ii) *p is a Mersenne prime and $d = 2$.*

COROLLARY 2.1.

$$\omega(\sigma(p^a)) \geqq \begin{cases} \tau(a+1) - 2\,, & \text{if } p = 2 \text{ and } 6 \mid a + 1 \\ \tau(a+1)\,, & \text{if } p > 2 \text{ is not Mersenne and} \\ & 2 \mid a + 1 \\ \tau(a+1) - 1\,, & \text{otherwise}\,. \end{cases}$$

The following is a weaker from of a lemma from [2].

LEMMA 2.1. *Suppose p, q are primes with $q > 2$ and x, y, b, c are natural numbers with $\sigma(q^x) = p^y$ and $q^b \mid \sigma(p^c)$. Then $q^{b-1} \mid c + 1$.*

3. The start of the proof. Suppose p, a, n is a solution of (1.1) where p is prime. Then there are integers s, t with

$$\sigma(n) = sp^a\ , \qquad \sigma(p^a) = tn\ .$$

As we remarked, we have already studied these equations in the case $s = 1$ (in [2]), so here we assume $s > 1$. We have

$$st = \frac{\sigma(p^a)}{p^a} \cdot \frac{\sigma(n)}{n}\ , \tag{3.1}$$

Considering the unique prime factorization of n, we write n_1 for the product of those prime powers q^b for which $\sigma(q^b)$ is divisible by a prime $\neq p$, and we write n_2 for the product of those prime powers q^b for which $\sigma(q^b)$ is a power of p. Then $(n_1, n_2) = 1$, $n_1 n_2 = n$, and $\sigma(n_2)$ is a power of p. Let ω_i be the number of distinct odd prime factors of n_i for $i = 1, 2$. Let ω_3 be the number of distinct prime factors of t which do not divide n. Hence

$$\omega(\sigma(p^a)) = \omega(tn) = \begin{cases} \omega_1 + \omega_2 + \omega_3\ , & \text{if } n \text{ is odd} \\ 1 + \omega_1 + \omega_2 + \omega_3\ , & \text{if } n \text{ is even}\ . \end{cases} \tag{3.2}$$

We write

$$n_1 = 2^{k_1} \prod_{i=1}^{\omega_1} p_i^{a_i}\ , \qquad n_2 = 2^{k_2} \prod_{i=1}^{\omega_2} q_i^{b_i}$$

where $k_1 k_2 = 0$ and the p_i and q_i are distinct odd primes.

4. The case $p > 2$. Since each $\sigma(q_i^{b_i})$ is a power of p, and since p is odd, we have each b_i even. Since also each $q_i^{b_i} \mid \sigma(p^a)$, Lemma 2.1 implies

$$\prod_{i=1}^{\omega_2} q_i \mid a + 1\ .$$

Suppose n is even. Then also $2 \mid a + 1$, so that $\tau(a + 1) \geqq 2^{\omega_2 + 1}$. It follows from (3.2) and Corollary 2.1 that

$$\omega_1 + \omega_3 \geqq 2^{\omega_2+1} - \omega_2 - 2\ . \tag{4.1}$$

Suppose $k_1 > 0$. Then $(\sigma(2^{k_1}), s) \geqq 3$ and for

$$1 \leqq i \leqq \omega_1\ , \quad (\sigma(p_i^{a_i}), s) \geqq 2\ .$$

Then $s \geqq 3 \cdot 2^{\omega_1}$. Also every prime counted by ω_3 is odd, so $t \geqq 3^{\omega_3}$. Hence from (3.1) we have

$$\begin{aligned}3\cdot\left(\frac{5}{4}\right)^{3\omega_1+4\omega_3} &< 3\cdot 2^{\omega_1}\cdot 3^{\omega_3}\\ &\leqq st=\frac{\sigma(p^a)}{p^a}\cdot\frac{\sigma(n)}{n}<\frac{p}{p-1}\cdot 2\cdot\prod_{i=1}^{\omega 1}\frac{p_i}{p_i-1}\cdot\prod_{i=1}^{\omega 2}\frac{q_i}{q_i-1}\\ &\leqq 3\cdot\left(\frac{5}{4}\right)^{\omega_1+\omega_2}\end{aligned}$$

so that

$$\omega_2 > 2\omega_1 + 4\omega_3 \geqq 2(\omega_1+\omega_3)\ .$$

Hence (4.1) implies that

$$\omega_2 > 2^{\omega_2+2} - 2\omega_2 - 4$$

which fails for all $\omega_2 \geqq 0$. This contradiction shows $k_1 = 0$.

Suppose $k_2 > 0$. Then $\sigma(2^{k_2})$ is a power of p, so that $\sigma(2^{k_2}) = p$ (Gerono [3]). Now $2 \mid a+1$, so $2^{k_2+1} = \sigma(p) \mid \sigma(p^a)$. Hence $2 \mid t$, so that $t \geqq 2\cdot 3^{\omega_3}$. Also $(\sigma(p_i^{a_i}), s) \geqq 2$, so $s \geqq 2^{\omega_1}$. Hence

$$\begin{aligned}\left(\frac{5}{4}\right)^{3\omega_1+4\omega_3} &< \frac{1}{2}st < \frac{p}{p-1}\cdot\prod\frac{p_i}{p_i-1}\cdot\prod\frac{q_i}{q_i-1} \leqq \frac{3}{2}\left(\frac{5}{4}\right)^{\omega_1+\omega_2}\\ &< \left(\frac{5}{4}\right)^{\omega_1+\omega_2+2}\end{aligned}$$

so that

$$\omega_2 > 2\omega_1 + 4\omega_3 - 2 \geqq 2(\omega_1+\omega_3) - 2\ . \tag{4.2}$$

It follows from (4.1) that

$$\omega_2 > 2^{\omega_2+2} - 2\omega_2 - 6\ ,$$

which implies $\omega_2 \leqq 1$. Then (4.2) implies $\omega_1 \leqq 1$. Since $s > 1$ and $2 \mid t$, we have

$$4 \leqq st < \frac{\sigma(2^{k_2})}{2^{k_2}}\cdot\frac{p}{p-1}\cdot\frac{p_1}{p_1-1}\cdot\frac{q_1}{q_1-1} < \frac{2pp_1q_1}{(p-1)(p_1-1)(q_1-1)}$$

so that $\max\{p, p_1, q_1\} = 13$. But $\sigma(2^{k_2}) = p$, so $k_2 \leqq 2$. Then

$$4 < \frac{\sigma(2^2)}{2^2}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6} < 4\ ,$$

so $k_2 = 0$.

Thus we have n odd, so $p^a n$ is an odd multiply perfect number. It follows from Hagis [5] and McDaniel [6] that

$$1 + \omega_1 + \omega_2 = 1 + \omega(n) = \omega(p^a n) \geqq 8\ . \tag{4.3}$$

From (3.2) and Corollary 2.1 we have

$$\omega_1 + \omega_3 \geqq 2^{\omega_2} - \omega_2 - 1 . \tag{4.4}$$

Now $s \geqq 2^{\omega_1}$, $t \geqq 2^{\omega_3}$ so that

$$\begin{aligned}\left(\frac{5}{4}\right)^{3\omega_1+3\omega_3} &< st < \frac{p}{p-1} \cdot \prod \frac{p_i}{p_i - 1} \cdot \prod \frac{q_i}{q_i - 1} \\ &\leqq \frac{3}{2}\left(\frac{5}{4}\right)^{\omega_1+\omega_2} < \left(\frac{5}{4}\right)^{\omega_1+\omega_2+2} .\end{aligned}$$

Hence

$$\omega_2 > 2\omega_1 + 3\omega_3 - 2 \geqq 2(\omega_1 + \omega_3) - 2 , \tag{4.5}$$

so that (4.4) implies

$$\omega_2 > 2^{\omega_2+1} - 2\omega_2 - 4 ,$$

which implies $\omega_2 \leqq 2$. Then (4.5) implies $w_1 \leqq 1$, contradicting (4.3).

5. The case $p = 2$. Since $\sigma(n_2)$ is a power of 2, it follows that n_2 is a product of distinct Mersenne primes (Sierpiński [7]), say

$$n_2 = \prod_{i=1}^{\omega_2} (2^{c_i} - 1)$$

where each c_i and $q_i = 2^{c_i} - 1$ is prime, and $c_1 < c_2 < \cdots < c_{\omega_2}$.

Suppose $n_1 = 1$. Then s is a power of 2, say $s = 2^c$. Then

$$2^{c+a} = \sigma(n) = \sigma(n_2) = 2^{\Sigma c_i}$$

so that $c + a = \sum c_i$. But $2^{c_i} - 1 \mid \sigma(2^a)$, so $c_i \mid a + 1$. Since $c_1, c_2, \cdots, c_2$ are distinct primes, $\prod c_i \mid a + 1$. Since $1 < s = 2^c$, we have $c \geqq 1$. Hence $\prod c_i \leqq a + 1 \leqq \sum c_i$, so

$$\prod_{i=1}^{\omega_2} c_i - \sum_{i=1}^{\omega_2} c_i \leqq 0 .$$

...ls only for $\omega_2 = 1$, which gives solution (1.3).

We now assume $n_1 > 1$. Then s is divisible by an odd prime; in fact, $s \geqq 3^{\omega_1} \geqq 3$. Also t is odd, so $t \geqq 3^{\omega_3}$. As above, $\prod c_i \mid a + 1$, so $\tau(a + 1) \geqq 2^{\omega_2}$. Hence from (3.2) and Corollary 2.1 we have

$$\omega_1 + \omega_3 \geqq 2^{\omega_2} - \omega_2 - 2 . \tag{5.1}$$

Also from (3.1) we have

$$\left(\frac{5}{4}\right)^{4\omega_1+4\omega_3-4} < 3^{\omega_1-1}\cdot 3^{\omega_3} \leqq \frac{1}{3}st = \frac{1}{3}\cdot\frac{\sigma(2^a)}{2^a}\cdot\frac{\sigma(n)}{n}$$
$$< \frac{1}{3}\cdot 2\cdot\prod\frac{p_i}{p_i-1}\cdot\prod\frac{q_i}{q_i-1} \leqq \left(\frac{5}{4}\right)^{\omega_1+\omega_2-1},$$

so that

$$\omega_2 > 3\omega_1 + 4\omega_3 - 3 \geqq 3(\omega_1+\omega_3) - 3\,. \tag{5.2}$$

Then (5.1) implies

$$\omega_2 > 3\cdot 2^{\omega_2} - 3\omega_2 - 9$$

so that $\omega_2 \leqq 2$. Then from (5.2) and the fact that $\omega_1 \geqq 1$, we have $\omega_1 = 1$, $\omega_3 = 0$, and $\omega_2 > 0$. Hence we have two choices for $\omega_1, \omega_2, \omega_3$: 1, 1, 0 and 1, 2, 0. Also since

$$5 > \frac{2}{1}\cdot\frac{3}{2}\cdot\frac{5}{4}\cdot\frac{7}{6} > \frac{\sigma(2^a)}{2^a}\cdot\frac{\sigma(n)}{n} = st$$

and since $s \geqq 3$, $s \neq 4$, we have $s = 3$, $t = 1$.

Suppose $\omega_2 = 1$. Then $\sigma(2^a) = p_1^{a_1}(2^{c_1}-1)$. Then c_1 is a proper divisor of $a+1$. But $\omega(\sigma(2^a)) = 2$, so Corollary 2.1 implies $a+1 = 6$ or $a+1 = c_1^2$. The first choice gives $n = 63$, but $\sigma(63) \neq 3\cdot 2^5$. Hence $a+1 = c_1^2$. Then Theorem 2.1 implies $\mathrm{ord}_{p_1}(2) = c_1^2$, so that $p_1 \equiv 1 \pmod{c_1^2}$. If $c_1 \geqq 3$, then $p_1 \geqq 19$, $q_1 = 2^{c_1} - 1 \geqq 7$, so that

$$3 = st < \frac{2}{1}\cdot\frac{7}{6}\cdot\frac{19}{18} < 3\,,$$

a contradiction. Hence $c_1 = 2$, $a+1 = 4$, $n = 15$, and we have solution (1.4).

Our last case is $\omega_2 = 2$. Then $\sigma(2^a) = p_1^{a_1}(2^{c_1}-1)(2^{c_2}-1)$, so that $c_1c_2 \mid a+1$. Now $\omega(\sigma(2^a)) = 3$, so that Corollary 2.1 implies $c_1c_2 = a+1$, where $c_1c_2 \neq 6$. We also have $\sigma(p_1^{a_1}(2^{c_1}-1)(2^{c_2}-1)) = 3\cdot 2^a$. Then $\sigma(p_1^{a_1})$ is 3 times a power of 2. Now $\sigma(p_1^{a_1}) \neq 3$, so $\sigma(p_1^{a_1})$ is even. Hence $2 \mid a_1 + 1$. Now Theorem 2.1 implies $\mathrm{ord}_{p_1}(2) = c_1c_2$, a composite number. Hence p_1 is not Mersenne. Also, $p_1 \equiv 1 \pmod{c_1c_2}$. From Corollary 2.1 and the fact that $\omega(\sigma(p_1^{a_1})) = 2$ we have $a_1 = 1$. Hence for some d we have $p_1 = 3\cdot 2^d - 1$. If $c_1 > 2$, then $q_1 = 2^{c_1} - 1 \geqq 7$, $q_2 = 2^{c_2} - 1 \geqq 31$, $p_1 \geqq 2c_1c_2 + 1 \geqq 31$. Then

$$3 = st < \frac{2}{1}\cdot\frac{31}{30}\cdot\frac{31}{30}\cdot\frac{7}{6} < 3\,,$$

so that we must have $c_1 = 2$. Then

$$2^{2c_2} - 1 = (3\cdot 2^d - 1)(2^2 - 1)(2^{c_2} - 1)$$

where $c_2 \geqq 3$. Looking at this equation mod 8, we obtain $3 \cdot 2^d - 1 \equiv 2^2 - 1 \pmod 8$. Hence $d = 2$, $p_1 = 11$. Then $a + 1 = 2c_2 = \mathrm{ord}_{p_1}(2) = 10$. This gives solution (1.6).

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UNIVERSITY OF GEORGIA