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On Locally Repeated Values of Certain Arithmetic Functions, IV

PAUL ERDŐS*
Mathematical Institute of the Hungarian Academy of Sciences, Reáltanoda u. 13-15, H-1364 Budapest, Hungary

CARL POMERANCE† carl@ada.math.uga.edu
Department of Mathematics, The University of Georgia, Athens, Georgia 30602

ANDRÁS SÁRKÖZY† sarkozy@cs.elte.hu
Mathematical Institute of the Hungarian Academy of Sciences, Reáltanoda u. 13-15, H-1364 Budapest, Hungary

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Abstract. Let $\omega(n)$ denote the number of prime divisors of n and let $\Omega(n)$ denote the number of prime power divisors of n. We obtain upper bounds for the lengths of the longest intervals below x where $\omega(n)$, respectively $\Omega(n)$, remains constant. Similarly we consider the corresponding problems where the numbers $\omega(n)$, respectively $\Omega(n)$, are required to be all different on an interval. We show that the number of solutions $g(n)$ to the equation $m + \omega(m) = n$ is an unbounded function of n, thus answering a question posed in an earlier paper in this series. A principal tool is a Turán-Kubilius type inequality for additive functions on arithmetic progressions with a large modulus.

Key words: Turán-Kubilius inequality, additive functions, prime divisors

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1. Introduction

Let $\omega(n)$ denote the number of distinct prime factors of n and let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity.

Let $g(n)$ denote the number of integers m with $m + \omega(m) = n$. In [3] we proved that $g(n) \geq 2$ infinitely often, and in [3], [4] and [5] we extended this result in various directions. However, as we wrote at the end of [5]: "Probably $g(n)$ is unbounded, but this ... seems difficult. The best we can do in this direction is that $g(n) \geq 2$ infinitely often—this is of course the main result of the first paper in this series."

In this paper we will first prove that

$$g(n) \gg (\log n)^{1/2} (\log \log n)^{-1}$$

infinitely often. More precisely we show the following result.

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Theorem 1. *There are absolute constants $c_1 > 0$ and x_0 such that for all $x > x_0$ there is an integer n with $n \le x$ and*

$$g(n) > c_1(\log x)^{1/2}(\log\log x)^{-1}.$$

For any arithmetic function $f(n)$ and $x > 1$, let $F(f, x)$ denote the greatest positive integer F such that there is a positive integer n with the properties that $n + F \le x$ and the values $f(n + 1), f(n + 2), \ldots, f(n + F)$ are all different. We can prove the following result on the localization of the repeated values of $h(n) := n + \omega(n)$.

Theorem 2. *There are absolute constants c_2 and x_0 such that for all $x > x_0$ we have*

$$F(h, x) < \exp(c_2(\log x)(\log\log x)^{-1/2}). \tag{1.1}$$

In the opposite direction we can show that

$$F(h, x) \gg (\log x)^{1/2}(\log\log x)^{-1}.$$

Indeed, this follows trivially from the following theorem.

Theorem 3. *For x sufficiently large there are positive integers n and k such that*

$$n + k \le x, \tag{1.2}$$

$$k > \frac{1}{11}(\log x)^{1/2}(\log\log x)^{-1} \tag{1.3}$$

and

$$\omega(n + 1) < \omega(n + 2) < \cdots < \omega(n + k). \tag{1.4}$$

By Theorem 3 we have

$$F(\omega, x) \gg (\log x)^{1/2}(\log\log x)^{-1}.$$

It can be shown by a similar argument that

$$F(\Omega, x) \gg (\log x)^{1/2}(\log\log x)^{-1}.$$

Trivially, for each $\epsilon > 0$,

$$F(\omega, x) < (1 + \epsilon)\frac{\log x}{\log\log x} \text{ (for } x > x_0(\epsilon))$$

and

$$F(\Omega, x) \le \frac{\log x}{\log 2} \text{ (for all } x > 1),$$

since for any $n \le x$, $\omega(n) < (1 + \epsilon)\log x/\log\log x$ and $\Omega(n) \le \log x/\log 2$. We conjecture that both $F(\omega, x)$ and $F(\Omega, x)$ are $o(\frac{\log x}{\log\log x})$. However, we do not have any reasonable upper bound for $F(\omega, x)$ and, indeed, we have not been able to prove even that there is some fixed $\epsilon > 0$ with

$$F(\omega, x) < (1-\epsilon)\frac{\log x}{\log\log x}$$

for all sufficiently large x.

On the other hand, it follows from Theorem 6 in [6] that

$$F(\Omega, x) = o(\log x).$$

We will improve on this by proving

Theorem 4. *For all $\epsilon > 0$ there is a number $x_0 = x_0(\epsilon)$ such that for $x > x_0$ we have*

$$F(\Omega, x) < (1+\epsilon)\frac{\log x}{\log\log x}.$$

For any arithmetic function $f(n)$ and $x > 1$, let $G(f, x)$ denote the greatest positive integer G such that there is a positive integer n with the properties that $n + G \le x$ and $f(n+1) = f(n+2) = \cdots = f(n+G)$. Erdős and Mirsky [2] proposed the study of the function $G(d, x)$, and later Heath-Brown [7] proved that $d(n) = d(n+1)$ (where d is the divisor function) and $\Omega(n) = \Omega(n+1)$ infinitely often, but no non-trivial upper bound has been given for $G(d, x)$ and $G(\Omega, x)$. On the other hand, it is not known whether $\omega(n) = \omega(n+1)$ holds infinitely often. We will prove

Theorem 5. *For all $\epsilon > 0$ there is a number $x_0 = x_0(\epsilon)$ such that for $x > x_0(\epsilon)$ we have*

$$G(\omega, x) < \exp((1/\sqrt{2}+\epsilon)(\log x \log\log x)^{1/2}) \tag{1.5}$$

and

$$G(\Omega, x) < \exp((\sqrt{\log 2}+\epsilon)(\log x)^{1/2}). \tag{1.6}$$

2. Lemmas

In this section we shall prove several lemmas needed in the proofs of the theorems. First we shall prove a Turán-Kubilius type inequality on arithmetic progressions:

Lemma 1. *Assume that $x \ge 1$, $m \in \mathbb{N}$,*

$$m \le x^{1/2}, \tag{2.1}$$

$h \in \mathbb{Z}$, and $f(n)$ is a non-negative additive arithmetic function such that

$$f(p^\alpha) = 0 \quad \textit{for } p \mid m,\ \alpha \in \mathbb{N}. \tag{2.2}$$

Let

$$K = max\{f(p^\alpha) : p^\alpha \le x\}, \qquad A = \sum_{p \le x} \frac{f(p)}{p}. \tag{2.3}$$

Then we have

$$\sum_{\substack{n\le x\\ n\equiv h(\mathrm{mod}\ m)}} (f(n)-A)^2 < c_3\frac{x}{m}(KA+K^2)$$

(*where* c_3 *is an absolute constant independent of* x, m, h *and* f).

Note that results of similar nature appear in [1] and [8], however, neither of these results is stated in the form needed by us. In particular, in both cases the modulus m must be much smaller than in (2.1) (it must be fixed or it may grow at most as fast as a power of $\log\log x$). We are able to cover the case $m > x^c$ at the expense of the appearance of the quantity K in the upper bound.

We might have applied Lemma 1 in [3], but we prefer to give the proof of the more general result above for possible future applications.

The assumption (2.1) can be replaced by $m \le x^{1-\delta}$ for any fixed δ, $0<\delta<1$, however, in this case the value of the constant c_3 in the last inequality depends on δ.

We remark that when f is completely additive we may change the definition of K in (2.3) to be the maximum of $f(p)$ for $p \le x$, at the expense of enlarging the absolute constant c_3 in the last inequality.

Note moreover that one can get rid of the assumption $f(n)\ge 0$ by writing a general real valued additive arithmetic function as the sum of a non-negative and a non-positive additive function, and similarly, the complex case can be handled by separating real and imaginary parts. (Of course, in these cases $f(p^\alpha)$ in (2.3) must be replaced with $|f(p^\alpha)|$.)

Proof of Lemma 1: Define the additive arithmetic function $f_1(n)$ by

$$f_1(p^\alpha)=\begin{cases} f(p^\alpha) & \text{for } p^\alpha \le x^{1/4},\\ 0 & \text{for } p^\alpha > x^{1/4}.\end{cases} \tag{2.4}$$

Clearly, for $n \le x$ there are at most 3 different prime powers p^α with $p^\alpha > x^{1/4}$, $p^\alpha \parallel n$. By (2.3), it follows that

$$|f(n)-f_1(n)| = \sum_{\substack{p^\alpha > x^{1/4}\\ p^\alpha \parallel n}} f(p^\alpha) \le 3K \quad \text{for all } n\le x. \tag{2.5}$$

Moreover, writing

$$A_1=\sum_{p^\alpha\le x}\frac{f_1(p^\alpha)}{p^\alpha},$$

clearly we have

$$\begin{aligned}|A-A_1| &\le \sum_{x^{1/4}<p\le x}\frac{f(p)}{p}+\sum_{p^\alpha\le x^{1/4},\alpha>1}\frac{f(p^\alpha)}{p^\alpha}\\ &\le K\sum_{x^{1/4}<p\le x}\frac{1}{p}+K\sum_{p^\alpha\le x^{1/4},\alpha>1}\frac{1}{p^\alpha}=O(K).\end{aligned} \tag{2.6}$$

(Here and throughout the proof of Lemma 1 the constants implied in the $O(\dots)$ terms are independent of $f(n)$ and the parameters x, m, h.)

Write

$$U = \sum_{\substack{n \le x \\ n \equiv h (\mathrm{mod}\ m)}} f_1(n)$$

and

$$V = \sum_{\substack{n \le x \\ n \equiv h (\mathrm{mod}\ m)}} f_1^2(n).$$

By (2.2) we have

$$U = \sum_{\substack{n \le x \\ n \equiv h (\mathrm{mod}\ m)}} \sum_{p^\alpha \| n} f_1(p^\alpha) = \sum_{\substack{p^\alpha \le x^{1/4} \\ (p,m)=1}} f_1(p^\alpha) \sum_{\substack{n \le x,\, p^\alpha \| n \\ n \equiv h (\mathrm{mod}\ m)}} 1. \tag{2.7}$$

By $(p, m) = 1$, $p^\alpha \le x^{1/4}$ and (2.1), the innermost sum is

$$\begin{aligned} \sum_{\substack{n \le x,\, p^\alpha \| n \\ n \equiv h (\mathrm{mod}\ m)}} 1 &= \sum_{\substack{k \le x/p^\alpha, (k,p)=1 \\ p^\alpha k \equiv h (\mathrm{mod}\ m)}} 1 = \sum_{\substack{k \le x/p^\alpha \\ p^\alpha k \equiv h (\mathrm{mod}\ m)}} 1 - \sum_{\substack{k \le x/p^\alpha,\, p|k \\ p^\alpha k \equiv h (\mathrm{mod}\ m)}} 1 \\ &= \frac{x}{mp^\alpha} + O\left(\frac{x}{mp^{\alpha+1}}\right). \end{aligned} \tag{2.8}$$

Thus by using (2.3) and (2.4), it follows from (2.7) that

$$U = \frac{x}{m} \sum_{\substack{p^\alpha \le x^{1/4} \\ (p,m)=1}} \frac{f_1(p^\alpha)}{p^\alpha} + O\left(\frac{x}{m} \sum_{p^\alpha \le x^{1/4}} \frac{f_1(p^\alpha)}{p^{\alpha+1}}\right) = \frac{x}{m}(A_1 + O(K)). \tag{2.9}$$

By (2.2) we have

$$\begin{aligned} V &= \sum_{\substack{n \le x \\ n \equiv h (\mathrm{mod}\ m)}} \left(\sum_{p^\alpha \| n} f_1(p^\alpha)\right)^2 = \sum_{\substack{n \le x \\ n \equiv h (\mathrm{mod}\ m)}} \sum_{p^\alpha \| n} \sum_{q^\beta \| n} f_1(p^\alpha) f_1(q^\beta) \\ &= \sum_{\substack{n \le x \\ n \equiv h (\mathrm{mod}\ m)}} \left(\sum_{p^\alpha \| n} f_1^2(p^\alpha) + \sum_{\substack{p^\alpha \| n \\ p \ne q}} \sum_{q^\beta \| n} f_1(p^\alpha) f_1(q^\beta) \right) \\ &= \sum_{\substack{p^\alpha \le x^{1/4} \\ (p,m)=1}} f_1^2(p^\alpha) \sum_{\substack{n \le x,\, p^\alpha \| n \\ n \equiv h (\mathrm{mod}\ m)}} 1 + \sum_{\substack{p^\alpha, q^\beta \le x^{1/4} \\ p \ne q, (pq,m)=1}} \sum f_1(p^\alpha) f_1(q^\beta) \sum_{\substack{n \le x,\, p^\alpha \| n, q^\beta \| n \\ n \equiv h (\mathrm{mod}\ m)}} 1 \\ &= S_1 + S_2, \text{ say.} \end{aligned} \tag{2.10}$$

By (2.3), the first term is

$$S_1 \le \sum_{p^\alpha \le x^{1/4}} f_1^2(p^\alpha)\left(\frac{x}{mp^\alpha}+1\right) = O\left(\frac{x}{m}KA_1 + K^2x^{1/4}\right). \tag{2.11}$$

The second term in (2.10) is

$$S_2 \le \sum\sum_{p^\alpha, q^\beta \le x^{1/4}} f_1(p^\alpha)\, f_1(q^\beta)\left(\frac{x}{mp^\alpha q^\beta}+1\right) = \frac{x}{m}A_1^2 + O(K^2x^{1/2}). \tag{2.12}$$

It follows from (2.10), (2.11) and (2.12) that

$$V \le \frac{x}{m}\big(A_1^2 + O(KA_1)\big) + O(K^2x^{1/2}). \tag{2.13}$$

By $f(n) \ge 0$ and (2.3) clearly we have

$$A_1 = \sum_{p^\alpha \le x} \frac{f_1(p^\alpha)}{p^\alpha} \le K \sum_{p^\alpha \le x^{1/4}} 1 \le Kx^{1/4}. \tag{2.14}$$

It follows from (2.1), (2.9), (2.13), (2.14) that

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv h(\mathrm{mod}\ m)}} (f_1(n)-A_1)^2 &= V - 2A_1U + A_1^2\left(\frac{x}{m}+O(1)\right)\\ &\ll \frac{x}{m}KA_1 + K^2x^{1/2} + A_1^2 \ll \frac{x}{m}(KA_1+K^2).\end{aligned} \tag{2.15}$$

Finally, by (2.5), (2.6) and the inequality

$$(a+b)^2 \le 2(a^2+b^2)$$

we have

$$\begin{aligned}(f(n)-A)^2 &= ((f_1(n)-A_1) + (f(n)-f_1(n)) + (A_1-A))^2\\ &\le 2(f_1(n)-A_1)^2 + O(K^2)\end{aligned} \tag{2.16}$$

(uniformly in n). It follows from (2.6), (2.15) and (2.16) that

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv h(\mathrm{mod}\ m)}} (f(n)-A)^2 &\le 2\sum_{\substack{n\le x\\ n\equiv h(\mathrm{mod}\ m)}} (f_1(n)-A_1)^2 + O\left(K^2 \sum_{\substack{n\le x\\ n\equiv h(\mathrm{mod}\ m)}} 1\right)\\ &= O\left(\frac{x}{m}(KA+K^2)\right)\end{aligned}$$

which completes the proof of Lemma 1. □

Lemma 2. *Let*

$$\omega_m(n) = |\{p : p \textit{ prime}\ ,\ p \nmid m,\ p \mid n\}| \tag{2.17}$$

and

$$\mathcal{S}(m, h, z, x) = \{n\colon n \le x,\ n \equiv h \pmod m,\ |\omega_m(n) - \log\log x| < z(\log\log x)^{1/2}\}.$$

There exist absolute constants c_4, x_0 *such that if* $x > x_0$, $m \in \mathbb{N}$,

$$m \le x^{1/2} \tag{2.18}$$

and $h \in \mathbb{Z}$, *then*

$$|\mathcal{S}(m, h, c_4, x)| > \frac{1}{2}\frac{x}{m}. \tag{2.19}$$

Proof of Lemma 2: We apply Lemma 1 with $\omega_m(n)$ in place of $f(n)$. Note that the number K in Lemma 1 is 1 and A, by (2.18), satisfies

$$A = \sum_{p \le x} - \sum_{p \mid m} \frac{1}{p} = \log\log x + O(\log\log\log x).$$

If c_4 is chosen large enough in terms of the constant c_3 in Lemma 1, our lemma now follows from a routine calculation.

Write

$$D(x, m, h) = \sum_{\substack{n \le x \\ n \equiv h \pmod m}} d(n). \tag{2.20}$$

□

Lemma 3. *If* $x \ge e^4$, $m \in \mathbb{N}$,

$$m \le x^{1/2}, \tag{2.21}$$

$h \in \mathbb{Z}$ *and* $(h, m) = 1$, *then*

$$D(x, m, h) < 2\frac{x \log x}{m}. \tag{2.22}$$

(Note that a similar, even sharper result is proved in [9], however, it is stated in a form slightly different from the one needed by us.)

Proof of Lemma 3: By (2.21) we have

$$\begin{aligned} D(x, m, h) &= \sum_{\substack{n \le x \\ n \equiv h \pmod m}} \sum_{d \mid n} 1 \le 2 \sum_{\substack{n \le x \\ n \equiv h \pmod m}} \sum_{\substack{d \mid n \\ d \le \sqrt{n}}} 1 \le 2 \sum_{\substack{d \le \sqrt{x} \\ (d,m)=1}} \sum_{\substack{n \le x, d \mid n \\ n \equiv h \pmod m}} 1 \\ &\le 2 \sum_{\substack{d \le \sqrt{x} \\ (d,m)=1}} \left(\frac{x}{dm} + 1\right) \le 2\frac{x}{m}(1 + \log\sqrt{x}) + 2\sqrt{x} \\ &\le 2\frac{x}{m}\left(2 + \frac{1}{2}\log x\right) < 2\frac{x \log x}{m}. \end{aligned}$$

□

Lemma 4. *There is a number x_0 such that if $x > x_0$, $m \in \mathbb{N}$,*

$$m \le x^{1/2} \tag{2.23}$$

and $h \in \mathbb{Z}$, then, writing

$$\mathcal{F} = \mathcal{F}(m,h,x) = \{n\colon n \le x,\ n \equiv h \pmod m),\ \omega_m(n) > 3\log\log x\}$$

(*where $\omega_m(n)$ is defined by* (2.17)), *we have*

$$|\mathcal{F}(m,h,x)| < \frac{x}{m\log x}. \tag{2.24}$$

Proof of Lemma 4: Write $(h,m) = g$, $h = gh_1$, $m = gm_1$. Then

$$\mathcal{F} = \{gn_1\colon n_1 \le x/g,\ n_1 \equiv h_1(\mathrm{mod}\, m_1),\ \omega_m(n_1) > 3\log\log x\}.$$

Clearly, for $gn_1 \in \mathcal{F}$ we have

$$d(n_1) \ge 2^{\omega(n_1)} \ge 2^{\omega_m(n_1)} > 2^{3\log\log x} = (\log x)^{\log 8}. \tag{2.25}$$

We apply Lemma 3 with x/g in place of x, m_1 in place of m, and h_1 in place of h. From (2.23), we have $m_1 \le x^{1/2}/g$, so the inequality (2.21) is satisfied with our new parameters. Lemma 4 now follows from (2.25), the fact that $\log 8 > 2$, and a simple calculation. □

3. Proofs of the theorems

Proof of Theorem 1: Let x be a large enough number, write

$$t_x = \left[\frac{1}{5}\left(\frac{\log x}{\log\log x}\right)^{1/2}\right] \text{ and } u_i = t_x - i \text{ for } i = 0, 1, \dots, t_x, \tag{3.1}$$

so that

$$v_x := \sum_{i=0}^{t_x} u_i = \left(\frac{1}{50} + o(1)\right)\frac{\log x}{\log\log x} \tag{3.2}$$

(as $x \to +\infty$). Let $\mathcal{P}_0$ denote the set of the first u_0 primes greater than t_x, and if $\mathcal{P}_{j-1}$ has been defined for some $1 \le j \le t_x$, then let $\mathcal{P}_j$ denote the set of the first u_j primes greater than the greatest prime in $\mathcal{P}_{j-1}$. Let $P = \prod_{j=0}^{t_x}\prod_{p\in\mathcal{P}_j} p$ so that, by (3.2), we have

$$\omega(P) = \sum_{j=0}^{t_x} |\mathcal{P}_j| = v_x = \left(\frac{1}{50} + o(1)\right)\frac{\log x}{\log\log x}. \tag{3.3}$$

By the prime number theorem, it follows from (3.1) and (3.3) that

$$P = x^{(1/50)+o(1)}. \tag{3.4}$$

Let r denote the least positive integer with

$$r+i \equiv 0 \left(\bmod \prod_{p\in\mathcal{P}_i} p \right) \text{ for } i=0,1,\ldots,t_x.$$

Clearly $p|(P,r+i)$ if and only if $p\in\mathcal{P}_i$ whence

$$\omega((r+i,P)) = u_i \text{ for } i=0,1,\ldots,t_x. \tag{3.5}$$

By (3.4), (2.18) in Lemma 2 holds with P in place of m. Thus using Lemma 2 with $m=P$ we obtain

$$\begin{aligned}\sum_{\substack{n\le x\\ n\equiv r(\bmod\ P)}} \sum_{\substack{0\le i\le t_x\\ n+i\in\mathcal{S}(P,r+i,c_4,x)}} 1 &= \sum_{0\le i\le t_x} \sum_{n+i\in\mathcal{S}(P,r+i,c_4,x)} 1 \\ &= \sum_{0\le i\le t_x} |\mathcal{S}(P,r+i,c_4,x)| > \sum_{0\le i\le t_x} \frac{1}{2}\frac{x}{P} > \frac{1}{2}\frac{x}{P}t_x.\end{aligned}$$

The summation $\sum_{n\le x, n\equiv r(\bmod P)}$ has at most $\frac{x}{P}+1 < 2\frac{x}{P}$ terms, thus it follows that there is an integer n with

$$\sum_{\substack{0\le i\le t_x\\ n+i\in\mathcal{S}(P,r+i,c_4,x)}} 1 > \left(\frac{1}{2}\frac{x}{P}t_x\right)\left(2\frac{x}{P}\right)^{-1} = \frac{1}{4}t_x.$$

Let $\mathcal{W}$ denote the set of the integers i with $0\le i\le t_x$, $n+i\in\mathcal{S}(P,r+i,c_4,x)$ so that

$$|\mathcal{W}| > \frac{1}{4}t_x.$$

Then for all $i\in\mathcal{W}$ we have

$$\begin{aligned}|h(n+i)-(n+t_x+\log\log x)| &= |i+\omega(n+i)-t_x-\log\log x| \\ &= |i+\omega((n+i,P))+\omega_P(n+i)-t_x-\log\log x| \\ &= |i+u_i+\omega_P(n+i)-t_x-\log\log x| \\ &= |\omega_P(n+i)-\log\log x| < c_4(\log\log x)^{1/2}.\end{aligned}$$

Thus for each of the $|\mathcal{W}|$ numbers $i\in\mathcal{W}$, the value of $h(n+i)$ belongs to the interval

$$(n+t_x+\log\log x - c_4(\log\log x)^{1/2},\ n+t_x+\log\log x + c_4(\log\log x)^{1/2})$$

which contains at most $2c_4(\log\log x)^{1/2}+1$ integers. Thus by (3.1), for $x>x_0$ at least one of these integers, say k, has at least

$$\frac{|\mathcal{W}|}{2c_4(\log\log x)^{1/2}+1} > \frac{t_x/4}{3c_4(\log\log x)^{1/2}} > c_5(\log x)^{1/2}(\log\log x)^{-1}$$

representations in the form $h(n+i)$ (with $i \in \mathcal{W}$) so that

$$g(k) > c_5 (\log x)^{1/2} (\log\log x)^{-1}.$$

This completes the proof of Theorem 1. □

Proof of Theorem 2: We have to prove that for all $x > x_0$ there are $m, n \in \mathbb{N}$ such that $x < m < n < x + \exp(c_2 (\log x)(\log\log x)^{-1/2})$ and $m + \omega(m) = n + \omega(n)$. The proof of this is similar to the proof of Theorem 1. This time we choose $t_x = c_6 (\log\log x)^{1/2}$ where c_6 is a large positive constant. We also need a short interval version of Lemma 1 (short: of the type $(x, x + \exp(c_2 (\log x)(\log\log x)^{-1/2}))$. Apart from these changes, the proof is nearly the same, thus we leave the details to the reader. □

Proof of Theorem 3: Let x be a large enough number, write

$$t_x = \left[\frac{1}{10} (\log x)^{1/2} (\log\log x)^{-1} \right],$$

$$u_i = i[10 \log\log x] \text{ for } i = 1, 2, \ldots, t_x$$

and

$$v_x = \sum_{i=1}^{t_x} u_i$$

so that

$$v_x = \left(\frac{1}{20} + o(1) \right) \frac{\log x}{\log\log x} \tag{3.6}$$

(as $x \to +\infty$). Let $\mathcal{P}_1$ denote the first u_1 primes greater than t_x, and if $\mathcal{P}_{i-1}$ has been defined, then let $\mathcal{P}_i$ denote the set of the first u_i primes greater than the greatest prime in $\mathcal{P}_{i-1}$. Let P_i denote the product of the primes in $\mathcal{P}_i$ and let $P = P_1 P_2 \cdots P_{t_x}$, so that, by (3.6), we have

$$\omega(P) = v_x = \left(\frac{1}{20} + o(1) \right) \frac{\log x}{\log\log x}. \tag{3.7}$$

By the prime number theorem, it follows that

$$P = x^{(1/20)+o(1)}. \tag{3.8}$$

Let r denote the least positive integer with

$$r + i \equiv 0 \pmod{P_i} \text{ for } i = 1, 2, \ldots, t_x.$$

Clearly

$$\omega((r+i, P)) = \omega(P_i) = u_i \text{ for } i = 1, 2, \ldots, t_x. \tag{3.9}$$

By (3.8), (2.23) in Lemma 4 holds with P in place of m. Thus using Lemma 4 with $m = P$ we obtain

$$\sum_{\substack{n \le x - t_x \\ n \equiv r (\text{mod } P)}} \sum_{\substack{1 \le i \le t_x \\ n+i \in \mathcal{F}(P, r+i, x)}} 1 \le \sum_{i=1}^{t_x} \sum_{n+i \in \mathcal{F}(P,r+i,x)} 1 < \sum_{i=1}^{t_x} \frac{x}{P \log x} = \frac{x t_x}{P \log x} < \frac{x}{3P}$$

for x large enough. Here the outer sum has at least $\frac{x}{P} - 2 > \frac{x}{2P}$ terms, thus at least one of the inner sums is <1. Since these sums are non-negative integers, it follows that at least one of them is 0, i.e., there is an integer n such that

$$n + t_x \le x, \tag{3.10}$$

$n \equiv r \pmod P$ and $n + i \notin \mathcal{F}(P, r+i, x)$ for $1 \le i \le t_x$ so that

$$\omega_P(n+i) \le 3 \log\log x \text{ for } i = 1, 2, \ldots, t_x. \tag{3.11}$$

By (3.9), for this n we have

$$\begin{aligned} \omega(n+i) &= \omega((n+i, P)) + \omega_P(n+i) = \omega((r+i, P)) + \omega_P(n+i) \\ &= u_i + \omega_P(n+i) = i[10 \log\log x] + \omega_P(n+i) \end{aligned}$$

whence, by (3.11),

$$i[10 \log\log x] \le \omega(n+i) \le i[10\log\log x] + 3 \log\log x \text{ for } i = 1, 2, \ldots, t_x.$$

Clearly, this implies (1.4) with t_x in place of k. Also (1.2) and (1.3) also hold by (3.10) and the definition of t_x. This completes the proof of Theorem 3. □

Proof of Theorem 4: We have to show that if $x > x_0(\epsilon)$, $n, k \in \mathbb{N}$,

$$n + k \le x \tag{3.12}$$

and

$$\Omega(n+i) \ne \Omega(n+j) \text{ for } 1 \le i < j \le k, \tag{3.13}$$

then

$$k < (1+\epsilon) \frac{\log x}{\log\log x}. \tag{3.14}$$

We may assume that k is large since otherwise there is nothing to be proved. Write $t = [k/\log k]$, and let p_i denote the ith prime. For each $i \in \{1, 2, \ldots, t\}$ remove that number from $\{n+1, n+2, \ldots, n+k\}$ which is divisible by the highest power of p_i (if there are several numbers divisible by the highest power, then remove the smallest of them). Denote the remaining set by $\mathcal{Y}$ so that

$$|\mathcal{Y}| = k - t = (1 + o(1))k. \tag{3.15}$$

Each positive integer y may be written in the form

$$y = \prod_{i=1}^{\infty} p_i^{\alpha(i,y)}.$$

If $y \in \mathcal{Y}$ and $i \le t$, then we have $p_i^{\alpha(i,y)} \le k$, for there is at most one number from $\{n+1, n+2, \ldots, n+k\}$ divisible by a power of p_i bigger than k, and if this number exists, it is not in $\mathcal{Y}$. Note too that

$$\prod_{i>t} p_i^{\alpha(i,y)} \le y \le n+k \le x,$$

so that

$$\sum_{i>t} \alpha(i,y) \le \frac{\log x}{\log p_t}.$$

It follows that if $y \in \mathcal{Y}$, then

$$\begin{aligned}\Omega(y) &= \sum_{i=1}^{\infty} \alpha(i,y) = \sum_{i=1}^{t} \alpha(i,y) + \sum_{i>t} \alpha(i,y) \le \sum_{i=1}^{t} \frac{\log k}{\log p_i} + \frac{\log x}{\log p_t} \\ &= (1+o(1))\frac{k}{\log k} + (1+o(1))\frac{\log x}{\log k}.\end{aligned}$$

Since all the values of $\Omega(y)$ are distinct for $y \in \mathcal{Y}$, we have by (3.15) that

$$k \le (1+o(1))\frac{k}{\log k} + (1+o(1))\frac{\log x}{\log k}.$$

This implies (3.14), and completes the proof of Theorem 4. $\square$

Proof of Theorem 5: First we will prove (1.5). We have to show that if

$$n + k \le x \tag{3.16}$$

and

$$\omega(n+1) = \omega(n+2) = \cdots = \omega(n+k), \tag{3.17}$$

then

$$k < \exp\left(\left(\frac{1}{\sqrt{2}} + \epsilon\right)(\log x \log\log x)^{1/2}\right). \tag{3.18}$$

We may assume that k is large since otherwise there is nothing to be proved. Let y denote the greatest positive integer with

$$P := \prod_{p \le y} p \le k$$

so that, by the prime number theorem,

$$y = (1 + o(1)) \log k$$

(as $k \to +\infty$) and

$$\pi(y) = (1 + o(1)) \frac{\log k}{\log \log k}. \tag{3.19}$$

Clearly there is an m with $n + 1 \le m \le n + k$ and $P \mid m$. Then by (3.19) we have

$$\omega(m) \ge \omega(P) = \pi(y) = (1 + o(1)) \frac{\log k}{\log \log k}. \tag{3.20}$$

It follows from (3.17) and (3.20) that

$$\sum_{i=1}^{k} \omega(n + i) = k\omega(m) \ge (1 + o(1)) \frac{k \log k}{\log \log k}. \tag{3.21}$$

On the other hand, we have

$$\sum_{i=1}^{k} \omega(n + i) = \sum_{i=1}^{k} \sum_{p \mid n+i} 1 = \sum_{p \le k} \sum_{\substack{1 \le i \le k \\ p \mid n+i}} 1 + \sum_{p > k} \sum_{\substack{1 \le i \le k \\ p \mid n+i}} 1. \tag{3.22}$$

Here the first term is

$$\sum_{p \le k} \sum_{\substack{1 \le i \le k \\ p \mid n+i}} 1 \le \sum_{p \le k} \left(\frac{k}{p} + 1 \right) = k \sum_{p \le k} \frac{1}{p} + \pi(k) = (1 + o(1)) k \log \log k. \tag{3.23}$$

It follows from (3.21), (3.22) and (3.23) that

$$\sum_{p > k} \sum_{\substack{1 \le i \le k \\ p \mid n+i}} 1 \ge (1 + o(1)) \frac{k \log k}{\log \log k} - (1 + o(1)) k \log \log k = (1 + o(1)) \frac{k \log k}{\log \log k}.$$

For every $p > k$ the innermost sum is 0 or 1. Thus

$$\left| \left\{ p\colon p > k,\ p \;\middle|\; \prod_{i=1}^{k} (n + i) \right\} \right| \ge (1 + o(1)) \frac{k \log k}{\log \log k}.$$

It follows that there is an integer t and primes $p_{i_1}, p_{i_2}, \ldots, p_{i_t}$ with

$$t = (1 + o(1)) \frac{k \log k}{\log \log k}, \tag{3.24}$$

$$k < p_{i_1} < p_{i_2} < \cdots < p_{i_t} \tag{3.25}$$

and

$$p_{i_1} p_{i_2} \dots p_{i_t} \;\Big|\; \prod_{i=1}^{k} (n+i). \tag{3.26}$$

(Here p_i denotes the ith prime.) Define u by

$$p_u \le k < p_{u+1}. \tag{3.27}$$

By the prime number theorem, it follows from (3.16), (3.24), (3.25), (3.26) and (3.27) that

$$\begin{aligned} x^k &\ge \prod_{i=1}^{k} (n+i) \ge p_{i_1} p_{i_2} \cdots p_{i_t} \ge p_{u+1} p_{u+2} \cdots p_{u+t} \\ &= \exp((1+o(1)) p_{u+t}) = \exp\left((1+o(1)) \frac{k \log^2 k}{\log\log k}\right), \end{aligned}$$

whence

$$\log x \ge (1+o(1)) \frac{\log^2 k}{\log\log k}.$$

This implies (3.18) which completes the proof of (1.5).

In order to prove (1.6), observe that assuming

$$\Omega(n+1) = \Omega(n+2) = \cdots = \Omega(n+k)$$

and writing $\ell = [\frac{\log k}{\log 2}]$, there is an integer m with $n+1 \le m \le n+k$ and $2^\ell \mid m$ so that

$$\Omega(n+1) = \cdots = \Omega(n+k) \ge \ell. \tag{3.28}$$

For every $p \le k$, remove the (least) number in $\{n+1, n+2, \dots, n+k\}$ divisible by the highest power of p, and denote the remaining set by $\mathcal{Y}$ so that

$$|\mathcal{Y}| = (1+o(1))k$$

and

$$\sum_{n+i \in \mathcal{Y}} \Omega(n+i) \ge \ell |\mathcal{Y}| = \left(\frac{1}{\log 2} + o(1)\right) k \log k. \tag{3.29}$$

On the other hand, we have

$$\sum_{n+i \in \mathcal{Y}} \Omega(n+i) = \sum_{p \le k} \sum_{\substack{n+i \in \mathcal{Y} \\ p^\alpha \| n+i}} \alpha + \sum_{p > k} \sum_{\substack{n+i \in \mathcal{Y} \\ p^\alpha \| n+i}} \alpha. \tag{3.30}$$

By the definition of $\mathcal{Y}$, for every prime $p \le k$ and any positive integer α, there are at most $[k/p^\alpha]$ members of $\mathcal{Y}$ divisible by p^α. Thus, the first term on the right of (3.30) is

$$\le \sum_{p\le k}\sum_{\alpha=1}^{\infty}\left[\frac{k}{p^\alpha}\right] < k\sum_{p\le k}\sum_{\alpha=1}^{\infty}\frac{1}{p^\alpha} = \sum_{p\le k}\frac{k}{p-1} = (1+o(1))k\log\log k.$$

Thus we obtain from (3.29) and (3.30) that

$$\sum_{p>k}\sum_{\substack{n+i\in\mathcal{Y}\\ p^\alpha\|n+i}}\alpha \ge \left(\frac{1}{\log 2}+o(1)\right)k\log k.$$

The rest of the proof is similar to the proof of (1.5); we leave the details to the reader. □

Note

This paper was written while A. Sárközy was visiting the University of Georgia.

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