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On the Distribution of Champs

Aleksandar Ivić and Carl Pomerance

Dedicated to the memory of our friend and teacher, Paul Erdős

ABSTRACT. Clearly every divisor of $(n-1)!$ is also a divisor of $n!$. Let $D(n)$ denote the number of divisors of $n!$ that do not divide $(n-1)!$. In an earlier paper by P. Erdős, S. W. Graham, and the authors, the concept of a "champ" was introduced. This is a number n the property that $D(m) < D(n)$ for all $m < n$. Under the assumption of the Riemann Hypothesis it was shown that the champs have asymptotic density zero. We are able here to remove this assumption, and use instead a result of G. Harman on the distribution of almost primes in short intervals. In the earlier paper it was shown that primes and doubles of primes are champs, and it was asked if there are infinitely many champs not of this form. This was proved conditionally on the prime k-tuples conjecture, by showing there are infinitely many champs of the form $3p$, with p prime. We show here, again conditional on the prime k-tuples conjecture, that for each fixed prime r, there are infinitely many champs of the form rp, with p prime.

1. Introduction

Let $D(n) = d(n!) - d((n-1)!)$, where $d(k)$ is the number of positive divisors of a natural number k. The function $D(n)$ represents then the number of divisors of $n!$ which are not divisors of $(n-1)!$. It was introduced in [**3**], where several problems involving $d(n!)$ were studied. In particular it seems interesting to study the so-called D-champions or simply *champs*, namely natural numbers $n\,(> 1)$ for which

$$D(m) < D(n) \qquad (m = 1, 2, \ldots, n-1). \tag{1}$$

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This is the final version of the paper.

The champs are in fact the analogues of *highly composite numbers*, introduced by S. Ramanujan (see [**7**]). These are natural numbers $n\,(>1)$ for which

$$d(m) < d(n) \qquad (m = 1, 2, \dots, n-1).$$

Thus highly composite numbers are d-champions. Their distribution was studied by several authors, most notably by J.-L. Nicolas (see his comprehensive paper [**6**]).

The object of this note is to study the distribution of champs, which is quite different from the distribution of highly composite numbers. It is not difficult to see that the numbers $p, 2p$ (p henceforth denotes primes) are champs, see [**3**]. However it seems difficult to decide whether there are infinitely many champs not of the form p or $2p$. Dr. J.-P. Massias of Université de Limoges kindly provided us with a table of champs up to 200 000, which he has calculated. It turns out that the champs in his table, which are not of the form p or $2p$, are all of the form $3p$, $4p$, $5p$, $6p$ or $7p$. Several much larger champs of the latter form were kindly calculated by Prof. J.-M. De Koninck of Université Laval (Québec), and more numerical evidence would be welcome to ascertain whether this phenomenon will continue to hold. We wish to continue here the discussion concerning the distribution of champs begun in [**3**]. We prove the following result.

THEOREM 1. *Assuming the prime k-tuples conjecture, for any prime r there are infinitely many champs of the form rq, q prime.*

The prime k-tuples conjecture is a famous conjecture from prime number theory (due to L. E. Dickson, see [**2**]) which generalizes the twin prime conjecture that there are infinitely many integers x such that both x and $x+2$ are prime. To formulate the prime k-tuples conjecture suppose that we have M linear functions in the variable x, each with integer coefficients and positive slope, such that there is no fixed integer $m > 1$ which divides the product of the M linear functions at every integer argument. Then there are infinitely many positive integers x at which the M linear functions are simultaneously prime. For example, for $M = 3$, the functions $x,\ 2x+1,\ 3x+2$ are "admissible", while the functions $x,\ 2x+1,\ 4x+1$ are not.

We also are interested in estimating $C(x)$, the number of champs not exceeding x. Since the numbers p, $2p$ are champs, one easily obtains from the prime number theorem that

$$C(x) \geq \left(\frac{3}{2} + o(1)\right)\frac{x}{\log x}. \tag{2}$$

As for the upper bound for $C(x)$, a (very strong) classical conjecture of H. Cramér [**1**] that

$$p_{n+1} - p_n \ll \log^2 p_n,$$

where p_n denotes the n-th prime, leads to the bound

$$C(x) \ll \frac{x \log\log x}{\log x}, \tag{3}$$

as shown in [**3**]. In particular (3) implies that

$$C(x) = o(x) \qquad (x \to \infty), \tag{4}$$

or in other words, the asymptotic density of the set of champs equals zero. In [**3**] the relation (4) was also shown to be true using a result of A. Selberg that most short intervals countain a prime. However, Selberg's theorem is conditional on the

Riemann Hypothesis. We shall show (4) to be true unconditionally, by using a result of G. Harman that asserts that most short intervals contain a number which is the product of two primes. Harman's theorem was built on earlier results of Y. Motohashi and D. Wolke. We record the following result.

THEOREM 2. *The set of champs has asymptotic density zero.*

2. The proof of Theorem 1

Before we give the proof of Theorem 1, we prove the following lemmas.

LEMMA 1. *Let k be a fixed positive integer and let q be a prime. Then we have*

$$\frac{d((kq)!)}{d((kq-1)!)} = \left(1+\frac{1}{k}\right)\left(1+\frac{1}{kq}\sum_{p^a\|k}(p-1)a\right) + O_k\left(\frac{\log q}{q^2}\right).$$

PROOF. We have

$$n! = \prod_{p\le n} p^{w_p(n)}, \quad w_p(n) := [\frac{n}{p}] + [\frac{n}{p^2}] + \cdots .$$

On replacing $[n/p^j]$ by n/p^j in the expression for $w_p(n)$ and summing over j, it follows that

$$w_p(n) = \frac{n}{p-1} + O(\frac{\log n}{\log p}). \tag{5}$$

On the other hand, if $q > k$ (as we may assume), $w_p(kq) = a + w_p(kq-1)$ if $p^a\|k$, $w_q(kq) = 1 + w_q(kq-1) = k$, and for all other primes p, $w_p(kq) = w_p(kq-1)$. Thus,

$$\frac{d((kq)!)}{d((kq-1)!)} = \left(1+\frac{1}{k}\right)\prod_{p^a\|k}\left(1+\frac{a}{w_p(kq-1)+1}\right).$$

But by (5),

$$\frac{a}{w_p(kq-1)+1} = \frac{a}{kq/(p-1)+O_k(\log q)} = \frac{(p-1)a}{kq} + O_k\left(\frac{\log q}{q^2}\right).$$

Thus, we have the lemma. □

LEMMA 2. *Let $F(n) := \sum_{p^a\|n}(p-1)a$. Then $F(n) \le n-1$ for all n.*

PROOF. The assertion follows from the fact that $F(p) = p-1$ for all primes p and the observation that F is completely additive. □

LEMMA 3. *Let $J \ge 2$ be a fixed positive integer. There is a number x_J such that if p is a prime with*

(i) $p \ge x_J$,

(ii) $p \equiv 1 \pmod{J!}$,

(iii) $q_j := (2p+j)/(j+2)$ *is prime for each integer j with* $0 \le j \le J-2$,

then for each j with $0 \le j \le J-2$ and $j+2$ prime, $(j+2)q_j = 2p+j$ is a champ.

PROOF. For each j with $0 \le j \le J-2$ define ε_j by the relation

$$\frac{d((2p+j)!)}{d((2p+j-1)!)} = \left(1+\frac{1}{j+2}\right)(1+\varepsilon_j).$$

By Lemma 1 we have

$$\varepsilon_j = \frac{F(j+2)}{2p+j} + O\left(\frac{\log p}{p^2}\right),$$

where F is as in Lemma 2. If $j+2$ is a prime and $i<j$, then

$$\begin{aligned}\varepsilon_j = \frac{j+1}{2p+j} + O\left(\frac{\log p}{p^2}\right) &= \frac{i+1}{2p+i} + \frac{(2p-1)(j-i)}{(2p+j)(2p+i)} + O\left(\frac{\log p}{p^2}\right)\\ &\ge \varepsilon_i + \frac{(2p-1)(j-i)}{(2p+j)(2p+i)} + O\left(\frac{\log p}{p^2}\right),\end{aligned}$$

by Lemma 2. Since

$$\frac{(2p-1)(j-i)}{(2p+j)(2p+i)} \asymp \frac{1}{p},$$

it follows that if p is sufficiently large then $\varepsilon_j > \varepsilon_i$. In other words, the champions for the sequence $\varepsilon_0, \varepsilon_1, \cdots, \varepsilon_{J-2}$ include those ε_j with $j+2$ prime.

We have that $2q_0 = 2p$ is a champ. For $0 \le i \le J-2$ we have

$$\begin{aligned}D(2p+i) &= \left(\left(1+\frac{1}{i+2}\right)(1+\varepsilon_i) - 1\right)d((2p+i-1)!)\\ &= \left(\frac{i+3}{i+2}\varepsilon_i + \frac{1}{i+2}\right)d((2p+i-1)!).\end{aligned}$$

Suppose $0 < j \le J-2$ and $j+2$ is prime. Then, for $0 \le i < j$,

$$\begin{aligned}D(2p+j) &= \left(\frac{j+3}{j+2}\varepsilon_j + \frac{1}{j+2}\right)d((2p+j-1)!)\\ &= \left(\frac{j+3}{j+2}\varepsilon_j + \frac{1}{j+2}\right)\frac{j+2}{j+1}\cdot\frac{j+1}{j}\cdot\ldots\cdot\frac{i+3}{i+2}(1+\varepsilon_{j-1})\cdots(1+\varepsilon_i)\\ &\quad\times d((2p+i-1)!)\\ &\ge \left(\frac{j+3}{j+2}\varepsilon_j + \frac{1}{j+2}\right)\frac{j+2}{i+2}d((2p+i-1)!)\\ &= \left(\frac{j+3}{i+2}\varepsilon_j + \frac{1}{i+2}\right)d((2p+i-1)!)\\ &> \left(\frac{i+3}{i+2}\varepsilon_j + \frac{1}{i+2}\right)d((2p+i-1)!)\\ &> \left(\frac{i+3}{i+2}\varepsilon_i + \frac{1}{i+2}\right)d((2p+i-1)!) = D(2p+i).\end{aligned}$$

Thus $D(2p+j) > D(2p+i)$ and since $2p$ is a champ, so too is $2p+j$ a champ, as asserted. □

PROOF OF THEOREM 1. Let r be a prime and let $J = r+2$. Consider the linear functions

$$q_j(x) = \frac{2J!}{j+2}x + 1 \qquad (0 \le j \le J-2).$$

They are "admissible" for the prime k-tuples conjecture since each has the value 1 when $x=0$. So by the prime k-tuples conjecture there are infinitely many positive integers x so that $q_j(x)$ is a prime for each j, $0 \le j \le J-2$. If we let $p = q_0(x)$, then $2p+j = 2J!x+j+2 = (j+2)q_j(x)$ for each j with $0 \le j \le J-2$. Thus all the hypotheses of Lemma 3 are satisfied when x is sufficiently large. So $(j+2)q_j(x)$ is a champ when $0 \le j \le J-2$, $j+2$ prime. Say we let $J = r$. By what we have just said there are infinitely many integers x with $q_j(x)$ prime for each j, $0 \le j \le J-2$. Thus there are infinitely many x with $q = q_{J-2}(x)$ prime and with rq a champ. Thus, Theorem 1 is proved. □

We remark that the proof in [**3**] that numbers of the form $2p$ are champs contains a small error. The proof depends on the inequality $D(m) \le \frac{1}{2}d(m!)$, and the argument that supports this inequality is suspect. We give an alternate approach: The inequality holds when m is a prime, in fact, it is an equality in this case. In addition, the inequality holds when $m = 4, 6, 8, 9$, or 10. Suppose $m \ge 12$ is composite. It is shown in (5) in [**3**] that $d(m!)/d((m-1)!) \le \exp(S(m)/m)$, where $S(m)$ is the sum of the prime factors of m, with multiplicity. It is easy to see that $S(m)/m \le 2/3$ when $m \ge 12$ is composite. Indeed if p is a prime factor of m we have

$$S(m) \le p + S(m/p) \le p + m/p \le 2 + m/2 \le 2m/3.$$

Thus, $d(m!)/d((m-1)!) \le e^{2/3} < 2$, which implies that $D(m) \le \frac{1}{2}d(m!)$, as claimed.

3. The proof of Theorem 2

The proof is similar to the proof of Theorem 6 in [**3**].

Let $P(n)$ denote the largest prime factor of an integer $n > 1$. We shall use Theorem 3 of [**3**] which says that

$$\frac{d(n!)}{d((n-1)!)} = 1 + \frac{P(n)}{n} + O(n^{-1/2}). \tag{6}$$

We first show that if

(a) $P(n) < n/\log^{20} n$,

(b) some integer k in the interval $[n - \log^8 n, n)$ has $P(k) > k/\log^8 n$,

(c) n is sufficiently large,

then n is not a champ. We next show that the set of integers satisfying these conditions has asymptotic density 1.

So suppose that (a), (b) and (c) hold and that n is a champ. From (6) we have

$$\begin{aligned} d((n-1)!) &= d((k-1)!) + D(k) + D(k+1) + \cdots + D(n-1) \\ &< d((k-1)!) + (n-k)D(n) \\ &= d((k-1)!) + (n-k)d((n-1)!)\left(\frac{d(n!)}{d((n-1)!)} - 1\right) \\ &\le d((k-1)!) + d((n-1)!)\log^8 n \left(\frac{1}{\log^{20} n} + O(n^{-1/2})\right). \end{aligned}$$

Since $d(m!)$ is an increasing function of m this gives

$$\frac{d(k!)}{d((k-1)!)} \le \frac{d((n-1)!)}{d((k-1)!)} = 1 + O\Big(\frac{1}{\log^{12} n}\Big). \tag{7}$$

But (6) and (b) imply that

$$\frac{d(k!)}{d((k-1)!)} \geq 1 + \frac{1}{\log^8 n} + O(n^{-1/2}), \tag{8}$$

and (7) and (8) contradict each other for sufficiently large n.

It is clear that the set of integers n which do not satisfy (a) has asymptotic density zero. In particular, if $x/\log x \leq n \leq x$ and (a) does not hold, then n is divisible by a prime p with $p > x/\log^{21} x$. But the number of integers up to x divisible by such a prime p is bounded by

$$\sum_{x/\log^{21} x \leq p \leq x} \frac{x}{p} \ll \frac{x \log\log x}{\log x}.$$

We now use the following result of G. Harman [4]. He proved that for each $\delta > 0$, for almost all integers n, the interval $[n - \log^{7+\delta} n, n]$ contains a number $k = pq$ with p, q prime and $p < \log^{7+\delta} n$. It thus follows that $P(k) = q$ is large. Thus, the exceptional set of integers n for which (a) and (b) do not both hold has $\ll x \log\log x/\log x + o(x)$ members up to x. Hence this exceptional set has asymptotic density zero. This proves the theorem.

By tracing through the argument in [4] and slightly de-optimizing the exponents, one can get a result with a smaller exceptional set, and which yields (3) unconditionally. In fact, one has that the number of integers $n \leq x$ for which $[n - \log^{10} n, n]$ does not contain a number $k = pq$ with p, q prime and $p < \log^{10} n$ is $\ll x/\log x$. Then changing the "20" in (a) to "21" and the "8" in (b) to "10", the above proof can be modified to give (3).

Note that the bounds in (2) and (3) differ only by a factor of $\log\log x$, but it is hard to make a guess which one of them lies closer to the truth. Although by Theorem 1 (assuming the prime k-tuples conjecture) it seems that there are many kinds of champs, the sequence of such champs (and any additional ones which may exist) may be rather thin, so that after all the bound (2) may be of the correct order of magnitude.

We finally remark that Mikawa [5] was able to reduce the constant "7" in Harman's theorem to "5". However, in Mikawa's proof the "almost-primes" that are produced do not have their largest prime factor large enough to help us with the distribution of champs.

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Katedra Matematike RGF-a, Universiteta u Beogradu, Djušina 7, 11000 Beograd, Serbia (Yugoslavia)
E-mail address: aleks@ivic.matf.bg.ac.yu, eivica@ubbg.etf.bg.ac.yu

Department of Mathematics, University of Georgia, Athens, GA 30602
E-mail address: carl@ada.math.uga.edu