

ON SOME PROBLEMS OF MAKOWSKI-SCHINZEL AND ERDŐS
CONCERNING THE ARITHMETICAL FUNCTIONS ϕ AND σ

BY

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Abstract. Let $\sigma(n)$ denote the sum of positive divisors of the integer n , and let ϕ denote Euler's function, that is, $\phi(n)$ is the number of integers in the interval $[1, n]$ that are relatively prime to n . It has been conjectured by Małowski and Schinzel that $\sigma(\phi(n))/n \geq 1/2$ for all n . We show that $\sigma(\phi(n))/n \rightarrow \infty$ on a set of numbers n of asymptotic density 1. In addition, we study the average order of $\sigma(\phi(n))/n$ as well as its range. We use similar methods to prove a conjecture of Erdős that $\phi(n - \phi(n)) < \phi(n)$ on a set of asymptotic density 1.

1. Introduction. In this paper, we investigate a couple of conjectures concerning inequalities involving the arithmetical functions ϕ , σ , and compositions of these. As usual, for a positive integer n we write $\phi(n)$ for the Euler function of n , and $\sigma(n)$ for the sum-of-divisors function of n . For any positive integer k and any positive number x we define $\log_k(x)$ recursively as the maximum of 1 and $\log(\log_{k-1}(x))$, where $\log_1 = \log$ is the natural logarithm. Throughout this paper, we use p , q and P to denote prime numbers, and c_1, c_2, \dots will denote positive computable constants.

The first conjecture we are looking at is due to Małowski and Schinzel (see [18]) and asserts that the inequality

$$(1) \quad \frac{\sigma(\phi(n))}{n} \geq \frac{1}{2}$$

holds for all positive integers n . It is known that

$$(2) \quad \limsup_n \frac{\sigma(\phi(n))}{n} = \infty \quad \text{and} \quad 0 < \liminf_n \frac{\sigma(\phi(n))}{n} \leq \frac{1}{2} + \frac{1}{2^{34} - 1}.$$

The first limit in (2) is due to Alaoglu and Erdős (see [1]). The positive lower bound for the second limit in (2) is due to the second author (see [20]) and the upper bound for the same limit appears in the original paper of Małowski and Schinzel [18]. It is known that (1) holds for positive integers n of various shapes (see, for example, [2], [6], [11], [18]), and in fact in [1] it

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is asserted that $\sigma(\phi(n))/n \rightarrow \infty$ on a set of asymptotic density 1, and that $\sigma(\phi(n))/\phi(n) \sim e^\gamma \log_3(n)$ on a set of asymptotic density 1.

Especially since the proofs of these results stated in [1] were not given, and since later researchers have labored to give somewhat weaker results (in [15] it is shown that (1) holds on a set of lower density at least 0.74), we think it is worthwhile to prove these assertions from [1], in a somewhat stronger form. For every positive integer n we write $S(n) := \sigma(\phi(n))/n$. Our first result in this paper gives the maximal, normal, and average orders of the function $S(n)$.

THEOREM 1. (i) *We have*

$$(3) \quad \limsup_n \frac{S(n)}{\log_2(n)} = e^\gamma.$$

(ii) *For each number u , $0 \leq u \leq 1$, the asymptotic density of the set of numbers n with*

$$(4) \quad S(n) > ue^\gamma \log_3(n)$$

exists, and this density function is strictly decreasing, varies continuously with u , and is 0 when $u = 1$.

(iii) *For every positive number x ,*

$$(5) \quad \frac{1}{x} \sum_{1 \leq n \leq x} S(n) = \frac{6e^\gamma}{\pi^2} \cdot \log_3(x) + O((\log_3(x))^{1/2}).$$

We remark that Warlimont (see [22]) has recently shown a result similar to part (iii) of Theorem 1 for the function $\phi(n)/\phi(\phi(n))$.

There are several results in the literature which assert that if $f(n)$ is, for example, either of the functions $\sigma(n)/n$ or $\phi(n)/n$, then the closure of the set $\{f(n)\}_{n \geq 1}$ is an interval. Our next result shows that the same is true for the function $f(n) = S(n)$.

THEOREM 2. *Let $\alpha = \liminf_n S(n)$. Then $\{S(n) \mid n \geq 1\}$ is dense in the interval $[\alpha, \infty]$.*

We now leave the function $S(n)$ and the next question we address is a conjecture of Erdős (see [8]) which asserts that the inequality

$$(6) \quad \phi(n - \phi(n)) < \phi(n)$$

holds on a set of positive integers n of asymptotic density 1 (i.e., for almost all positive integers n), but the inequality

$$(7) \quad \phi(n - \phi(n)) > \phi(n)$$

holds for infinitely many positive integers n . Some infinite families of positive integers n for which inequality (7) holds were pointed out in [16]. In that paper, it was also shown that (6) holds for a set of positive integers n of lower

density at least 0.54. In this note, we prove that (6) holds indeed for almost all positive integers n . In fact, we prove a stronger statement, namely:

THEOREM 3. (i) *Let $\varepsilon(x)$ be any positive function of the positive variable x which tends to zero when x tends to infinity. The set of integers $n > 1$ for which the inequality*

$$(8) \quad \phi(n - \phi(n)) < \phi(n) - n \cdot \varepsilon(n)$$

fails has asymptotic density 0.

(ii) *The set of positive integers n for which the inequality*

$$(9) \quad \left| \frac{\phi(n)}{n} - \frac{\phi(n - \phi(n))}{n - \phi(n)} \right| < \frac{2 \log_3(n)}{\log_2(n)}$$

fails has asymptotic density 0.

For example, (9) implies that for any fixed $\varepsilon > 0$ the set of n for which

$$\left| \frac{\phi(n)}{n} - \frac{\phi(n - \phi(n))}{n - \phi(n)} \right| < \varepsilon$$

fails has asymptotic density 0. In particular, the two functions $\phi(n)/n$ and $\phi(n - \phi(n))/(n - \phi(n))$ are asymptotically equal on a set of n of asymptotic density 1.

For $n > 1$, let $f(n) = \phi(n - \phi(n))/\phi(n)$. It can be shown, using the method of proof of Theorem 2, that the set of numbers $f(n)$ is dense in the interval $[0, \infty]$. This result shows, in particular, that if c is any positive number, then the inequality

$$(10) \quad \phi(n - \phi(n)) > c\phi(n)$$

holds for infinitely many positive integers n , which is a statement much stronger than the fact that (7) holds for infinitely many positive integers n . We do not give further details here.

2. Preliminary results. In 1928, Schoenberg (see [21]) proved that the function $\phi(n)/n$ has a distribution. That is, $D(u)$ defined as the asymptotic density of the set of n with $\phi(n)/n \geq u$ exists for every u . In addition, $D(u)$ is continuous and strictly decreasing on $[0, 1]$. Clearly, $D(0) = 1$ and $D(1) = 0$. From these considerations, we immediately derive the following:

LEMMA 1. *Let $\varepsilon : (0, \infty) \rightarrow (0, 1)$ be any function such that $\varepsilon(x)$ tends to zero as x tends to infinity. Then, for a set of n of asymptotic density 1,*

$$(11) \quad \frac{\phi(n)}{n} > \varepsilon(n).$$

Proof. This is almost obvious. Indeed, let ε be an arbitrarily small positive number. For large x we have $\varepsilon(x) < \varepsilon$. Thus, the set of positive integers n for which inequality (11) holds contains a set of n of asymptotic density

at least $D(\varepsilon)$. Since this holds for every $\varepsilon > 0$, it follows that inequality (11) holds for a set of n of asymptotic density $D(0) = 1$.

Lemma 1 may also be proved by using the average order of $\phi(n)/n$, namely,

$$(12) \quad \sum_{n \leq x} \frac{\phi(n)}{n} = \frac{6}{\pi^2}x + O(\log(x)),$$

but we shall have other uses for the distribution function $D(u)$ later.

The next result plays a key role in the proofs of both Theorems 1 and 3 and is essentially due to Erdős, Granville, Pomerance and Spiro (see [9]). However, since this result was not explicitly stated in [9], we state it below and provide a short proof.

LEMMA 2. *There exists a computable positive constant c_1 such that on a set of n of asymptotic density 1, $\phi(n)$ is divisible by all prime powers p^a with $p^a < c_1 \log_2(n)/\log_3(n)$.*

Proof. We use the notations from [9]. For any positive integer m and any positive number x we let

$$(13) \quad S(x, m) = \sum_{\substack{q \leq x \\ m|(q-1)}} \frac{1}{q}.$$

From Theorem 3.4 in [9], we know that there exist computable positive numbers c_1, x_0 such that the inequality

$$(14) \quad S(x, m) > \frac{c_1 \log_2(x)}{\phi(m)}$$

holds for all $x > x_0$ and all $m \leq \log(x)$. Let $g(x) = c_1 \log_2(x)/\log_3(x)$. From the proof of Theorem 4.1 in [9], we have, uniformly for all m ,

$$(15) \quad \sum_{\substack{n \leq x \\ m \nmid \phi(n)}} 1 < \frac{c_2 x}{\exp(S(x, m))}.$$

Assume now that p^a is any prime power with $p^a < g(x)$. By increasing x_0 if necessary, we assume that the inequality $\log(x) > g(x)$ holds for $x > x_0$. By inequalities (14) and (15), it follows that for such p^a ,

$$(16) \quad \sum_{\substack{n \leq x \\ p^a \nmid \phi(n)}} 1 < \frac{c_2 x}{\exp(S(x, p^a))} < \frac{c_2 x}{\exp(\log_3(x))} = \frac{c_2 x}{\log_2(x)}.$$

Now let $M(x)$ be the least common multiple of all the prime powers $p^a < g(x)$. Inequality (16) shows that the number of $n \leq x$ for which $\phi(n)$ is not a

multiple of $M(x)$ is at most

$$(17) \quad \sum_{p^a < g(x)} \sum_{\substack{n \leq x \\ p^a \nmid \phi(n)}} 1 < \frac{c_2 x}{\log_2(x)} \pi_0(g(x)) < \frac{2c_2 x g(x)}{\log_2(x) \log(g(x))} < \frac{c_3 x}{(\log_3(x))^2},$$

where $\pi_0(y)$ denotes the number of prime powers $p^a \leq y$ with $a \geq 1$. Inequality (17) shows that but for $O(x/(\log_3(x))^2) = o(x)$ positive integers $n \leq x$, $\phi(n)$ is a multiple of $M(x)$. Lemma 2 is therefore proved.

For the remainder of this paper, we let $g(n) = c_1 \log_2(n)/\log_3(n)$ and denote by $M(n)$ the least common multiple of all prime powers $p^a < g(n)$. Here, c_1 is the constant appearing in the statement of Lemma 2.

We shall also make use of the following result:

LEMMA 3. *On a set of positive integers n of asymptotic density 1 the following inequalities hold:*

$$(18) \quad h_1(n) := \sum_{\substack{p > g(n) \\ p|n}} \frac{1}{p} < \frac{\log_3(n)}{\log_2(n)},$$

$$(19) \quad h_2(n) := \sum_{\substack{p > g(n) \\ p|(n-\phi(n)), p \nmid n}} \frac{1}{p} < \frac{\log_3(n)}{\log_2(n)},$$

$$(20) \quad h_3(n) := \sum_{\substack{p > \log_2(n) \\ p|\phi(n)}} \frac{1}{p} < \frac{\log_4(n)}{\log_3(n)}.$$

Proof. The fact that inequality (18) holds for a set of n of asymptotic density 1 follows from an averaging argument. Indeed, if T is any slowly increasing function of x (like $T(x) = g(x)$, for example), then

$$(21) \quad \sum_{1 \leq n \leq x} \sum_{\substack{T(n) < p \\ p|n}} \frac{1}{p} = O\left(\frac{x}{T(x) \log(T(x))}\right)$$

(see also [9], page 199). Taking $T(x) = g(x)$ in formula (21), we get

$$\sum_{n \leq x} h_1(n) \leq \frac{c_4 x}{\log_2(x)}.$$

In particular, but for a set of $n \leq x$ of cardinality $O(x/\log_3(x)) = o(x)$, inequality (18) holds.

We now turn to the second part of the lemma. For $n \leq x$,

$$(22) \quad h_2(n) \leq \sum_{p \leq x} \frac{1}{p} \leq \log_2(x) + c_5,$$

though it is not hard to get a sharper inequality here. Let

$$\eta = \eta(x) = \frac{\log_4(x)}{3 \log_3(x)}.$$

For any positive integer $n > 1$, we let $P(n)$ be the largest prime divisor of n . By de Bruijn [4], the number of $n \leq x$ with $P(n) \leq x^\eta$ is $x/(\log_2(x))^{3+o(1)}$, and so is at most $x/(\log_2(x))^{2.5}$ for all sufficiently large values of x . It is easy to see that the number of $n \leq x$ with $P(n) > x^\eta$ and $P(n)^2 \nmid n$ is at most $x^{1-\eta}$. Let A be the set of numbers n with

$$(23) \quad x^{1/2} < n \leq x, \quad P(n) > x^\eta, \quad P(n)^2 \nmid n.$$

It follows from (22) and the above estimates that

$$(24) \quad \sum_{n \leq x, n \notin A} h_2(n) \leq \frac{x}{\log_2(x)}$$

for all sufficiently large values of x .

Next, for $n \in A$, let $h_2(n) = h_{2,0}(n) + h_{2,1}(n)$, where

$$(25) \quad h_{2,0}(n) = \sum_{\substack{x^{\eta/2} \geq p > g(n) \\ p|(n-\phi(n)), p \nmid n}} \frac{1}{p}, \quad h_{2,1}(n) = \sum_{\substack{p > x^{\eta/2} \\ p|(n-\phi(n)), p \nmid n}} \frac{1}{p}.$$

For $m \leq x$, the number of primes $p \mid m$ with $p > x^{\eta/2}$ is at most $2/\eta$. Thus, for $n \in A$, $h_{2,1}(n) \leq (2/\eta)x^{-\eta/2}$, and so

$$(26) \quad \sum_{n \in A} h_{2,1}(n) \leq \frac{x}{\log(x)}$$

for x sufficiently large.

For an estimation of $h_{2,0}(n)$ we again use an averaging argument. We have

$$(27) \quad \sum_{n \in A} h_{2,0}(n) \leq \sum_{g(x^{1/2}) < p \leq x^{\eta/2}} \frac{1}{p} \sum_{\substack{n \not\equiv 0 \pmod{p} \\ n \equiv \phi(n) \pmod{p} \\ n \in A}} 1.$$

We now fix a prime number p in the interval $(g(x^{1/2}), x^{\eta/2}]$ and we find an upper bound for the inner sum appearing in (27). Assume that $n \in A$ is such that $p \mid (n - \phi(n))$. Write $n = Pm$ where $P = P(n)$. Then $n - \phi(n) = Pm - (P - 1)\phi(m) = P(m - \phi(m)) + \phi(m)$. Thus,

$$(28) \quad P(m - \phi(m)) \equiv -\phi(m) \pmod{p}.$$

Notice that p does not divide $m - \phi(m)$. Indeed, if $p \mid (m - \phi(m))$, then congruence (28) implies that $p \mid \phi(m)$, therefore $p \mid m$, which contradicts the fact that $p \nmid n$. Let a_m be the integer in the interval $[0, p - 1]$ given by $a_m \equiv -\phi(m)(m - \phi(m))^{-1} \pmod{p}$. Congruence (28) implies that $P \equiv a_m$

(mod p). From (23) we deduce that $m \leq x^{1-\eta}$. Thus, summing up first over all the possible values of $P(n)$ when m is fixed, we get

$$(29) \quad \sum_{\substack{n \not\equiv 0 \pmod{p} \\ n \equiv \phi(n) \pmod{p} \\ n \in A}} 1 \leq \sum_{1 \leq m \leq x^{1-\eta}} \sum_{\substack{1 \leq P \leq x/m \\ P \equiv a_m \pmod{p}}} 1 = \sum_{1 \leq m \leq x^{1-\eta}} \pi\left(\frac{x}{m}, a_m, p\right).$$

By a well known result of Montgomery and Vaughan (see [19]),

$$(30) \quad \pi\left(\frac{x}{m}, a_m, p\right) \leq \frac{2x}{m(p-1) \log(x/(mp))}.$$

Since $p \leq x^{\eta/2}$, we have

$$(31) \quad \sum_{1 \leq m \leq x^{1-\eta}} \frac{2x}{m(p-1) \log(x/(mp))} \leq \frac{2x}{(p-1)(\eta/2) \log(x)} \sum_{1 \leq m \leq x^{1-\eta}} \frac{1}{m} < \frac{5x}{\eta p}$$

for x sufficiently large. Hence (29)–(31) imply that

$$\sum_{\substack{n \not\equiv 0 \pmod{p} \\ n \equiv \phi(n) \pmod{p} \\ n \in A}} 1 < \frac{5x}{\eta p},$$

so that from (27), we have

$$(32) \quad \sum_{n \in A} h_{2,0}(n) \leq \sum_{p > g(x^{1/2})} \frac{5x}{p^2 \eta} \leq \frac{6x}{\eta g(x^{1/2}) \log(g(x^{1/2}))} \leq \frac{20x \log_3(x)}{c_1 \log_2(x) \log_4(x)}$$

for all sufficiently large values of x . With (24) and (26), (32) implies that

$$\sum_{n \leq x} h_2(n) \leq \frac{c_6 x \log_3(x)}{\log_2(x) \log_4(x)}.$$

In particular, but for $O(x/\log_4(x)) = o(x)$ values of $n \leq x$, inequality (19) holds.

The third part of Lemma 3 follows immediately from the inequality

$$\sum_{\substack{n \leq x \\ p | \phi(n)}} 1 = O\left(\frac{x \log_2 x}{p}\right)$$

(uniformly for every prime p and x larger than some x_1 that is independent of p ; see Theorem 3.5 in [9]). It follows that the average of $h_3(n)$ for $n \leq x$ is $O(1/\log_3(x))$, and thus, we get (20) for a set of asymptotic density 1.

Lemma 3 is therefore proved.

3. Proofs of the theorems

Proof of Theorem 1. (i) We first observe that since

$$\frac{c_7 n}{\log_2(n)} < \phi(n) \leq n$$

for all integers n (see [17], Theorem 328), it follows that

$$\lim_n \frac{\log_2(\phi(n))}{\log_2(n)} = 1.$$

Thus,

$$(33) \quad \limsup_n \frac{\sigma(\phi(n))}{n \log_2(n)} \leq \limsup_n \frac{\sigma(\phi(n))}{\phi(n) \log_2(\phi(n))} \leq \limsup_n \frac{\sigma(n)}{n \log_2(n)} = e^\gamma$$

(see Theorem 323 in [17]). For the reverse inequality, let n_k be a sequence of integers which attains the last lim sup, that is,

$$(34) \quad \lim_k \frac{\sigma(n_k)}{n_k \log_2(n_k)} = e^\gamma.$$

Let p_k be the least prime with $p_k \equiv 1 \pmod{n_k}$. By Linnik's theorem we have $p_k < n_k^{c_8}$, so that $\log_2(p_k) \sim \log_2(n_k)$. Thus,

$$\frac{S(p_k)}{\log_2(p_k)} = \frac{\sigma(p_k - 1)}{p_k \log_2(p_k)} \sim \frac{\sigma(p_k - 1)}{(p_k - 1) \log_2(n_k)} \geq \frac{\sigma(n_k)}{n_k \log_2(n_k)}.$$

With (33) and (34), we thus have part (i) of Theorem 1.

(ii) Here we use the notations from the proof of Lemma 2. By Lemma 2, for a set of positive integers n of asymptotic density 1, $\phi(n)$ is a multiple of $M(n)$. For each prime $p < g(n)$, let p^{a_p} be the power of p in the prime factorization of $M(n)$. For such n , we have the inequality

$$(35) \quad \frac{\sigma(\phi(n))}{\phi(n)} \geq \prod_{p < g(n)} \left(1 + \frac{1}{p} + \dots + \frac{1}{p^{a_p}} \right) \sim \prod_{p < g(n)} \frac{p}{p-1} \sim e^\gamma \log_3(n)$$

as $n \rightarrow \infty$. In addition, we have

$$(36) \quad \frac{\sigma(\phi(n))}{\phi(n)} < \prod_{p|\phi(n)} \frac{p}{p-1} \leq \left(\prod_{p \leq \log_2(n)} \frac{p}{p-1} \right) \left(\prod_{\substack{p|\phi(n) \\ p > \log_2(n)}} \frac{p}{p-1} \right) \\ \sim e^\gamma \log_3(n) \cdot \exp \left(\sum_{\substack{p|\phi(n) \\ p > \log_2(n)}} \frac{1}{p} \right).$$

We now use (20). On a set of asymptotic density 1, the argument of exp in (36) is $o(1)$, so that with (35), which also holds on a set of asymptotic

density 1, we have

$$\frac{\sigma(\phi(n))}{\phi(n)} \sim e^\gamma \log_3(n).$$

To see that (ii) of Theorem 1 holds, we write

$$S(n) = \frac{\sigma(\phi(n))}{\phi(n)} \cdot \frac{\phi(n)}{n}.$$

The density function referred to in (ii) of Theorem 1 is then exactly the function $D(u)$ in Schoenberg's theorem for $\phi(n)/n$.

(iii) We first show that the mean value of $S(n)$ is at least what is claimed. Let c_1 be the constant appearing in Lemma 2, and for large x let $A_0(x)$ be the set of those n with $\sqrt{x} \leq n < x$ such that $\phi(n)$ is a multiple of $M(n)$. By the arguments from the proof of Lemma 2, we know that the cardinality of the complement of $A_0(x)$ in the interval $[1, x]$ is at most $O(x/(\log_3(x))^2)$. For $n \in A_0(x)$ and a_p as in (35), we have

$$\begin{aligned} (37) \quad \frac{\sigma(\phi(n))}{\phi(n)} &\geq \frac{\sigma(M(n))}{M(n)} = \prod_{p < g(n)} \left(1 + \frac{1}{p} + \dots + \frac{1}{p^{a_p}}\right) \\ &= \prod_{p < g(n)} \frac{p}{p-1} \left(1 - \frac{1}{p^{a_p+1}}\right) \\ &\geq \left(\prod_{p < g(n)} \frac{p}{p-1}\right) \left(1 - \frac{1}{g(n)}\right)^{\pi(g(n))} \\ &= (e^\gamma \log(g(n)) + O(1)) \left(1 + O\left(\frac{1}{\log(g(n))}\right)\right) \\ &\geq e^\gamma \log_3(x) - c_9. \end{aligned}$$

Thus,

$$\begin{aligned} (38) \quad \frac{1}{x} \sum_{1 \leq n \leq x} S(n) &\geq \frac{1}{x} \sum_{n \in A_0(x)} \frac{\sigma(\phi(n))}{\phi(n)} \cdot \frac{\phi(n)}{n} \\ &\geq (e^\gamma \log_3(x) - c_9) \cdot \frac{1}{x} \sum_{n \in A_0(x)} \frac{\phi(n)}{n}. \end{aligned}$$

Using estimate (12) and the fact that

$$x - |A_0(x)| = O\left(\frac{x}{(\log_3(x))^2}\right),$$

we get

$$(39) \quad \frac{1}{x} \sum_{n \in A_0(x)} \frac{\phi(n)}{n} > \frac{6}{\pi^2} - \frac{c_{10}}{(\log_3(x))^2}.$$

Combining (38) with (39), we further get

$$(40) \quad \frac{1}{x} \sum_{1 \leq n < x} S(n) > \frac{6e^\gamma}{\pi^2} \log_3(x) - c_{11},$$

which is a lower bound better than the one asserted.

It remains to get an upper bound for the mean value of $S(n)$. Let x be a large real number. We split the positive integers in the interval $[1, x]$ into the following four subsets:

$$(41) \quad A(x) = \{n \mid 1 \leq n < \sqrt{x}\},$$

$$(42) \quad B(x) = \left\{ n \mid \sqrt{x} \leq n \leq x, h_3(n) < \frac{1}{\sqrt{\log_3(x)}} \right\},$$

$$(43) \quad C(x) = \left\{ n \mid \sqrt{x} \leq n \leq x, h_3(n) \geq \frac{1}{\sqrt{\log_3(x)}} \text{ and } \omega(\phi(n)) < (\log_2(x))^3 \right\},$$

and finally let $D(x)$ be the complement of $A(x) \cup B(x) \cup C(x)$ in the interval $[1, x]$. Here, $h_3(n)$ is as in Lemma 3 and $\omega(n)$ denotes the number of different primes that divide n .

We first comment on the sizes of the cardinalities of the four sets defined above. By the proof of Lemma 3, the cardinality of $C(x)$ is at most $O(x/\sqrt{\log_3(x)})$. Clearly, the cardinality of $A(x)$ is exactly $\lfloor \sqrt{x} \rfloor$. We now show that $D(x)$ is small as well. More precisely, we show that the cardinality of $D(x)$ is at most $O(x/\log_2(x))$. To see why this is so, notice that

$$(44) \quad \omega(\phi(n)) \leq \Omega(\phi(n)) \leq \Omega(n) + \sum_{p|n} \Omega(p-1),$$

where $\Omega(m)$ is the number of prime factors of m , counted with multiplicity. Thus,

$$(45) \quad \sum_{1 \leq n \leq x} \omega(\phi(n)) \leq \sum_{1 \leq n \leq x} \Omega(n) + \sum_{1 \leq n \leq x} \sum_{p|n} \Omega(p-1) = \sum_1 + \sum_2.$$

It is well known that

$$(46) \quad \sum_1 = \sum_{1 \leq n \leq x} \Omega(n) \sim x \log_2(x).$$

For the second sum appearing in formula (45), we interchange the order of summation to get

$$(47) \quad \sum_2 = \sum_{1 \leq p \leq x} \Omega(p-1) \cdot \left\lfloor \frac{x}{p} \right\rfloor < x \sum_{1 \leq p \leq x} \frac{\Omega(p-1)}{p} \sim \frac{1}{2} x (\log_2(x))^2.$$

The rightmost approximation appearing in formula (47) is a result due to Erdős and Pomerance (see Lemma 2.3 in [10]). Formulae (46) and (47) imply that the cardinality of $D(x)$ is indeed at most $O(x/\log_2(x))$.

The above considerations show that $B(x)$ contains all integers up to x , except for $O(x/\sqrt{\log_3(x)})$ of them.

If E is any one of the letters A, B, C, D , we define

$$(48) \quad S_E = \frac{1}{x} \sum_{n \in E(x)} S(n).$$

It suffices to bound each one of the numbers S_E for $E \in \{A, B, C, D\}$. Notice that since

$$(49) \quad \frac{\sigma(\phi(n))}{n} < c_{12} \log_2(x)$$

for all $n \leq x$ (see (34)), it follows that

$$(50) \quad S_A < c_{12} \frac{|A(x)|}{x} \log_2(x) = O\left(\frac{\log_2(x)}{\sqrt{x}}\right) = o(1),$$

and similarly

$$(51) \quad S_D < c_{12} \frac{|D(x)|}{x} \log_2(x) = O(1).$$

For the remaining numbers, write

$$(52) \quad \phi(n) = n_1 \cdot n_2,$$

where

$$(53) \quad n_1 = \prod_{\substack{p^a \parallel \phi(n) \\ p \leq \log_2(n)}} p^a,$$

$$(54) \quad n_2 = \prod_{\substack{p^a \parallel \phi(n) \\ p > \log_2(n)}} p^a.$$

Clearly,

$$(55) \quad \frac{\sigma(\phi(n))}{\phi(n)} = \frac{\sigma(n_1)}{n_1} \cdot \frac{\sigma(n_2)}{n_2} < \frac{n_1}{\phi(n_1)} \cdot \frac{n_2}{\phi(n_2)}.$$

Notice that

$$(56) \quad \frac{n_1}{\phi(n_1)} \leq \prod_{p < \log_2(x)} \left(1 - \frac{1}{p}\right)^{-1} < e^\gamma \log_3(x) + c_{13}$$

for all $n \leq x$, and

$$(57) \quad \frac{n_2}{\phi(n_2)} \leq \exp(c_{14} h_3(n))$$

for all $n > \sqrt{x}$, where c_{14} can be taken to be any constant larger than 1 provided that x is large enough. By combining (55)–(57), it follows that if $n \in B(x)$, then

$$(58) \quad \frac{\sigma(\phi(n))}{\phi(n)} < (e^\gamma \log_3(x) + c_{13}) \exp\left(\frac{c_{14}}{\sqrt{\log_3(x)}}\right) < e^\gamma \log_3(x) + c_{15} \sqrt{\log_3(x)}.$$

Hence,

$$(59) \quad S_B < (e^\gamma \log_3(x) + c_{15} \sqrt{\log_3(x)}) \cdot \frac{1}{x} \sum_{1 \leq n \leq x} \frac{\phi(n)}{n} < \frac{6e^\gamma}{\pi^2} \log_3(x) + c_{16} \sqrt{\log_3(x)},$$

where the last inequality follows from (12).

Finally, when $n \in C(x)$, let $t(x) = \lfloor (\log_2(x))^3 \rfloor$. For a given $n \in C(x)$, let $p_1 < \dots < p_{t(x)}$ be the first $t(x)$ primes with p_1 larger than $\log_2(n)$. Then obviously

$$(60) \quad h_3(n) \leq \sum_{j=1}^{t(x)} \frac{1}{p_j} < \sum_{c_{17} \log_2(x) < p < c_{18} (\log_2(x))^3 \log_3(x)} \frac{1}{p} < c_{19}.$$

Hence, by (57) and (60), we have

$$(61) \quad \frac{n_2}{\phi(n_2)} < c_{20}$$

when $n \in C(x)$. Combining (55) with (56) and (61), we get

$$(62) \quad \frac{\sigma(\phi(n))}{\phi(n)} < c_{21} \log_3(x)$$

whenever $n \in C(x)$. Thus,

$$(63) \quad S_C < c_{21} \frac{|C(x)|}{x} \log_3(x) < c_{22} \sqrt{\log_3(x)},$$

where the last inequality in (63) follows from the fact that $|C(x)| = O(x/\sqrt{\log_3(x)})$. The assertion (iii) follows now by combining inequalities (50), (51), (59) and (63).

This concludes the proof of Theorem 1.

Proof of Theorem 2. By inequality (2), we know that $0 < \alpha \leq 1/2 + 1/(2^{34} - 1)$. We shall make use of the following theorem due to Chen (see [5], or Chapter 11 of [14], or Lemma 1.2 in [13]):

CHEN'S THEOREM. *For each even natural number m and $x \geq x_0(m)$ there exists a prime number $s \in (x/2, x]$ with $s \equiv 1 \pmod{m}$ such that $(s - 1)/m$ has at most two prime factors each of which exceeds $x^{1/10}$.*

We shall distinguish two cases:

CASE 1: *There exists a sequence of integers $(m_n)_{n \geq 1}$ such that $\lim_n S(m_n) = \alpha$ and $\lim_n \text{ord}_2(\phi(m_n)) = \infty$. Fix any real number $\beta \geq 1$. Let ε be any small positive number, and let m be a positive integer such that*

$$(64) \quad |S(m) - \alpha| < \varepsilon \quad \text{and} \quad \text{ord}_2(\phi(m)) > 1/\varepsilon.$$

After fixing such an m , let t be an odd number such that

$$(65) \quad t > 1/\varepsilon, \quad |\sigma(t)/t - \beta| < \varepsilon, \quad \text{gcd}(t, \phi(m)) = 1.$$

It is easy to see that one can find such a number t . We now use Chen's theorem to find a prime number $s > m$ such that

$$(66) \quad s - 1 = 2tl$$

where l has at most two prime factors larger than $\max(\phi(m), t, 1/\varepsilon)$ each. Write

$$(67) \quad \phi(m) = 2^\gamma m_1, \quad m_1 \equiv 1 \pmod{2}.$$

Then

$$S(m) = \frac{\sigma(\phi(m))}{m} = \frac{\sigma(2^\gamma m_1)}{m} = \frac{(2^{\gamma+1} - 1)\sigma(m_1)}{m}.$$

Notice that m and s are coprime, t and l are also coprime (because the smallest prime factor of l is larger than t), and tl is coprime to $\phi(m)$; therefore

$$(68) \quad S(ms) = \frac{\sigma(\phi(ms))}{ms} = \frac{\sigma(2^{\gamma+1} m_1 tl)}{m(2tl + 1)} = \frac{(2^{\gamma+2} - 1)\sigma(m_1)\sigma(t)\sigma(l)}{m(2tl + 1)}.$$

Thus,

$$(69) \quad \frac{S(ms)}{S(m)} = \frac{2^{\gamma+2} - 1}{2(2^{\gamma+1} - 1)} \cdot \frac{\sigma(t)}{t} \cdot \frac{\sigma(l)}{l} \cdot \frac{1}{1 + (2tl)^{-1}}.$$

Formula (69) together with the previous assumptions on the parameters γ, t and l implies that

$$(70) \quad \left| \frac{S(ms)}{S(m)} - \beta \right| < c_{23}\varepsilon,$$

where c_{23} depends only on β . Inequalities (64) and (70) imply that

$$(71) \quad |S(ms) - \alpha\beta| < c_{24}\varepsilon,$$

where c_{24} depends only on β (because α is absolute). Since ε can be taken to be arbitrarily small after β was fixed, it follows that $\alpha\beta$ is a cluster point of $\{S(n) \mid n \geq 1\}$, which settles this case.

CASE 2: There exist two constants K_1 and K_2 such that $|S(n) - \alpha| < K_1$ implies $\text{ord}_2(\phi(n)) < K_2$. Notice first of all that since

$$(72) \quad S(p^2m) = \frac{\sigma(\phi(p^2m))}{p^2m} = \frac{\sigma(p\phi(pm))}{p^2m} > \frac{\sigma(\phi(pm))}{pm} = S(pm),$$

it follows that there exists a sequence $(m_n)_{n \geq 1}$ of squarefree integers such that $S(m_n)$ converges to α . Moreover, in this case we know that $\text{ord}_2(\phi(m_n)) < K_2$ when n is large enough. In particular, $\omega(m_n) < K_2 + 1$ when n is large enough.

Define k to be the smallest positive integer such that there exists a sequence $(m_n)_{n \geq 1}$ of squarefree numbers for which $S(m_n)$ converges to α and such that $\omega(m_n) = k$ for all $n \geq 1$. For each n let

$$(73) \quad m_n = p_1(n) \cdots p_k(n),$$

where $p_1(n) < \cdots < p_k(n)$. Write $m_n = p_k(n)m'_n$. Notice first of all that $k > 1$. Indeed, if $k = 1$, it follows that α is a cluster point of $\{S(p) \mid p \text{ prime}\}$. However, for $p \geq 7$,

$$S(p) = \frac{\sigma(p-1)}{p} \geq \frac{1}{p} \left(1 + 2 + \frac{p-1}{2} + (p-1) \right) = \frac{3(p+1)}{2p} \geq \frac{3}{2},$$

which would imply that $\alpha \geq 3/2$, contradicting the result of Małkowski and Schinzel (see (2)). Thus, $k > 1$. Since $p_k(n) = P(m_n)$ tends to infinity with n , we get

$$(74) \quad \begin{aligned} \alpha &= \lim_n S(m_n) = \lim_n \frac{\sigma(\phi(m'_n)(p_k(n) - 1))}{m'_n p_k(n)} \\ &\geq \liminf_n \frac{p_k(n) - 1}{p_k(n)} S(m'_n) = \liminf_n S(m'_n). \end{aligned}$$

From the definition of k , it follows that $(m'_n)_{n \geq 1}$ cannot have infinitely many distinct terms and is, therefore, bounded. These arguments show that one may assume that $m_n = ap_k(n)$ where $a > 1$ is a fixed integer. Notice that $a > 2$. Indeed, if $a = 2$, then, for $p \geq 7$, one has

$$S(2p) = \frac{\sigma(p-1)}{2p} \geq \frac{1}{2p} \left(1 + 2 + \frac{p-1}{2} + p - 1 \right) = \frac{3(p+1)}{4p} > \frac{3}{4},$$

therefore $\alpha \geq 3/4$, contradicting again the result of Małkowski and Schinzel. Thus, $a > 2$, therefore $\phi(a) > 1$. We next show that

$$(75) \quad \alpha = \min \left(\frac{\sigma(\phi(a)d)}{ad} \mid d \text{ an even squarefree divisor of } \phi(a) \right).$$

To see why α is at least what is claimed by formula (75), let p be a large prime and write

$$p - 1 = d_p n_p,$$

where $\gcd(d_p, n_p) = 1$ and d_p is a number whose prime factors are exactly the prime factors of $\gcd(p - 1, \phi(a))$. Notice that d_p is even. Then

$$(76) \quad S(ap) = \frac{\sigma(\phi(a)d_p n_p)}{a(d_p n_p + 1)} = \frac{\sigma(\phi(a)d_p)}{ad_p} \cdot \frac{\sigma(n_p)}{n_p} \cdot \frac{1}{1 + (p - 1)^{-1}}.$$

Since $\sigma(n_p) \geq n_p$, the above argument shows that

$$(77) \quad \alpha \geq \liminf'_m \frac{\sigma(\phi(a)m)}{am},$$

where the \liminf' means that we are allowing m to run only over those even positive integers whose prime divisors are among the prime divisors of $\phi(a)$. If d is the largest squarefree divisor of m , then

$$\frac{\sigma(\phi(a)m)}{am} \geq \frac{\sigma(\phi(a)d)}{ad},$$

so α is at least as large as the number appearing on the right hand side of (75). To see that α is at most that number, choose d to be any even squarefree divisor of $\phi(a)$ and use Chen's theorem to construct large primes s such that $l = (s - 1)/d$ is an integer composed of at most two primes, each of them large. Now

$$(78) \quad S(as) = \frac{\sigma(\phi(a)dl)}{a(dl + 1)} = \frac{\sigma(\phi(a)d)}{ad} \cdot \frac{\sigma(l)}{l} \cdot \frac{1}{1 + (s - 1)^{-1}},$$

and notice that the right hand side of (78) tends to $\sigma(\phi(a)d)/(ad)$ when s tends to infinity through such numbers. This proves formula (75).

Finally, having (75) at hand one can again use Chen's theorem to prove our Theorem 2. Indeed, assume that d is an even squarefree divisor of $\phi(a)$ realizing the minimum of the expression appearing on the right hand side of (75). Let β be an arbitrary real number ≥ 1 . Fix $\varepsilon > 0$ arbitrarily small and choose a number t coprime to $\phi(a)$ (in particular, to $\phi(a)d$) such that both

$$(79) \quad t > 1/\varepsilon \quad \text{and} \quad |\sigma(t)/t - \beta| < \varepsilon.$$

Now use Chen's theorem to construct a prime number s such that $l = (s - 1)/(td)$ is an integer composed of at most two primes larger than $\max(t\phi(a), 1/\varepsilon)$ each. Now t and l are coprime and tl is coprime to $\phi(a)d$, therefore

$$(80) \quad S(sa) = \frac{\sigma(\phi(a)tdl)}{a(tdl + 1)} = \alpha \cdot \frac{\sigma(t)}{t} \cdot \frac{\sigma(l)}{l} \cdot \frac{1}{1 + (s - 1)^{-1}}.$$

It is easily seen that formulae (79) and (80) together with our assumptions on s and l imply that

$$(81) \quad |S(sa) - \alpha\beta| < c_{25}\varepsilon,$$

where c_{25} depends only on β . Since β was first fixed and then ε was chosen arbitrarily small, it follows that $\alpha\beta$ is a cluster point of $\{S(n) \mid n \geq 1\}$. Finally, since $\beta \geq 1$ was arbitrary, it follows that $\{S(n) \mid n \geq 1\}$ is dense in $[\alpha, \infty]$.

REMARK. Instead of using Chen's theorem, we could have used a weaker fact, namely that for every even m there exist infinitely many primes p such that $(p-1)/m$ is an integer whose smallest prime factor is at least $\log p$. This statement follows from simpler sieve methods such as Brun's or Selberg's. Of course, when p goes to infinity through such primes, $S(p)$ approaches $\sigma(m)/m$, which is enough for the arguments employed in the proof of our Theorem 2 to go through.

Proof of Theorem 3. We use the notations from the proofs of Lemmas 2 and 3. Let A be the set of all integers n which are not primes and for which $M(n) \mid \phi(n)$ and inequalities (18) and (19) hold. By Lemmas 2 and 3, A has asymptotic density 1. We now show that inequality (9) holds for all values of $n \in A$ which are large enough. We start by finding suitable upper and lower bounds on $\phi(n - \phi(n))$. Notice that

$$\begin{aligned} (82) \quad \phi(n - \phi(n)) &= \sum_{\substack{m < n - \phi(n) \\ (m, n - \phi(n)) = 1}} 1 \\ &\leq \sum_{\substack{m < n - \phi(n) \\ (m, n) = 1}} 1 + \sum_{\substack{m < n - \phi(n) \\ (m, n - \phi(n)) = 1, (m, n) > 1}} 1 \\ &= \sum_1 + \sum_2. \end{aligned}$$

Clearly,

$$\begin{aligned} (83) \quad \sum_1 &= (n - \phi(n)) - \sum_{p|n} \left\lfloor \frac{n - \phi(n)}{p} \right\rfloor + \sum_{pq|n, p < q} \left\lfloor \frac{n - \phi(n)}{pq} \right\rfloor - \dots \\ &< (n - \phi(n)) \frac{\phi(n)}{n} + \tau(n). \end{aligned}$$

Here, $\tau(n)$ is the number-of-divisors function of n . For the second sum, we use the fact that $M(n) \mid \phi(n)$. Hence, if m is a number such that $(m, n) > 1$, but $(m, n - \phi(n)) = 1$, it follows that m has to be a multiple of a prime $p \mid n$ with $p > g(n)$. Of course, for such n , one has

$$\begin{aligned} (84) \quad \sum_2 &< \sum_{\substack{p|n \\ p > g(n)}} \left\lfloor \frac{n - \phi(n)}{p} \right\rfloor < (n - \phi(n)) \sum_{\substack{p|n \\ p > g(n)}} \frac{1}{p} \\ &= (n - \phi(n)) h_1(n) < \frac{(n - \phi(n)) \log_3(n)}{\log_2(n)}. \end{aligned}$$

Putting together inequalities (82)–(84), we get

$$(85) \quad \frac{\phi(n - \phi(n))}{n - \phi(n)} - \frac{\phi(n)}{n} < \frac{\tau(n)}{n - \phi(n)} + \frac{\log_3(n)}{\log_2(n)} < \frac{2 \log_3(n)}{\log_2(n)}.$$

For the rightmost inequality in (85) we used the fact that $n - \phi(n) \geq n^{1/2}$, which holds because n is not prime, and the fact that $\tau(n) < n^{1/4}$, which holds for all n large enough, thus

$$\frac{\tau(n)}{n - \phi(n)} < \frac{1}{n^{1/4}} < \frac{\log_3(n)}{\log_2(n)}$$

for $n \in A$ large enough. Inequality (85) proves half of (9). For the other half, we write

$$(86) \quad \begin{aligned} \phi(n - \phi(n)) &= \sum_{\substack{m < n - \phi(n) \\ (m, n - \phi(n)) = 1}} 1 \\ &\geq \sum_{\substack{m < n - \phi(n) \\ (m, n) = 1}} 1 - \sum_{\substack{m < n - \phi(n) \\ (m, n - \phi(n)) > 1, (m, n) = 1}} 1 \\ &= \sum_1 - \sum_3. \end{aligned}$$

From elementary arguments similar to the ones employed above, we get

$$(87) \quad \sum_1 > (n - \phi(n)) \cdot \frac{\phi(n)}{n} - \tau(n)$$

and

$$(88) \quad \begin{aligned} \sum_3 &< (n - \phi(n)) \sum_{\substack{p > g(n) \\ p | (n - \phi(n)), p \nmid n}} \frac{1}{p} \\ &= (n - \phi(n)) h_2(n) < \frac{(n - \phi(n)) \log_3(n)}{\log_2(n)}. \end{aligned}$$

Putting together inequalities (86)–(88), we get

$$\frac{\phi(n - \phi(n))}{n - \phi(n)} - \frac{\phi(n)}{n} > - \left(\frac{\tau(n)}{n - \phi(n)} + \frac{\log_3(n)}{\log_2(n)} \right) > - \frac{2 \log_3(n)}{\log_2(n)}$$

for $n \in A$ large enough. With (85) we thus have (9).

To see (8), say $\varepsilon(n)$ tends to 0 arbitrarily slowly as $n \rightarrow \infty$, and let

$$\delta(n) = (\varepsilon(n) + 2 \log_3(n) / \log_2(n))^{1/2}.$$

Thus, $\delta(n) \rightarrow 0$ as $n \rightarrow \infty$. By Lemma 1, we may assume that $\phi(n) > \delta(n)n$, and by (9), which we have just proved, we may assume that

$$(89) \quad \phi(n - \phi(n)) < \frac{\phi(n)}{n} (n - \phi(n)) + \frac{2 \log_3(n)}{\log_2(n)} (n - \phi(n)).$$

But for $\phi(n) > \delta(n)n$, the right side of (89) is less than $\phi(n) - \varepsilon(n)n$, which gives (8).

This completes the proof of Theorem 3.

4. Comments and problems. Let $u(n) = n - \phi(n)$. In many respects, the function $u(n)$ resembles the sum of aliquot divisors of n , namely, the function $s(n) = \sigma(n) - n$. It seems interesting to iterate the function u at a starting value of n until 0 is reached. Let u_k be the k th iterate. Let $k(n)$ be the first positive integer k for which $u_k(n) = 0$. It is easy to see that

$$(90) \quad k(n) < c_{26} \log(n) \log_2(n)$$

for all n large enough, where c_{26} can be taken to be any constant strictly larger than e^γ , but we suspect that $k(n) = o(\log(n) \log_2(n))$. It would be interesting to investigate both the average and the normal behavior of the function $k(n)$.

Note also that $u_2(n)$ is defined for all $n > 1$ and that $u_2(n) = 0$ if and only if n is prime, so that $u_2(n)$ is defined and $u_2(n) > 0$ for a set of asymptotic density 1. Note, too, that (ii) of Theorem 3 says that $n/u(n)$ is asymptotically equal to $u(n)/u_2(n)$ on a set of asymptotic density 1. Our Theorem 3 is analogous to some results from [7] and [9], where it was shown that $s(n)/n$ is asymptotically equal to $s_2(n)/s(n)$ on a set of n of asymptotic density 1. In analogy with Conjecture 3 in [9] which deals with higher iterates of the function $s(n)$, we conjecture that for each fixed number k , $u_k(n)$ is defined, $u_k(n) > 0$, and $n/u(n)$ is asymptotically equal to $u_{k-1}(n)/u_k(n)$ on a set of asymptotic density 1. That at least $u_k(n)$ is defined, $u_k(n) > 0$, and

$$(91) \quad \frac{u_{k-1}(n)}{u_k(n)} < (1 + \varepsilon) \frac{n}{u(n)}$$

on a set of asymptotic density 1, for any fixed $\varepsilon > 0$, follows by the same methods as in [7].

While as we said above, we know by (ii) of Theorem 3 that $n/u(n)$ and $u_1(n)/u_2(n)$ are asymptotically equal on a set of asymptotic density 1, what is still in doubt is which one is larger, and the same question can also be formulated for the pair of asymptotically equal functions $n/s(n)$ and $s_1(n)/s_2(n)$. Computations revealed that 550177 numbers n smaller than 10^6 satisfy

$$(92) \quad nu_2(n) < u_1(n)^2$$

and 608799 numbers n smaller than 10^6 satisfy

$$(93) \quad ns_2(n) < s_1(n)^2.$$

Of these, 371154 numbers satisfy both (92) and (93). Notice that when

n is prime, both (92) and (93) hold. Of course, the set of all primes is of asymptotic density 0 but this is less noticeable computationally in small ranges. In particular, (92) holds for 471679 of the 921500 composite numbers n smaller than 10^6 , or about 51%, while (93) holds for 530301 composites, or about 58%. Both inequalities hold for 292656 composites, or about 32%. Based on our computations, it may be reasonable to conjecture that both inequalities (92) and (93) hold on a set of asymptotic density $1/2$ and that they are independent, that is, that they both hold on a set of asymptotic density $1/4$.

Recall that an integer n is called a *cototient* if it is in the range of the function u , that is, $n = m - \phi(m)$ for some integer m . It is not known if the set of cototients has an asymptotic density nor if the upper density of the set is < 1 . In fact, until a few years ago it was not even known that there are infinitely many non-cototients, until an infinite family of such was pointed out by Browkin and Schinzel in [3] (see also [12] for more examples of such infinite families of non-cototients). In analogy with the notion of a cototient, let us call a positive integer n a *strong cototient* if the equation $u_k(x) = n$ has a positive solution x for every $k \geq 1$. Does the set of strong cototients have a density and if so, what is it? Clearly, since $u_k(p^{m+k}) = p^m$ holds for all $m \geq 0$, $k \geq 1$ and $p \geq 2$ prime, it follows that all prime powers are strong cototients. Moreover, by looking at the values of $u(pq)$ with p and q odd primes, Goldbach's conjecture would imply that all odd integers are cototients, therefore strong cototients.

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