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ON A TILING PROBLEM OF R.B. EGGLETON

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A tiling of the plane with polygonal tiles is said to be strict if any point common to two tiles is a vertex of both or a vertex of neither. A triangle is said to be rational if its sides have rational length. Recently R.B. Eggleton asked if it is possible to strictly tile the plane with rational triangles using precisely one triangle from each congruence class. In this paper we constructively prove the existence of such a tiling by a suitable modification of the technique suggested by Eggleton. The theory of rational points on elliptic curves, in particular, the Nagell–Lutz theorem, plays a crucial role in completing the proof.

1. Introduction

A tiling of the plane with polygonal tiles is said to be *strict* if any point common to two tiles is either a vertex of both or of neither. A triangle is said to be *rational* if its sides have rational length. In [2], Eggleton asked if it is possible to strictly tile the plane with rational triangles using precisely one triangle from each congruence class. In this paper we constructively prove the existence of such a tiling.

Eggleton [2] suggested a method of proof that reduces the geometric problem to proving that if a, b, c are rationals with $0 < a \leqslant b$ and $0 \leqslant c$, then there are infinitely many rational solutions to

$$x - a/x = y - b/y + c. \tag{1}$$

However there are choices of a, b, c for which (1) does not have infinitely many rational solutions. Huff [3] proved that the only rational t for which $(t, 0)$ is rationally distant from both $(0, 1)$ and $(0, 2)$ is $t = 0$. This implies (compare with the proof of Theorem 2.1 in Section 2) the only rational solutions of (1) for $a = 1/4$, $b = 1$, $c = 0$ are the four solutions given by $x = \pm 1/2$, $y = \pm 1$. It can also be shown that (1) has no rational solutions for $a = b = c = 1$ and for $a = 1$, $b = 2$, $c = 0$.

Our main idea is to use the theory of rational points on elliptic curves, in particular the Nagell–Lutz theorem, to show that with an extra condition on the parameters a, b, c, equation (1) will have infinitely many rational solutions. Then we shall show that this extra condition is weak enough to solve Eggleton's tiling problem affirmatively by a suitable modification of the method of proof suggested by him.

The tiling we obtain is locally finite, in that every compact set intersects at most finitely many tiles. In the more traditional tiling problems where only finitely many congruence classes of tiles are used, the locally finite condition is automatically satisfied. But this is not the case in our problem. We remark that it is possible to obtain a solution to Eggleton's tiling problem without using the Nagell–Lutz theorem, but then sacrificing the locally finite property. In a forthcoming joint paper with Eggleton, we shall use an analogue of this non-locally finite tiling together with the Nagell–Lutz theorem to obtain a "strict" tiling of three-space with rational-edged tetrahedra using precisely one from each isometry class.

I take pleasure in acknowledging the referee's many helpful suggestions for improving the exposition of this paper. In addition, the referee corrected a minor error in the proof of Theorem 2.2.

2. Preliminaries

If A, B, C are non-collinear points on the plane, by ABC we shall mean the triangle with vertices A, B, C. By $\overline{AB}$ we shall mean the line segment with endpoints A, B. By AB we shall mean the length of $\overline{AB}$. By $x(A)$ we shall mean the x-coordinate of A.

We record the following two facts noted by Eggleton [1]:

Property 1. If d is the length of an altitude of a rational triangle, then d^2 is rational, and every positive rational arises as the value of d^2 for some rational triangle.

Property 2. An altitude of a rational triangle intersects the extended side perpendicular to it at a point rationally distant from both endpoints of the side.

The following two theorems will be relevant:

Theorem 2.1. *Let $a > 0$ be rational. The set of rationals t such that $(t, 0)$ is rationally distant from $(0, \sqrt{a})$ is dense in the real numbers.*

Proof. If d, t are rationals such that $d^2 - t^2 = a$, there is a nonzero rational x such that $d + t = 2x$ and $d - t = a/2x$, so $t = x - a/4x$. Conversely $(x - a/4x, 0)$ is rationally distant from $(0, \sqrt{a})$ for every nonzero rational x. But the set of rationals of the form $x - a/4x$, x rational, is dense in the real numbers.

Theorem 2.2. *Let a, b, c be positive rationals written in reduced form as $a = \alpha_1/\beta_1$, $b = \alpha_2/\beta_2$, $c = \gamma_1/\gamma_2$. Suppose*

$$\gamma_1 \nmid (\alpha_1\beta_2 - \alpha_2\beta_1). \tag{2}$$

Then the set of rationals t such that $(t, 0)$ is rationally distant from both $(0, \sqrt{a})$ and $(c, \sqrt{b})$ is dense in the real numbers.

We postpone the proof of Theorem 2.2 to Section 4.

3. The tiling

The triangles are arranged into a number of horizontal strips as in [1]. Each triangle has a side on one boundary of its strip, with the opposite vertex on the opposite boundary. The side of the triangle lying on the strip boundary is a maximal side of the triangle, so the width of the strip is a minimal altitude for each triangle in the strip. Two abutting triangles in a strip form a trapezoid. The x-coordinate of each vertex of every triangle is rational (compare with Property 2).

We denote the horizontal lines which form the edges of the strips, in order of increasing height, by $\ldots, l_{-2}, l_{-1}, l_0, l_1, l_2, \ldots$ where l_0 is the x-axis. Let $\sqrt{a_n}$ be the distance between l_n and l_{n-1} (compare with Property 1). The numbers a_n are positive rationals and $a_n \neq a_{n-1}$. Written in reduced form, we have $a_n = \alpha_n / \beta_n$ where α_n, β_n are relatively prime positive integers.

Two points P, Q, one on l_{n-1} and the other on l_{n+1}, will be called *well-placed* if $x(P) \neq x(Q)$ are rational and γ, the numerator of $x(P) - x(Q)$ in reduced form, is such that

$$\gamma \nmid (\alpha_n \beta_{n+1} - \alpha_{n+1} \beta_n).$$

For each n there are only finitely many divisors of $\alpha_n \beta_{n+1} - \alpha_{n+1} \beta_n$, so we have the following useful result.

Lemma 3.1. *Let l, l' denote l_{n-1}, l_{n+1} in some order. Fixing P on l with $x(P)$ rational, the set of points Q on l' with $x(Q)$ rational, $x(Q) \neq x(P)$, such that P, Q are not well-placed is a bounded discrete subset of l'.*

In what follows we have implicitly in mind a one-to-one correspondence between the set of congruence classes of rational triangles and the natural numbers, so phrases such as "a triangle in the first congruence class not already considered" have meaning.

We begin our tiling by placing a triangle in the first congruence class in the strip bounded by l_0 and l_1, with a maximal side on l_1 and opposite vertex at the origin O. Denote this triangle by ABO (see Fig. 1). Note that Property 2 implies $x(A)$ and $x(B)$ are rational. By "the triangle directly above ABO" we mean that triangle, yet to be chosen, in the strip bounded by l_1 and l_2 which shares the side $\overline{AB}$. By "the triangle directly below ABO" we mean that triangle, yet to be chosen, in the strip bounded by l_{-1} and l_0 which has a side on l_{-1} and a vertex at O (in our tiling there will be just one such triangle). Continuing in this fashion we can describe a doubly

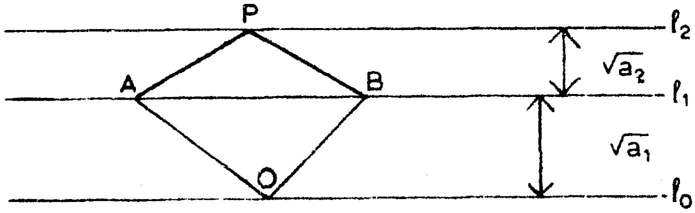

Fig. 1.

infinite sequence of triangles, yet to be chosen, arranged in a "column" above and below ABO. We call this the *critical column*.

On each line l_n the critical column will have two vertices if n is odd, one vertex if n is even. We shall construct the critical column so that for each k the rightmost vertices on l_{2k-1} and l_{2k+1} are well-placed, as are the leftmost vertices on these same lines.

As we build the critical column we partially fill in the strips to either side with triangles, creating a "diamond" of triangles. As construction of the critical column continues, this "diamond" expands, filling the plane.

We can now describe how the tiling continues after the placement of ABO. First we construct the triangle directly above ABO as follows (see Figure 1). Consider the perpendicular bisector of $\overline{AB}$, extended above l_1. As a point P on this bisector descends, approaching l_1, the common lengths AP and BP pass through a continuum of values. Hence every rational in this continuum is attained. Thus we may choose P so that ABP is rational, $AP = BP \leqslant AB$, ABP is incongruent to ABO, and $a_2 \neq a_1$. The two triangles in Fig. 1 constitute our first "diamond".

Next we wish to find a place for a triangle in the first congruence class not already represented by a tile. In Fig. 2 this triangle is denoted CDE, where $\overline{DE}$ is a maximal side, and $x(E) > x(D)$. We reserve the freedom of sliding CDE to the left or right in its strip.

We begin by letting a_0, a_{-1} be arbitrary positive rationals subject to the restraints $a_1 \neq a_0 \neq a_{-1} \neq a_{-2}$ and $\sqrt{a_0} + \sqrt{a_{-1}} \geqslant 1$. (Note that $\sqrt{a_{-2}}$ is the length of a minimal

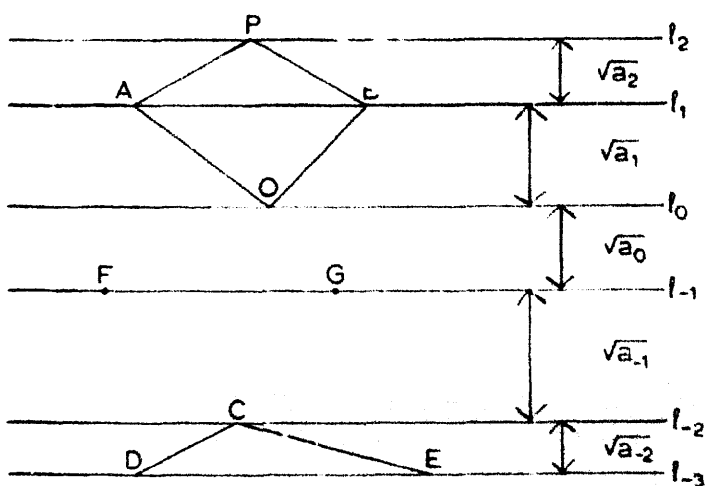

Fig. 2.

altitude of CDE. By Lemma 3.1, we can choose C on l_{-2} so that C and O are well-placed and $x(C)$ is rational. This fixes D and E on l_{-3} since the triangle CDE was already chosen. By Theorem 2.2 and Lemma 3.1 we can choose F, G on l_{-1} so that $x(F)$, $x(G)$, CF, CG, FO, GO are all rational; A, F and D, F are well-placed as are B, G and E, G; $x(G) - x(F) \geq CF, CG, FO, GO$; and CFG, FGO are incongruent to all previously chosen triangles.

We next choose H (Fig. 3) so that DEH is an appropriate isosceles triangle (as we chose P).

Since B, G are well-placed, by Theorem 2.2 we can choose a point I on l_0 so that $x(I), BI, GI$ are rational; $x(I) - x(O) \geq BO, BI, GO, GI$; and BIO, GIO are incongruent to all previously chosen triangles.

Next by Theorem 2.2 and Lemma 3.1 we can choose J on l_{-2} so that $x(J), GJ, EJ$ are rational; I, J are well-placed; $x(J) - x(C) \geq CG, CE, GJ, EJ$; and CGJ, CEJ are incongruent to all previously chosen triangles.

Since I, J are well-placed, by Theorem 2.2 we can choose K on l_{-1} so that $x(K), IK, JK$ are rational; $x(K) - x(G) \geq GI, GJ, IK, JK$; and GIK, GJK are incongruent to all previously chosen triangles.

In a similar fashion we sequentially choose L, M, N to fill in the corresponding "half-diamond" on the left side of Fig. 3. We have ensured that our second "diamond" has height greater than 1.

It should now be clear how we proceed. We next consider a triangle in the first congruence class not already represented by a tile and find a place for it in the critical column in the third strip above ABP, proceeding as with the placement of CDE. After putting an isosceles triangle above this triangle we fill out an enlarged "diamond". We can ensure that this third "diamond" will have height greater than 2. Continuing in this fashion, extending the critical column by a height greater than

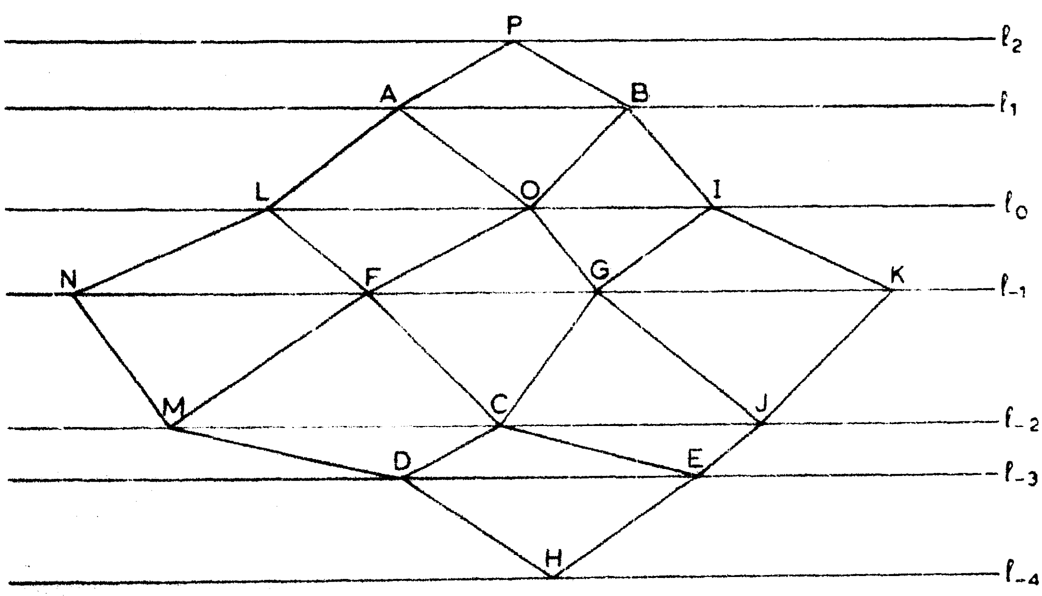

Fig. 3.

1 alternately above and below the previous "diamond" and filling in to create a new "diamond", we obtain our tiling. Indeed, as our diamond expands, the number of triangles in a strip increases without bound, and each triangle has an edge of its strip boundary that is longer than the strip width. Hence each strip is completely tiled. Moreover since we ensure the critical column continues to grow in height by an amount greater than 1 in both directions after every two "diamonds" are completed (after the first), our horizontal strips fill the plane. Infinitely often we repeat the step of placing in the critical column a triangle in the first congruence class not already represented by a tile, so we are assured of using at least one triangle from every congruence class. At each step no triangle is used as a tile if it is congruent to a triangle already used, so we are assured of using at most one triangle from each congruence class. This proves:

Theorem 3.2. *There is a locally finite strict tiling of the plane using precisely one triangle from each congruence class of rational triangles.*

4. The proof of Theorem 2.2

Recall that a, b, c are positive rationals, with reduced form $a = \alpha_1/\beta_1$, $b = \alpha_2/\beta_2$, $c = \gamma_1/\gamma_2$, such that

$$\gamma_1 \nmid (\alpha_1\beta_2 - \alpha_2\beta_1). \tag{2}$$

As in the proof of Theorem 2.1, t is rational and $(t, 0)$ is rationally distant from both $(0, \sqrt{a})$ and $(c, \sqrt{b})$ if and only if there are rationals x, y with t being the common value of

$$x - a/4x = y - b/4y + c. \tag{3}$$

Multiply (3) by $xy(x-y)$, gather the third and fourth order terms, and complete the square in their factor complementary to xy. Multiply by $4\alpha_1^2\beta_1^4\beta_2^6\gamma_2^5 y/x^3$ and make the following change of variables:

$$u = -\alpha y/x, \qquad w = \beta_1\beta_2\gamma_2 u(2y - 2x + c), \tag{4}$$

where $\alpha = \alpha_1\beta_1\beta_2^2\gamma_2^2$. Let $\beta = \alpha_2\beta_1^2\beta_2\gamma_2^2$ and $\gamma = \beta_1\beta_2\gamma_1$. Note that if (x, y) is a real solution of (3), then (u, w) is a real solution of

$$w^2 = u^3 + (\gamma^2 + \alpha + \beta)u^2 + \alpha\beta u. \tag{5}$$

Conversely, if (u, w) is a real solution of (5) with $u \neq 0, -\alpha, -\beta$, then (x, y), given by the inverse of (4), is a real solution of (3). The inverse of (4) is

$$x = \frac{c\alpha u - \alpha_1\beta_2\gamma_2 w}{2(u^2 + \alpha u)}, \quad y = \frac{ux}{-\alpha}. \tag{6}$$

To prove Theorem 2.2, it will be sufficient to prove that the rational solutions of (5) are dense in the real solutions. For then it will follow that the rational solutions of (3) are dense in the real solutions, since the change of variables relating (3) and (5) is birational and bicontinuous. But (3) has a real solution (x, y) for every $x \neq 0$, so the set of $t = x - a/4x$ for which (x, y) is a rational solution of (3) will be dense in the set $\{x - a/4x \mid x \neq 0\}$, which is all the real numbers.

Let P be the integer point $(-\alpha, \alpha\gamma)$. We easily verify that P is a solution of (5). It corresponds to the solution $x = y = (b - a)/4c$ of (3) given in [2]. We compute $2P$ on (5); that is, the reflection in the u-axis of the second intersection point of the tangent to (5) at P with (5) (see Mordell [5] or Tate [6]). We get

$$2P = \left(\left(\frac{\beta - \alpha}{2\gamma}\right)^2, \left(\frac{\beta - \alpha}{2\gamma}\right)^3 + \gamma\left(\frac{\beta - \alpha}{2\gamma}\right)^2 + \alpha\left(\frac{\beta - \alpha}{2\gamma}\right)\right).$$

We note that the coordinates of $2P$ are not integers, since $\gamma \nmid (\beta - \alpha)$. Indeed this follows from (2) and the fact that $(\gamma_1, \gamma_2) = 1$.

We now use the Nagell–Lutz theorem (again see [5] or [6]), which shows how to determine all rational points of finite order on a non-singular elliptic curve in normal form. Such points must have integer coordinates, and the second coordinate must be zero or a divisor of the discriminant. Hence it follows that $2P$ is not of finite order, so (5) has infinitely many rational solutions.

Note now that the graph of the real points of (5) has two branches: a bounded branch and an unbounded branch. The bounded branch is "even" in that a non-tangent intersecting line meets the branch twice; the unbounded branch is "odd" in that a non-tangent intersecting line meets the branch once or three times. By a theorem of Hurwitz (Theorem 13 of [4]), if an elliptic curve whose real points have an "even" branch and an "odd" branch has infinitely many rational points, then the rational points are dense on the "odd" branch, and on the "even" branch they are either dense or entirely absent. The rational point P lies on the "even" branch of (5). Hence the rational solutions of (5) are dense in the real solutions of (5). (Note that the group of real points of (5) is topologically isomorphic to the compact abelian group $S^1 \times Z_2$ where S^1 is the circle group. Under this isomorphism the rational points of (5) get sent to an infinite subgroup of $S^1 \times Z_2$ which is not contained in S^1. Hence the rational points are dense.) This completes our proof of Theorem 2.2.

5. Comments

With slight modifications in the proof of Theorem 3.2, we can prove:

Theorem 5.1. *Let T be any set of positive real numbers whose squares are rational. If T has more than one element, then the plane can be strictly tiled with precisely one triangle from each congruence class of rational triangles with minimal altitude length in T.*

Corollary 5.2. *The plane can be strictly tiled with precisely one triangle from each congruence class of rational triangles with rational area.*

We do not know if Theorem 5.1 remains true if T has just one element.

Let δ_n be a cardinal number with $1 \leq \delta_n \leq \aleph_0$. Having in mind a specific one-to-one correspondence between the congruence classes of rational triangles and the natural numbers, we can prove:

Theorem 5.3. *The plane can be strictly tiled using precisely δ_n triangles from the nth congruence class of rational triangles,* $n = 1, 2, 3, \ldots$.

Theorem 5.3 can be generalized along the line of Theorem 5.1.

We can also prove:

Theorem 5.4. *The plane can be strictly tiled with precisely one triangle from each congruence class of algebraic triangles (triangles with sides whose lengths are algebraic numbers).*

Theorem 5.4 can be generalized along the lines of Theorem 5.1 and 5.3. The corresponding assertions with the field of real algebraic numbers replaced by any subfield are also true.

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