LEMMA. *Nondistinct solutions of* $S = \mathrm{Circ}(1,1,1,0,\dots,0)X$ *result* (i) *if two consecutive elements in* S *are equal*; *or* (ii) *if* S *contains five consecutive elements of the form abcba.*

Proof. The equation $x_j + x_{j+1} + x_{j+2} = x_{j+1} + x_{j+2} + x_{j+3}$ implies $x_j = x_{j+3}$, confirming (i). To confirm (ii), suppose, without loss of generality, that

$$x_1 + x_2 + x_3 = a,\ x_2 + x_3 + x_4 = b,\ x_4 + x_5 + x_6 = b,\ x_5 + x_6 + x_7 = a.$$

These equations imply $x_1 - x_4 = a - b = x_7 - x_4$, so $x_1 = x_7$.

The key question regarding the flim-flam game of Example (4) is: What can a player's secret number possibly be? Obviously it can't be, say, 14 because everyone has a 12 on their clockface and, next to it, at least 1 and 2. The bidding should begin at 16. The bids 17, 18, 19, 20, ... pass uneventfully.

Let σ denote any secret number. To obtain a lower bound on σ we compute $\bar{s}$, the average value of the triple-sums s_j:

$$\bar{s} = \frac{1}{12}\sum_{k=1}^{12} s_k = \frac{1}{4}\sum_{k=1}^{12} x_k = \frac{1}{4}\sum_{k=1}^{12} k = 19.5.$$

Thus, $\sigma \geqslant 20 > \bar{s}$. In actuality, σ can never equal 20. Twelve integers, the largest of them 20, average out to 19.5 only if at least 6 of them equal 20. If exactly 6 of the s_j equal 20, then the others must all be 19. In that case, either two consecutive s_j are the same, or else the 20's and 19's alternate. Both possibilities are violations, in view of the above Lemma. If more than 6 of the s_j equal 20, we have the same violation because (the pigeonhole principle!) at least two consecutive entries in S must be 20.

Thus, $\sigma \geqslant 21$. Meanwhile, my accomplice is ready with a circular permutation for which σ attains its lower bound. It is 1, 8, 10, 3, 5, 9, 4, 6, 11, 2, 7, 12.

The only remaining question: What is the probability that someone in the audience has also stumbled onto a circular permutation of $1, 2, \dots, 12$ for which $\sigma = 21$? Frankly, I don't know the probability, but I believe that it is very small. After all, my accomplice and I have never lost a game.

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MULTIPLICATIVE RELATIONS FOR SUMS OF INITIAL kTH POWERS

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Is the curious and pretty identity

$$(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$

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an anomaly or are there more such? Specifically, let f_k denote the function whose value at the positive integer n is

$$f_k(n) = 1^k + 2^k + \cdots + n^k.$$

We ask for a list of all identities of the form

$$\prod_{i=1}^{r} f_{h(i)}^{a(i)} = \prod_{j=1}^{s} f_{k(j)}^{b(j)}, \tag{1}$$

where the exponents are positive integers and the subscripts are distinct nonnegative integers. We shall show the only such identity is $f_1^2 = f_3$ and powers of this equation.

It is well known that f_k is a polynomial of degree $k + 1$. In fact,

$$f_k(n-1) = \frac{1}{k+1}\sum_{j=1}^{k+1}\binom{k+1}{j}B_{k+1-j}n^j,$$

where the $\{B_i\}$ are the Bernoulli numbers. A tidy way of defining these ubiquitous numbers is via the identity

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty}\frac{B_i x^i}{i!},$$

so that B_i is the value of the ith derivative of $x/(e^x - 1)$ at $x = 0$. See [**5**], pp. 230–246, for proofs and a complete discussion.

Suppose that (1) holds. Without loss of generality we may assume that

$$h(1) < h(2) < \cdots < h(r) \quad \text{and} \quad k(1) < k(2) < \cdots < k(s);$$

moreover, that $h(r) < k(s)$. Let us first examine the case in which $k(s)$, the largest of the subscripts, is at most 3. Then the identity in (1) may involve only the polynomials

$$f_0(n) = n, \quad f_1(n) = \frac{1}{2}n(n+1),$$

$$f_2(n) = \frac{1}{6}n(n+1)(2n+1), \quad f_3(n) = \frac{1}{4}n^2(n+1)^2.$$

Because $f_2(n)$ has the irreducible factor $2n + 1$ which appears in none of the other three polynomials, $f_2(n)$ is not involved in the identity. By examining the exponent on the factor $n + 1$, we see that the only possible identity is $f_1^{2a} = f_3^a$ for some positive integer a.

Now suppose that $k(s) > 3$. By evaluating the polynomial identity in (1) at $n = 2$ we obtain the integer equation

$$\prod_{i=1}^{r}[1 + 2^{h(i)}]^{a(i)} = \prod_{j=1}^{s}[1 + 2^{k(j)}]^{b(j)}. \tag{2}$$

From a theorem of Bang [**2**], there is a prime divisor p of $1 + 2^{k(s)}$ such that 2 belongs to the exponent $2k(s)$ in the integers modulo p. Because $h(i) < k(s)$ for each i, we see that p cannot be a divisor of the left-hand side in (2). Thus there can be no identity of the form in (1) with $k(s) > 3$. This completes the proof of our theorem.

The result of Bang to which we refer deals with "primitive" prime factors of expressions of the form $a^n - 1$ (we were concerned above with $2^{2k(s)} - 1$). This result was later generalized by Zsigmondy [**9**] to expressions of the form $a^n - b^n$:

ZSIGMONDY'S THEOREM. *If a, b, and n are integers with $a > b > 0$, $\gcd(a, b) = 1$, and $n > 2$, then there is a prime divisor p of $a^n - b^n$ such that p is not a divisor of $a^k - b^k$ for any integer k with $1 \leqslant k < n$, except for the case $a = 2$, $b = 1$, $n = 6$.*

Both Bang's theorem and Zsigmondy's theorem have been rediscovered many times in the last century. A partial list of references is given in [**3**], p. 361. It should also be noted that Zsigmondy's theorem has itself been generalized to algebraic number fields—a special case of this situation implies that Fibonacci numbers have primitive prime factors. A recent reference on generalizations of Zsigmondy's theorem is Stewart [**7**], from which earlier references may be tracked down.

Using Zsigmondy's theorem, we can handle the following generalization of (1). If $c, d \geqslant 1$, $k, n \geqslant 0$ are integers, let $f(c, d, k)$ be the function whose value at n is

$$f(c,d,k;n) = \sum_{i=0}^{n-1} (c+di)^k. \tag{3}$$

Thus $f(1,1,k) = f_k$. For a fixed pair c, d we can ask if there are any multiplicative identities

$$\prod_{i=1}^{r} f(c,d,h(i))^{a(i)} = \prod_{j=1}^{s} f(c,d,k(j))^{b(j)}, \tag{4}$$

where the $\{h(i)\}$ and $\{k(j)\}$ are distinct nonnegative integers. We leave it to the reader to verify the pleasant exercise that the only solutions of (4) are powers of the equations

$$f(1,1,1)^2 = f(1,1,3),$$

and

$$f(1,2,0)^2 = f(1,2,1).$$

Note that this result is a grand generalization of MONTHLY Problem E 2951 [**8**], which asked for a catalog of all equations

$$f(1,2,h)^a = f(1,2,k)^b,$$

and also the work of Edmonds [**4**] and Allison [**1**], who considered identities of the form

$$f(1,1,h)^a = f(1,1,k)^b.$$

This latter equation was also considered in MONTHLY Problem E 2136 [**6**].

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