LEMMA. Nondistinct solutions of $S = \text{Circ}(1, 1, 1, 0, \ldots, 0) \times X$ result (i) if two consecutive elements in $S$ are equal; or (ii) if $S$ contains five consecutive elements of the form abcba.

Proof. The equation $x_j + x_{j+1} + x_{j+2} = x_{j+1} + x_{j+2} + x_{j+3}$ implies $x_j = x_{j+3}$, confirming (i). To confirm (ii), suppose, without loss of generality, that

$$x_1 + x_2 + x_3 = a, x_2 + x_3 + x_4 = b, x_4 + x_5 + x_6 = b, x_2 + x_6 + x_7 = a.$$ 

These equations imply $x_1 - x_4 = a - b = x_7 - x_4$, so $x_1 = x_7$.

The key question regarding the film-flam game of Example (4) is: What can a player's secret number possibly be? Obviously it can't be, say, 14 because everyone has a 12 on their clockface and, next to it, at least 1 and 2. The bidding should begin at 16. The bids 17, 18, 19, 20, ... pass uneventfully.

Let $\sigma$ denote any secret number. To obtain a lower bound on $\sigma$ we compute $\bar{s}$, the average value of the triple-sums $s_j$:

$$\bar{s} = \frac{1}{12} \sum_{k=1}^{12} s_k = \frac{1}{4} \sum_{k=1}^{12} x_k = \frac{1}{4} \sum_{k=1}^{12} k = 19.5.$$

Thus, $\sigma \geq 20 > \bar{s}$. In actuality, $\sigma$ can never equal 20. Twelve integers, the largest of them 20, average out to 19.5 only if at least 6 of them equal 20. If exactly 6 of the $s_j$ equal 20, then the others must all be 19. In that case, either two consecutive $s_j$ are the same, or else the 20's and 19's alternate. Both possibilities are violations, in view of the above Lemma. If more than 6 of the $s_j$ equal 20, we have the same violation because (the pigeonhole principle!) at least two consecutive entries in $S$ must be 20.

Thus, $\sigma \geq 21$. Meanwhile, my accomplice is ready with a circular permutation for which $\sigma$ attains its lower bound. It is 1, 8, 10, 3, 5, 9, 4, 6, 11, 2, 7, 12.

The only remaining question: What is the probability that someone in the audience has also stumbled onto a circular permutation of 1, 2, ..., 12 for which $\sigma = 21$? Frankly, I don't know the probability, but I believe that it is very small. After all, my accomplice and I have never lost a game.

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References


MULTIPLICATIVE RELATIONS FOR SUMS OF INITIAL $k$TH POWERS

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Is the curious and pretty identity

$$(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$

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an anomaly or are there more such? Specifically, let \( f_k \) denote the function whose value at the positive integer \( n \) is

\[
f_k(n) = 1^k + 2^k + \cdots + n^k.
\]

We ask for a list of all identities of the form

\[
\prod_{i=1}^{r} f_{a(i)}^{k(i)} = \prod_{j=1}^{s} f_{b(j)}^{k(j)},
\]

where the exponents are positive integers and the subscripts are distinct nonnegative integers. We shall show the only such identity is \( f_1^2 = f_3 \) and powers of this equation.

It is well known that \( f_k \) is a polynomial of degree \( k + 1 \). In fact,

\[
f_k(n - 1) = \frac{1}{k + 1} \sum_{j=1}^{k+1} \binom{k + 1}{j} B_{k+1-j} n^j,
\]

where the \( \{B_j\} \) are the Bernoulli numbers. A tidy way of defining these ubiquitous numbers is via the identity

\[
\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i x^i}{i!},
\]

so that \( B_i \) is the value of the \( i \)th derivative of \( x/(e^x - 1) \) at \( x = 0 \). See [5], pp. 230–246, for proofs and a complete discussion.

Suppose that (1) holds. Without loss of generality we may assume that

\[ h(1) < h(2) < \cdots < h(r) \quad \text{and} \quad k(1) < k(2) < \cdots < k(s); \]

moreover, that \( h(r) < k(s) \). Let us first examine the case in which \( k(s) \), the largest of the subscripts, is at most 3. Then the identity in (1) may involve only the polynomials

\[
f_0(n) = n, \quad f_1(n) = \frac{1}{2} n(n + 1),
\]

\[
f_2(n) = \frac{1}{6} n(n + 1)(2n + 1), \quad f_3(n) = \frac{1}{4} n^2(n + 1)^2.
\]

Because \( f_2(n) \) has the irreducible factor \( 2n + 1 \) which appears in none of the other three polynomials, \( f_2(n) \) is not involved in the identity. By examining the exponent on the factor \( n + 1 \), we see that the only possible identity is \( f_1^{2^a} = f_3^2 \) for some positive integer \( a \).

Now suppose that \( k(s) > 3 \). By evaluating the polynomial identity in (1) at \( n = 2 \) we obtain the integer equation

\[
\prod_{i=1}^{r} [1 + 2^{k(i)}]^{a(i)} = \prod_{j=1}^{s} [1 + 2^{k(j)}]^{b(j)}.
\]

From a theorem of Bang [2], there is a prime divisor \( p \) of \( 1 + 2^{k(s)} \) such that 2 belongs to the exponent \( 2k(s) \) in the integers modulo \( p \). Because \( h(i) < k(s) \) for each \( i \), we see that \( p \) cannot be a divisor of the left-hand side in (2). Thus there can be no identity of the form in (1) with \( k(s) > 3 \). This completes the proof of our theorem.

The result of Bang to which we refer deals with “primitive” prime factors of expressions of the form \( a^n - 1 \) (we were concerned above with \( 2^{2k(s)} - 1 \)). This result was later generalized by Zsigmondy [9] to expressions of the form \( a^n - b^n \):

**ZSIGMONDY’S THEOREM.** If \( a, b, \) and \( n \) are integers with \( a > b > 0 \), \( \gcd(a, b) = 1 \), and \( n > 2 \), then there is a prime divisor \( p \) of \( a^n - b^n \) such that \( p \) is not a divisor of \( a^k - b^k \) for any integer \( k \) with \( 1 \leq k < n \), except for the case \( a = 2, b = 1, n = 6 \).
Both Bang's theorem and Zsigmondy's theorem have been rediscovered many times in the last century. A partial list of references is given in [3], p. 361. It should also be noted that Zsigmondy's theorem has itself been generalized to algebraic number fields—a special case of this situation implies that Fibonacci numbers have primitive prime factors. A recent reference on generalizations of Zsigmondy's theorem is Stewart [7], from which earlier references may be tracked down.

Using Zsigmondy's theorem, we can handle the following generalization of (1). If \( c, d \geq 1 \), \( k, n \geq 0 \) are integers, let \( f(c, d, k) \) be the function whose value at \( n \) is

\[
(3) \quad f(c, d, k; n) = \sum_{i=0}^{n-1} (c + di)^k.
\]

Thus \( f(1, 1, k) = f_k \). For a fixed pair \( c, d \) we can ask if there are any multiplicative identities

\[
(4) \quad \prod_{i=1}^{r} f(c, d, h(i))^{a(i)} = \prod_{j=1}^{s} f(c, d, k(j))^{b(j)},
\]

where the \( \{h(i)\} \) and \( \{k(j)\} \) are distinct nonnegative integers. We leave it to the reader to verify the pleasant exercise that the only solutions of (4) are powers of the equations

\[
f(1, 1, 1)^2 = f(1, 1, 3),
\]

and

\[
f(1, 2, 0)^2 = f(1, 2, 1).
\]

Note that this result is a grand generalization of MONTHLY Problem E 2951 [8], which asked for a catalog of all equations

\[
f(1, 2, h)^a = f(1, 2, k)^b,
\]

and also the work of Edmonds [4] and Allison [1], who considered identities of the form

\[
f(1, 1, h)^a = f(1, 1, k)^b.
\]

This latter equation was also considered in MONTHLY Problem E 2136 [6].

We take this opportunity to thank Paul Bateman, Bruce Berndt, H. W. Gould, and William J. LeVeque for their assistance in locating references concerning the equation in (1). In light of the many rediscoveries of the Bang-Zsigmondy result, it would be ironic if our results here had been anticipated. In such an event, which is perhaps not unlikely, the responsibility is of course solely ours.

References