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ON THE AVERAGE NUMBER OF GROUPS OF SQUARE-FREE ORDER

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ABSTRACT. Let $G(n)$ denote the number of (nonisomorphic) groups of order n. It is shown here that for large x

$$x^{1.68} \leq \sum_{n\leq x}{}' G(n) \leq x^2 \cdot \exp\{-(1+o(1)) \log x \log\log\log x / \log\log x\},$$

where $\sum'$ denotes a sum over square-free n. Under an unproved hypothesis on the distribution of primes p with all primes in $p-1$ small, it is shown that the upper bound is tight.

1. Introduction. Let $G(n)$ denote the number of isomorphism classes among the groups of order n. For n square-free, there is a relatively simple formula for $G(n)$ due to Hölder [**9**]. First let

$$f(n,m) = \prod_{q|m} (n, q-1).$$

(Throughout the paper, the letters p, q will denote primes.) Then

$$G(n) = \sum_{d|n} \prod_{p|d} \frac{f(p, n/d) - 1}{p-1}, \quad n \text{ square-free.} \tag{1.1}$$

With this elegant formula, the techniques of number theory can be applied to give several interesting results about $G(n)$ for n square-free. For example, in Murty and Murty [**11**], it is shown that

$$\sum_{n\leq x} \mu^2(n) \log G(n) = (c_1 + o(1)) x \log\log x$$

for a certain positive constant c_1. (The square of the Möbius function $\mu^2(n)$ is the characteristic function of the square-free integers.) Thus the geometric mean of $G(n)$ for square-free $n \leq x$ is about $(\log x)^{\pi^2 c_1/6}$. In Erdös, Murty, and Murty [**4**] it is shown that $\mu^2(n) \log G(n)/\log\log n$ has a continuous, strictly increasing distribution function on $[0, \infty)$.

The maximal order of $\mu^2(n)G(n)$ is somewhat different. Murty and Srinivasan [**12**] recently showed that for all n,

$$\mu^2(n) G(n) \leq n/(\log n)^{A \log_3 n} \tag{1.2}$$

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for a certain positive constant A, where $\log_k n$ denotes the k-fold iterated natural logarithm. They also showed that the estimate (1.2) is essentially best possible, for there are infinitely many n with

$$\mu^2(n)G(n) \geq n/(\log n)^{B \log_3 n}$$

for some positive constant B.

This paper will be concerned with the average order of $\mu^2(n)G(n)$. From the distribution function result cited above it follows that for any K, the set of n with $\mu^2(n)G(n) > (\log n)^K$ has positive asymptotic density. From this fact and from (1.2) it follows that for any fixed K and all large x (depending on the choice of K),

$$x(\log x)^K \leq \sum_{n \leq x} \mu^2(n)G(n) \leq x^2/(\log x)^{A \log_3 x}.$$

Below it is shown that for large x,

$$x^{1.68} \leq \sum_{n \leq x} \mu^2(n)G(n) \leq x^2/\exp\{(1+o(1))\log x \log_3 x/\log_2 x\}. \tag{1.3}$$

Moreover, the equation

$$\sum_{n \leq x} \mu^2(n)G(n) = x^2/\exp\{(1+o(1))\log x \log_3 x/\log_2 x\} \tag{1.4}$$

is shown modulo a reasonable conjecture on the distribution of primes p with all primes in $p-1$ small. The "$o(1)$" appearing in (1.3) and (1.4) is partially expanded to

$$\frac{1}{\log_3 x}\left(\log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O\left(\left(\frac{\log_4 x}{\log_3 x}\right)^2\right)\right).$$

Let $C(n)$ denote the number of isomorphism classes among all groups of order n which have all of their Sylow subgroups cyclic. Then for n square-free, $G(n) = C(n)$. In [**11**], a formula that generalizes the Hölder formula (1.1) is given. Namely it is shown that

$$C(n) = \sum_{d \| n} \prod_{p^a \| d} \sum_{i=1}^{a} \frac{f(p^i, n/d) - f(p^{i-1}, n/d)}{p^{i-1}(p-1)}, \tag{1.5}$$

where we write $m \| n$ if $m | n$ and $(m, n/m) = 1$. In [**12**], (1.2) is actually shown via the stronger result

$$C(n) \leq n/(\log n)^{A \log_3 n}. \tag{1.6}$$

Below, the upper bound in (1.3) is shown by proving the stronger result that the same upper bound holds for $\sum_{n \leq x} C(n)$.

In [**11**], it is shown that $C(n) \leq f(n, n)$ for all n. Thus the upper bound result of this paper could possibly be achieved by proving a similar result for $\sum_{n \leq x} f(n, n)$. Although superficially simpler, this sum does not appear so easy to estimate. Probably the same upper bound could be worked out, but I have not completed all of the details. It should be remarked that essentially the same upper bound is established in [**5**] for the sum $\sum'_{n \leq x} f(n-1, n)$, where the dash indicates the sum is over composite integers.

If we drop the restriction that n is square-free, the behavior of $G(n)$ changes markedly. In fact

$$x^{C\log^2 x} \le \sum_{n\le x} G(n) \le x^{D\log^2 x}$$

holds for certain positive constants C, D. The lower bound follows from restricting n to prime powers and using results of Higman [**8**] and Sims [**17**] on p-groups. The upper bound follows from work of Neumann [**13**] and the recent classification of the finite simple groups. In fact, from these papers it follows that C can be chosen as any number smaller than $2/(27\log^2 2)$ and D can be chosen as any number greater than $1/(2\log^2 2)$.

2. The upper bound.

THEOREM 2.1. *There is a constant c_2 such that for all large x,*
(2.1)

$$\frac{1}{x}\sum_{n\le x} C(n) \le x\cdot\exp\left\{-\frac{\log x}{\log_2 x}\left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + c_2\left(\frac{\log_4 x}{\log_3 x}\right)^2\right)\right\}.$$

PROOF. From (1.5) we have (ϕ denotes Euler's function)

$$\begin{aligned} C(n) &= \sum_{d\|n}\frac{1}{\phi(d)}\prod_{p^a\|d}\sum_{i=1}^{a} p^{a-i}(f(p^i, n/d) - f(p^{i-1}, n/d)) \\ &\le \sum_{\substack{d\|n \\ p|d\to f(p,n/d)>1}} \frac{1}{\phi(d)}\prod_{p^a\|d}\sum_{i=1}^{a} p^{a-i} f(p^i, n/d). \end{aligned}$$

Note that

$$\begin{aligned} \prod_{p^a\|d}\sum_{i=1}^{a} p^{a-i} f(p^i, n/d) &\le d\sum_{\delta|d}\prod_{p^i\|\delta} p^{-i} f(p^i, n/d) \\ &= d\sum_{\delta|d}\delta^{-1} f(\delta, n/d) \le \sigma(d) f(d, n/d), \end{aligned}$$

where σ is the sum of the divisors function. Thus

$$C(n) \le \sum_{\substack{d\|n \\ p|d\to f(p,n/d)>1}} \frac{\sigma(d)}{\phi(d)} f\left(d, \frac{n}{d}\right). \tag{2.2}$$

From the Hardy-Ramanujan inequality [**7**], the number of integers $n \le x$ with $\omega(n) > 2\log x \log_3 x/(\log_2 x)^2$ (where $\omega(n)$ is the number of distinct prime factors of n) is less than

$$x\cdot\exp(-1.5\log x\log_3 x/\log_2 x)$$

for large x. Thus by virtue of (1.6) (or the easier inequality $C(n) \le f(n,n) \le \phi(n)$ from [**11**]), we may ignore such n in the sum (2.1). Let $\sum^*$ denote a sum over integers n with

$$\omega(n) \le 2\log x\log_3 x/(\log_2 x)^2. \tag{2.3}$$

From (2.2) we have

$$\sum_{n\le x}^{*} C(n) \le \sum_{d\le x}^{*} \frac{\sigma(d)}{\phi(d)} \sum_{\substack{m\le x/d \\ p|d\to f(p,m)>1}}^{*} f(d,m). \tag{2.4}$$

Let $\alpha(n)$ denote the largest square-free divisor of n. Then if m is such that $p|d$ implies $f(p,m)>1$, then

$$\alpha(f(d,m)) = \alpha(d).$$

If k is any integer with $\alpha(k)=\alpha(d)$, let

$$N_{k,d}(y) = \sum_{\substack{m\le y \\ f(d,m)=k}}^{*} 1.$$

Thus (2.4) is now transformed to

$$\sum_{n\le x}^{*} C(n) \le \sum_{d\le x}^{*} \frac{\sigma(d)}{\phi(d)} \sum_{\substack{k\le x/d \\ \alpha(k)=\alpha(d)}} kN_{k,d}\left(\frac{x}{d}\right). \tag{2.5}$$

We now turn our attention to bounding $N_{k,d}(y)$. If m is such that $f(d,m)=k$, we write $m=m_1m_2$ where m_1 is the product of the distinct primes $q|m$ with $(d,q-1)>1$. Say the prime factorization of m_1 is $q_1\cdots q_s$. Let $k_j=(d,q_j-1)$ for $j=1,\ldots,s$. Thus $\prod_{j=1}^{s} k_j = k$ and each $k_j>1$. That is, the multiset $\{k_1,\ldots,k_s\}$ is a factorization of k. (By a factorization of an integer k we mean an unordered multiset of integers exceeding 1 whose product is k.) Let $\mathcal{F}(k)$ denote the set of all factorizations of k. For each factorization $\mathcal{F}$ in $\mathcal{F}(k)$, let $N_{\mathcal{F},d}(y)$ denote the number of m counted by $N_{k,d}(y)$ which give rise to the factorization $\mathcal{F}$ as described above. Thus

$$N_{k,d}(y) = \sum_{\mathcal{F}\in\mathcal{F}(k)} N_{\mathcal{F},d}(y). \tag{2.6}$$

For $\mathcal{F}=\{k_1,\ldots,k_s\}$, we have

$$N_{\mathcal{F},d}(y) \le y \sum_{m_1\le y} \frac{1}{m_1},$$

where m_1 runs over integers of the form $q_1\cdots q_s$, the q's being distinct primes with $(d,q_j-1)=k_j$ for $j=1,\ldots,s$. Thus

$$N_{\mathcal{F},d}(y) \le y \prod_{j=1}^{s} \sum_{\substack{q\le y \\ q\equiv 1 \bmod k_j}} \frac{1}{q} \le y \prod_{j=1}^{s} \frac{\log\log y + c_3\log k_j}{\phi(k_j)} \tag{2.7}$$

for some absolute constant c_3 (see Norton [**14**] or Pomerance [**15**]).

Note that $s=\omega(m_1)\le\omega(m)$ so that if m is counted by $N_{k,d}(y)$, we have by (2.3) that

$$s \le 2\log x \log_3 x/(\log_2 x)^2. \tag{2.8}$$

We now majorize the product $\prod_{j=1}^{s}(\log\log y + c_3 \log k_j)$ (for $k, y \le x$) by breaking it into two parts corresponding to $k_j < \exp((\log_2 x)^3)$ and $k_j \ge \exp((\log_2 x)^3)$. The first part is at most, by (2.8),

$$(\log_2 x + c_3(\log_2 x)^3)^s \le \exp\left\{7\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right\}$$

for large x. The number of factors in the second part is $O(\log x/(\log_2 x)^3)$, so the second part is majorized by

$$(\log_2 x + c_3 \log x)^{O(\log x/(\log_2 x)^3)} = \exp\left\{O\left(\frac{\log x}{(\log_2 x)^2}\right)\right\}.$$

Therefore, for large x and $k, y \le x$,

$$\prod_{j=1}^{s}(\log\log y + c_3 \log k_j) \le \exp\left\{8\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right\}.$$

Next note that by (2.8),

$$k\Big/\prod_{j=1}^{s}\phi(k_j) = \prod_{j=1}^{s} k_j/\phi(k_j) \le (c_4 \log_2 x)^s$$
$$\le \exp\{3\log x(\log_3 x)^2/(\log_2 x)^2\}$$

for large x.

Putting these estimates into (2.7), we have

$$N_{\mathcal{F},d}(y) \le \frac{y}{k}\exp\left\{11\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right\}.$$

Thus from (2.6) we have

$$kN_{k,d}(y) \le yf(k)\exp\left\{11\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right\},$$

where $f(k)$ is the cardinality of $\mathcal{F}(k)$, that is, the number of factorizations of k.

From (2.5) we now have

$$\sum_{n\le x}{}^{*} C(n) \le x\sum_{d\le x}{}^{*}\frac{\sigma(d)}{d\phi(d)}\left(\sum_{\substack{k\le x/d\\ \alpha(k)=\alpha(d)}} f(k)\right)\exp\left\{11\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right\}. \tag{2.9}$$

But note that

$$\sum_{d\le x}\frac{\sigma(d)}{d\phi(d)} = O(\log x).$$

Also note that since d satisfies (2.3), for large x

$$\sum_{\substack{k\le x/d\\ \alpha(k)=\alpha(d)}} 1 \le \psi\left(\frac{x}{d\alpha(d)}, \frac{3\log x\log_3 x}{\log_2 x}\right) \le \psi\left(x, \frac{3\log x\log_3 x}{\log_2 x}\right),$$

where $\psi(x, y)$ denotes the number of integers up to x, none of whose primes exceed y. Indeed, if $\alpha(k) = \alpha(d)$, then $\alpha(d)|k$ and all of the primes in $k/\alpha(d)$ are among the primes in d. Thus the number of such $k \le x/d$ is at most the number of integers below $x/d\alpha(d)$ all of whose primes are among the first $\omega(d)$ primes. But using (2.3) the $\omega(d)$th prime is less than $3\log x \log_3 x/\log_2 x$ for large x.

From de Bruijn [**2**],

$$\psi\left(x, \frac{3\log x \log_3 x}{\log_2 x}\right) \le \exp\left\{4\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right\}$$

for large x. Therefore, from (2.9) we have

$$\sum_{n\le x}{}^{*} C(n) \le x\left(\max_{k\le x} f(k)\right)\exp\left\{16\frac{\log x(\log_3 x)^2}{(\log_2 x)^2}\right\}$$

for large x. We now use Theorem 5.1 in [**3**] which asserts that

$$f(k) \le k\cdot\exp\left\{-\frac{\log k}{\log_2 k}\left(\log_3 k + \log_4 k + \frac{\log_4 k - 1}{\log_3 k} + c_5\left(\frac{\log_4 k}{\log_3 k}\right)^2\right)\right\}$$

for all large k and some constant c_5. The theorem now follows where we may choose c_2 as any constant with $c_2 < c_5$.

3. The lower bound.

THEOREM 3.1. *For all large x,*

$$\sum_{n\le x}\mu^2(n)G(n) > x^{1.68}.$$

PROOF. A key ingredient in the proof is a new result which comes out of the work of Balog [1], Fouvry [6], and Rousselet [16]:

THEOREM (BALOG-FOUVRY-ROUSSELET). *There is a constant $0 < c_6 < 0.32$ such that uniformly for all $y \ge x^{c_6}$ the number N of primes $p \le x$ with all primes in $p-1$ below y satisfies $N >> x/\log^2 x$.*

(The notation $f(x) >> g(x)$ is equivalent to $g(x) = O(f(x))$.) For a short discussion on the background of this kind of result, see [**5**]. The number of integers up to x divisible by a square-full number exceeding $\log^5 x$ is $O(x/\log^{5/2} x)$. We thus have the following corollary of the Balog-Fouvry-Rousselet theorem.

COROLLARY. *Uniformly for all $y \ge x^{c_6}$, the number N_1 of primes $p \le x$ with all primes in $p-1$ below y and with all square-full divisors of $p-1$ below $\log^5 x$ satisfies $N_1 >> x/\log^2 x$.*

Now choose numbers $c_7, \varepsilon > 0$ with

$$\frac{1}{c_6} > c_7 > \frac{1+\varepsilon}{0.32}.$$

Let $\mathbf{P}$ denote the set of primes p with

(i) $(\log x)^{c_7-\varepsilon} \le p \le (\log x)^{c_7}$,

(ii) all primes in $p-1$ are below $\log x/\log\log x$,

(iii) all square-full divisors of $p-1$ are below $(c_7\log\log x)^5$.

By the corollary, the cardinality of $\mathbf{P}$ satisfies

$$|\mathbf{P}| >> (\log x)^{c_7}/(\log\log x)^2.$$

Let

$$k = \left[\frac{\log x - 2\log x/\log\log x}{c_7 \log\log x}\right],$$

so that the product m of any k primes in $\mathbf{P}$ uniformly satisfies

$$x^{(c_7-\varepsilon)/c_7+o(1)} \le m \le x^{1-2/\log\log x}. \tag{3.1}$$

Finally, let $\mathbf{S}$ denote the set of all integers $m\alpha(\phi(m))$, where m is the product of k distinct primes in $\mathbf{P}$ and, as in §2, the function α gives the largest square-free divisor of its argument. By (ii) above

$$\alpha(\phi(m)) \le \prod_{q<\log x/\log\log x} q < x^{2/\log\log x} \tag{3.2}$$

for large x. Thus from (3.1), $x^{1-\varepsilon/c_7+o(1)} \le n < x$ uniformly for $n \in \mathbf{S}$.

We now show that if $n \in \mathbf{S}$, then $G(n)$ is very large. Indeed, from (1.1), if $n = m\alpha(\phi(m)) \in \mathbf{S}$ and $d = \alpha(\phi(m))$, then

$$\begin{aligned} G(n) &\ge \prod_{p|d} \frac{f(p,m)-1}{p-1} = \frac{1}{\phi(d)} \prod_{p|d} (f(p,m)-1) \\ &\ge \frac{1}{\phi(d)} \prod_{p|d} \frac{p-1}{p} f(p,m) = \frac{1}{d} f(d,m) \\ &= \frac{1}{d} \prod_{q|m} (d, q-1) \ge \frac{\phi(m)}{d(c_7 \log\log x)^{5k}} \\ &\ge x^{1-\varepsilon/c_7+o(1)} \end{aligned} \tag{3.3}$$

uniformly, using (3.1), (3.2), and property (iii) above. Therefore

$$\sum_{n\le x} \mu^2(n) G(n) \ge x^{1-\varepsilon/c_7+o(1)} |\mathbf{S}|$$

and it remains for us to estimate the cardinality of $\mathbf{S}$. But this is easy since

$$|\mathbf{S}| = \binom{|\mathbf{P}|}{k} \ge \left(\frac{|\mathbf{P}|}{k}\right)^k \ge x^{1-1/c_7+o(1)}.$$

Therefore

$$\sum_{n\le x} \mu^2(n) G(n) \ge x^{2-(1+\varepsilon)/c_7+o(1)}.$$

But by the choice of ε and c_7, we have $2 - (1+\varepsilon)/c_7 > 1.68$, which proves the theorem.

4. The conditional lower bound. In this section we give a stronger result than Theorem 3.1, but it depends on an unproved hypothesis. Recall that $\psi(x,y)$ denotes the number of integers $n \le x$ with all primes in n not exceeding y.

CONJECTURE. *For each $\varepsilon > 0$, the number $N(x,y)$ of primes p in $[x/2, x]$ with $p-1$ square-free and all primes in $p-1$ not exceeding y satisfies $N(x,y) >> \psi(x,y)/\log x$ uniformly for $y > \exp((\log x)^\varepsilon)$.*

It might seem more appropriate to compare $N(x,y)$ with $\psi_0(x,y)$, the number of square-free $n \le x$ with no prime in n exceeding y. However, Ivić and Tenenbaum [**10**] recently showed that

$$\psi_0(x,y) \sim \frac{6}{\pi^2}\psi(x,y) \quad \text{as } x \to \infty \text{ and } \frac{\log y}{\log\log x} \to \infty$$

and that for any $\varepsilon > 0$, $\psi_0(x,y) >> \psi(x,y)$ uniformly for $y > (\log x)^{2+\varepsilon}$. In any event we shall only be interested in the conjecture for $y \approx \exp(\sqrt{\log x})$.

THEOREM 4.1. *If the conjecture is true, there is a constant c_8 such that*

$$\sum_{n \le x} \mu^2(n)G(n)$$
$$> x^2 \exp\left\{-\frac{\log x}{\log_2 x}\left(\log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + c_8\left(\frac{\log_4 x}{\log_3 x}\right)^2\right)\right\}.$$

PROOF. The proof parallels that of Theorem 3.1, but we use the conjecture rather than the Balog-Fouvry-Rousselet theorem. Let $\mathbf{P}$ denote the set of primes p with

(i) $p \in [\frac{1}{2}e^{(\log_2 x)^2}, e^{(\log_2 x)^2}]$,
(ii) every prime in $p-1$ is below $\log x/(\log_2 x)^2$,
(iii) $p-1$ is square-free.

By the conjecture,

$$|\mathbf{P}| >> \psi(e^{(\log_2 x)^2}, \log x/(\log_2 x)^2)/(\log_2 x)^2.$$

From [**3**], we thus have

$$|\mathbf{P}| > \exp\left\{(\log_2 x)^2 - \log_2 x\left(\log_3 x + \log_4 x - 1 + \frac{\log_4 x - 1}{\log_3 x} + c_9\left(\frac{\log_4 x}{\log_3 x}\right)^2\right)\right\}$$

for some constant c_9. Let

$$k = \left[\frac{\log x - 2\log x/(\log_2 x)^2}{(\log_2 x)^2}\right].$$

If m is the product of k primes in $\mathbf{P}$, then

$$x^{1-3(\log_2 x)^{-2}} < m \le x^{1-2(\log_2 x)^{-2}} \tag{4.1}$$

for large x. Let $\mathbf{S}$ denote the set of all $m\alpha(\phi(m))$, where m runs over the integers composed of k distinct primes in $\mathbf{P}$. Then

$$\alpha(\phi(m)) \le \prod_{p < \log x/(\log_2 x)^2} p < x^{2(\log_2 x)^{-2}} \tag{4.2}$$

for large x, so that if $n \in \mathbf{S}$, then

$$x^{1-3(\log_2 x)^{-2}} < n < x.$$

Write $n \in \mathbf{S}$ in the form md, where m is the product of k distinct primes in $\mathbf{P}$ and $d = \alpha(\phi(m))$. Then from (3.3),

$$G(n) \ge \frac{1}{d}f(d,m) = \frac{1}{d}\phi(m) > x^{1-6(\log_2 x)^{-2}}$$

for large x by (4.1) and (4.2). Thus for large x,

$$\sum_{n \le x} \mu^2(n)G(n) > x^{1-6(\log_2 x)^{-2}}|\mathbf{S}|.$$

But

$$|\mathbf{S}| = \binom{|\mathbf{P}|}{k} > x \cdot \exp\left\{-\frac{\log x}{\log_2 x}\left(\log_3 x + \log_4 x + \frac{\log_4 - 1}{\log_3 x} + c_{10}\left(\frac{\log_4 x}{\log_3 x}\right)^2\right)\right\}$$

for any $c_{10} > c_9$ and all large x depending on the choice of c_{10}. Thus the theorem is proved for any $c_8 > c_{10}$.

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