ON THE AVERAGE NUMBER OF GROUPS OF SQUARE-FREE ORDER

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ABSTRACT. Let \( G(n) \) denote the number of (nonisomorphic) groups of order \( n \). It is shown here that for large \( x \)

\[
x^{1.68} \leq \sum'_{n \leq x} G(n) \leq x^2 \cdot \exp\left\{ (1 + o(1)) \log x \log \log \log x / \log \log x \right\},
\]

where \( \sum' \) denotes a sum over square-free \( n \). Under an unproved hypothesis on the distribution of primes \( p \) with all primes in \( p - 1 \) small, it is shown that the upper bound is tight.

1. Introduction. Let \( G(n) \) denote the number of isomorphism classes among the groups of order \( n \). For \( n \) square-free, there is a relatively simple formula for \( G(n) \) due to Hölder [9]. First let

\[
f(n, m) = \prod_{q \mid m} (n, q - 1).
\]

(Throughout the paper, the letters \( p, q \) will denote primes.) Then

\[
(1.1) \quad G(n) = \sum_{d \mid n} \prod_{p \mid d} \frac{f(p, n/d) - 1}{p - 1}, \quad n \text{ square-free}.
\]

With this elegant formula, the techniques of number theory can be applied to give several interesting results about \( G(n) \) for \( n \) square-free. For example, in Murty and Murty [11], it is shown that

\[
\sum_{n \leq x} \mu^2(n) \log G(n) = (c_1 + o(1)) x \log \log x
\]

for a certain positive constant \( c_1 \). (The square of the Möbius function \( \mu^2(n) \) is the characteristic function of the square-free integers.) Thus the geometric mean of \( G(n) \) for square-free \( n \leq x \) is about \((\log x)^{\pi^2 c_1 / 6}\). In Erdös, Murty, and Murty [4] it is shown that \( \mu^2(n) \log G(n) / \log \log n \) has a continuous, strictly increasing distribution function on \([0, \infty)\).

The maximal order of \( \mu^2(n)G(n) \) is somewhat different. Murty and Srinivasan [12] recently showed that for all \( n \),

\[
(1.2) \quad \mu^2(n)G(n) \leq n / (\log n)^A \log_3 n
\]

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for a certain positive constant $A$, where $\log_k n$ denotes the $k$-fold iterated natural logarithm. They also showed that the estimate (1.2) is essentially best possible, for there are infinitely many $n$ with

$$
\mu^2(n)G(n) \geq n/(\log n)^B \log n
$$

for some positive constant $B$.

This paper will be concerned with the average order of $\mu^2(n)G(n)$. From the distribution function result cited above it follows that for any $K$, the set of $n$ with $\mu^2(n)G(n) > (\log n)^K$ has positive asymptotic density. From this fact and from (1.2) it follows that for any fixed $K$ and all large $x$ (depending on the choice of $K$),

$$
x(\log x)^K \leq \sum_{n \leq x} \mu^2(n)G(n) \leq x^2 / (\log x)^A \log x.
$$

Below it is shown that for large $x$,

$$
x^{1.68} \leq \sum_{n \leq x} \mu^2(n)G(n) \leq x^2 / \exp\{(1 + o(1)) \log x \log_3 x / \log_2 x\}.
$$

Moreover, the equation

$$
\sum_{n \leq x} \mu^2(n)G(n) = x^2 / \exp\{(1 + o(1)) \log x \log_3 x / \log_2 x\}
$$

is shown modulo a reasonable conjecture on the distribution of primes $p$ with all primes in $p-1$ small. The "$o(1)$" appearing in (1.3) and (1.4) is partially expanded to

$$
\frac{1}{\log x} \left( \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + O\left(\left(\frac{\log_4 x}{\log_3 x}\right)^2\right) \right).
$$

Let $C(n)$ denote the number of isomorphism classes among all groups of order $n$ which have all of their Sylow subgroups cyclic. Then for $n$ square-free, $G(n) = C(n)$. In [11], a formula that generalizes the Hölder formula (1.1) is given. Namely it is shown that

$$
C(n) = \sum_{d \mid n} \prod_{d \mid n} \sum_{a=1}^{\phi(p)} f(p^i, n/d) - f(p^{i-1}, n/d) / p^{i-1}(p-1),
$$

where we write $m \mid n$ if $m \mid n$ and $(m, n/m) = 1$. In [12], (1.2) is actually shown via the stronger result

$$
C(n) \leq n / (\log n)^A \log_3 n.
$$

Below, the upper bound in (1.3) is shown by proving the stronger result that the same upper bound holds for $\sum_{n \leq x} C(n)$.

In [11], it is shown that $C(n) \leq f(n, n)$ for all $n$. Thus the upper bound result of this paper could possibly be achieved by proving a similar result for $\sum_{n \leq x} f(n, n)$. Although superficially simpler, this sum does not appear so easy to estimate. Probably the same upper bound could be worked out, but I have not completed all of the details. It should be remarked that essentially the same upper bound is established in [5] for the sum $\sum'_{n \leq x} f(n-1, n)$, where the dash indicates the sum is over composite integers.
If we drop the restriction that \( n \) is square-free, the behavior of \( G(n) \) changes markedly. In fact
\[
x^{C \log^2 x} \leq \sum_{n \leq x} G(n) \leq x^{D \log^2 x}
\]
holds for certain positive constants \( C, D \). The lower bound follows from restricting \( n \) to prime powers and using results of Higman [8] and Sims [17] on \( p \)-groups. The upper bound follows from work of Neumann [13] and the recent classification of the finite simple groups. In fact, from these papers it follows that \( C \) can be chosen as any number smaller than \( 2/(27 \log^2 2) \) and \( D \) can be chosen as any number greater than \( 1/(2 \log^2 2) \).

2. The upper bound.

**Theorem 2.1.** There is a constant \( c_2 \) such that for all large \( x \),
\[
\frac{1}{x} \sum_{n \leq x} C(n) \leq x \cdot \exp \left\{ - \frac{\log x}{\log_2 x} \left( \log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + c_2 \left( \frac{\log_4 x}{\log_3 x} \right)^2 \right) \right\}.
\]

**Proof.** From (1.5) we have (\( \phi \) denotes Euler’s function)
\[
C(n) = \sum_{d \mid n} \frac{1}{\phi(d)} \prod_{p^a \mid d} \sum_{i=1}^{a} p^{-i} (f(p^i, n/d) - f(p^{i-1}, n/d))
\leq \sum_{d \mid n} \frac{1}{\phi(d)} \prod_{p^a \mid d} \sum_{i=1}^{a} p^{-i} f(p^i, n/d).
\]

Note that
\[
\prod_{p^a \mid d} \sum_{i=1}^{a} p^{-i} f(p^i, n/d) \leq d \sum_{\delta | d} \prod_{p^a \mid \delta} p^{-i} f(p^i, n/d)
= d \sum_{\delta | d} \delta^{-1} f(\delta, n/d) \leq \sigma(d) f(d, n/d),
\]
where \( \sigma \) is the sum of the divisors function. Thus
\[
(2.2) \quad C(n) \leq \sum_{d \mid n, \, \phi(d) > 1} \frac{\sigma(d)}{\phi(d)} f \left( d, \frac{n}{d} \right).
\]

From the Hardy-Ramanujan inequality [7], the number of integers \( n \leq x \) with \( \omega(n) > 2 \log x \log_3 x / (\log_2 x)^2 \) (where \( \omega(n) \) is the number of distinct prime factors of \( n \)) is less than
\[
x \cdot \exp(-1.5 \log x \log_3 x / \log_2 x)
\]
for large \( x \). Thus by virtue of (1.6) (or the easier inequality \( C(n) \leq f(n, n) \leq \phi(n) \) from [11]), we may ignore such \( n \) in the sum (2.1). Let \( \sum^* \) denote a sum over integers \( n \) with
\[
(2.3) \quad \omega(n) \leq 2 \log x \log_3 x / (\log_2 x)^2.
\]
From (2.2) we have

\[
\sum_{n \leq x}^* C(n) \leq \sum_{d \leq x}^* \frac{\sigma(d)}{\phi(d)} \sum_{m \leq x/d \atop p \mid d \Rightarrow f(p, m) > 1} f(d, m).
\]

Let \(\alpha(n)\) denote the largest square-free divisor of \(n\). Then if \(m\) is such that \(p \mid d\) implies \(f(p, m) > 1\), then

\[
\alpha(f(d, m)) = \alpha(d).
\]

If \(k\) is any integer with \(\alpha(k) = \alpha(d)\), let

\[
N_{k, d}(y) = \sum_{m \leq y \atop f(d, m) = k}^* 1.
\]

Thus (2.4) is now transformed to

\[
\sum_{n \leq x}^* C(n) \leq \sum_{d \leq x}^* \frac{\sigma(d)}{\phi(d)} \sum_{k \leq x/d \atop \alpha(k) = \alpha(d)} kN_{k, d}\left(\frac{x}{d}\right).
\]

We now turn our attention to bounding \(N_{k, d}(y)\). If \(m\) is such that \(f(d, m) = k\), we write \(m = m_1m_2\) where \(m_1\) is the product of the distinct primes \(q\mid m\) with \((d, q_j - 1) > 1\). Say the prime factorization of \(m_1\) is \(q_1 \cdots q_s\). Let \(k_j = (d, q_j - 1)\) for \(j = 1, \ldots, s\). Thus \(\prod_{j=1}^s k_j = k\) and each \(k_j > 1\). That is, the multiset \(\{k_1, \ldots, k_s\}\) is a factorization of \(k\). (By a factorization of an integer \(k\) we mean an unordered multiset of integers exceeding 1 whose product is \(k\).) Let \(\mathcal{F}(k)\) denote the set of all factorizations of \(k\). For each factorization \(\mathcal{F}\) in \(\mathcal{F}(k)\), let \(N_{\mathcal{F}, d}(y)\) denote the number of \(m\) counted by \(N_{k, d}(y)\) which give rise to the factorization \(\mathcal{F}\) as described above. Thus

\[
N_{k, d}(y) = \sum_{\mathcal{F} \in \mathcal{F}(k)} N_{\mathcal{F}, d}(y).
\]

For \(\mathcal{F} = \{k_1, \ldots, k_s\}\), we have

\[
N_{\mathcal{F}, d}(y) \leq y \sum_{m_1 \leq y} \frac{1}{m_1},
\]

where \(m_1\) runs over integers of the form \(q_1 \cdots q_s\), the \(q_i\)'s being distinct primes with \((d, q_j - 1) = k_j\) for \(j = 1, \ldots, s\). Thus

\[
N_{\mathcal{F}, d}(y) \leq y \prod_{j=1}^s \sum_{q \leq y \atop \text{q } \equiv 1 \mod k_j} \frac{1}{q} \leq y \prod_{j=1}^s \frac{\log \log y + c_3 \log k_j}{\phi(k_j)}
\]

for some absolute constant \(c_3\) (see Norton [14] or Pomerance [15]).

Note that \(s = \omega(m_1) \leq \omega(m)\) so that if \(m\) is counted by \(N_{k, d}(y)\), we have by (2.3) that

\[
(2.8) \quad s \leq 2\log x\log_3 x/\left(\log_2 x\right)^2.
\]
We now majorize the product \( \prod_{j=1}^{s} (\log \log y + c_3 \log k_j) \) (for \( k, y \leq x \)) by breaking it into two parts corresponding to \( k_j \leq \exp((\log x)^3) \) and \( k_j \geq \exp((\log x)^3) \). The first part is at most, by (2.8),

\[
(\log_2 x + c_3 (\log_2 x)^3)^s \leq \exp \left\{ \frac{7 \log x (\log_3 x)^2}{(\log_2 x)^2} \right\}
\]

for large \( x \). The number of factors in the second part is \( O(\log x/(\log_2 x)^3) \), so the second part is majorized by

\[
(\log_2 x + c_3 \log x)^{O(\log x/(\log_2 x)^3)} = \exp \left\{ O \left( \frac{\log x}{(\log_2 x)^2} \right) \right\}.
\]

Therefore, for large \( x \) and \( k, y \leq x \),

\[
\prod_{j=1}^{s} (\log \log y + c_3 \log k_j) \leq \exp \left\{ \frac{8 \log x (\log_3 x)^2}{(\log_2 x)^2} \right\}.
\]

Next note that by (2.8),

\[
k^{-1} \prod_{j=1}^{s} \phi(k_j) = \prod_{j=1}^{s} k_j / \phi(k_j) \leq (c_4 \log_2 x)^s \leq \exp\{3 \log x (\log_3 x)^2/(\log_2 x)^2\}
\]

for large \( x \).

Putting these estimates into (2.7), we have

\[
N_{\tau, d}(y) \leq \frac{y}{k} \exp \left\{ \frac{11 \log x (\log_3 x)^2}{(\log_2 x)^2} \right\}.
\]

Thus from (2.6) we have

\[
k N_{k, d}(y) \leq y f(k) \exp \left\{ \frac{11 \log x (\log_3 x)^2}{(\log_2 x)^2} \right\},
\]

where \( f(k) \) is the cardinality of \( \mathcal{F}(k) \), that is, the number of factorizations of \( k \).

From (2.5) we now have

\[
(2.9) \quad \sum_{n \leq x} \sum_{d \leq x}^* \frac{\sigma(d)}{d \phi(d)} \left( \sum_{k \leq x/d \atop \alpha(k) = \alpha(d)} f(k) \right) \exp \left\{ \frac{11 \log x (\log_3 x)^2}{(\log_2 x)^2} \right\}.
\]

But note that

\[
\sum_{d \leq x} \frac{\sigma(d)}{d \phi(d)} = O(\log x).
\]

Also note that since \( d \) satisfies (2.3), for large \( x \)

\[
\sum_{k \leq x/d \atop \alpha(k) = \alpha(d)} 1 \leq \psi \left( \frac{x}{d \alpha(d)}, \frac{3 \log x \log_3 x}{\log_2 x} \right) \leq \psi \left( x, \frac{3 \log x \log_3 x}{\log_2 x} \right),
\]

where \( \psi(x, y) \) denotes the number of integers up to \( x \), none of whose primes exceed \( y \). Indeed, if \( \alpha(k) = \alpha(d) \), then \( \alpha(d)|k \) and all of the primes in \( k/\alpha(d) \) are among
the primes in $d$. Thus the number of such $k \leq x/d$ is at most the number of integers below $x/\omega(d)$ all of whose primes are among the first $\omega(d)$ primes. But using (2.3) the $\omega(d)$th prime is less than $3 \log x \log_3 x / \log_2 x$ for large $x$.

From de Bruijn [2],

$$
\psi \left( x, \frac{3 \log x \log_3 x}{\log_2 x} \right) \leq \exp \left\{ 4 \frac{\log x (\log_3 x)^2}{(\log_2 x)^2} \right\}
$$

for large $x$. Therefore, from (2.9) we have

$$
\sum_{n \leq x}^* C(n) \leq x \left( \max_{k \leq x} f(k) \right) \exp \left\{ 16 \frac{\log x (\log_3 x)^2}{(\log_2 x)^2} \right\}
$$

for large $x$. We now use Theorem 5.1 in [3] which asserts that

$$
f(k) \leq k \cdot \exp \left\{ -\frac{\log k}{\log_2 k} \left( \log_3 k + \log_4 k + \frac{\log_4 k - 1}{\log_3 k} + c_5 \left( \frac{\log_4 k}{\log_3 k} \right)^2 \right) \right\}
$$

for all large $k$ and some constant $c_5$. The theorem now follows where we may choose $c_2$ as any constant with $c_2 < c_5$.

3. The lower bound.

**Theorem 3.1.** For all large $x$,

$$
\sum_{n \leq x} \mu^2(n) G(n) > x^{1.68}.
$$

**Proof.** A key ingredient in the proof is a new result which comes out of the work of Balog [1], Fouvry [6], and Rousset [16]:

**Theorem (Balog-Fouvry-Rousset).** There is a constant $0 < c_6 < 0.32$ such that uniformly for all $y \geq x^{c_6}$ the number $N$ of primes $p \leq x$ with all primes in $p - 1$ below $y$ satisfies $N \gg x^{1/2} / \log^2 x$.

(The notation $f(x) \gg g(x)$ is equivalent to $g(x) = O(f(x))$.) For a short discussion on the background of this kind of result, see [5]. The number of integers up to $x$ divisible by a square-full number exceeding $\log^2 x$ is $O(x / \log^{5/2} x)$. We thus have the following corollary of the Balog-Fouvry-Rousset theorem.

**Corollary.** Uniformly for all $y \geq x^{c_6}$, the number $N_1$ of primes $p \leq x$ with all primes in $p - 1$ below $y$ and with all square-full divisors of $p - 1$ below $\log^5 x$ satisfies $N_1 \gg x / \log^2 x$.

Now choose numbers $c_7, \varepsilon > 0$ with

$$
\frac{1}{c_6} > c_7 > \frac{1 + \varepsilon}{0.32}.
$$

Let $P$ denote the set of primes $p$ with

(i) $(\log x)^{c_7 - \varepsilon} \leq p \leq (\log x)^{c_7},$

(ii) all primes in $p - 1$ are below $\log x / \log \log x,$

(iii) all square-full divisors of $p - 1$ are below $(c_7 \log \log x)^5.$

By the corollary, the cardinality of $P$ satisfies

$$
|P| \gg (\log x)^{c_7} / (\log \log x)^2.
$$
Let
\[ k = \left\lfloor \frac{\log x - 2 \log x / \log \log x}{c_7 \log \log x} \right\rfloor, \]
so that the product \( m \) of any \( k \) primes in \( \mathbf{P} \) uniformly satisfies
\[ x^{(c_7 - \varepsilon)/c_7 + o(1)} \leq m \leq x^{1 - 2/\log \log x}. \tag{3.1} \]

Finally, let \( \mathbf{S} \) denote the set of all integers \( m \alpha(\phi(m)) \), where \( m \) is the product of \( k \) distinct primes in \( \mathbf{P} \) and, as in §2, the function \( \alpha \) gives the largest square-free divisor of its argument. By (ii) above
\[ \alpha(\phi(m)) \leq \prod_{q < \log x / \log \log x} q < x^{2/\log \log x} \tag{3.2} \]
for large \( x \). Thus from (3.1), \( x^{1 - \varepsilon/c_7 + o(1)} \leq n < x \) uniformly for \( n \in \mathbf{S} \).

We now show that if \( n \in \mathbf{S} \), then \( G(n) \) is very large. Indeed, from (1.1), if \( n = m \alpha(\phi(m)) \in \mathbf{S} \) and \( d = \alpha(\phi(m)) \), then
\[ G(n) \geq \prod_{p \mid d} \frac{f(p, m)}{p - 1} = \frac{1}{\phi(d)} \prod_{p \mid d} (f(p, m) - 1) \]
\[ \geq \frac{1}{\phi(d)} \prod_{p \mid d} \frac{p - 1}{p} f(p, m) = \frac{1}{d} f(d, m) \]
\[ = \frac{1}{d} \prod_{q \mid m} (d, q - 1) \geq \frac{\phi(m)}{d(c_7 \log \log x)^{5k}} \]
\[ \geq x^{1 - \varepsilon/c_7 + o(1)} \tag{3.3} \]
uniformly, using (3.1), (3.2), and property (iii) above. Therefore
\[ \sum_{n \leq x} \mu^2(n) G(n) \geq x^{1 - \varepsilon/c_7 + o(1)} |\mathbf{S}| \]
and it remains for us to estimate the cardinality of \( \mathbf{S} \). But this is easy since
\[ |\mathbf{S}| = \left( \frac{|\mathbf{P}|}{k} \right)^k \geq x^{1 - 1/c_7 + o(1)}. \]
Therefore
\[ \sum_{n \leq x} \mu^2(n) G(n) \geq x^{2 - (1 + \varepsilon)/c_7 + o(1)}. \]
But by the choice of \( \varepsilon \) and \( c_7 \), we have \( 2 - (1 + \varepsilon)/c_7 > 1.68 \), which proves the theorem.

4. The conditional lower bound. In this section we give a stronger result than Theorem 3.1, but it depends on an unproved hypothesis. Recall that \( \psi(x, y) \) denotes the number of integers \( n \leq x \) with all primes in \( n \) not exceeding \( y \).

**Conjecture.** *For each \( \varepsilon > 0 \), the number \( N(x, y) \) of primes \( p \) in \([x/2, x]\) with \( p - 1 \) square-free and all primes in \( p - 1 \) not exceeding \( y \) satisfies \( N(x, y) \geq \psi(x, y)/\log x \) uniformly for \( y > \exp((\log x)^\varepsilon) \).*
It might seem more appropriate to compare $N(x,y)$ with $\psi_0(x,y)$, the number of square-free $n \leq x$ with no prime in $n$ exceeding $y$. However, Ivić and Tenenbaum [10] recently showed that

$$\psi_0(x,y) \sim \frac{6}{\pi^2} \psi(x,y) \quad \text{as } x \to \infty \quad \text{and} \quad \frac{\log y}{\log \log x} \to \infty$$

and that for any $\varepsilon > 0$, $\psi_0(x,y) \gg \psi(x,y)$ uniformly for $y > (\log x)^{2+\varepsilon}$. In any event we shall only be interested in the conjecture for $y \approx \exp(\sqrt{\log x})$.

**Theorem 4.1.** If the conjecture is true, there is a constant $c_8$ such that

$$\sum_{n \leq x} \mu^2(n) G(n)$$

$$\geq x^2 \exp \left\{-\frac{\log x}{\log_2 x} \left( \log_3 x + \log_4 x + \frac{\log_4 x - 1}{\log_3 x} + c_8 \left( \frac{\log_4 x}{\log_3 x} \right)^2 \right) \right\}.$$

**Proof.** The proof parallels that of Theorem 3.1, but we use the conjecture rather than the Balog-Fouvry-Rousselet theorem. Let $\mathbf{P}$ denote the set of primes $p$ with

(i) $p \in [\frac{1}{2} e^{(\log_2 x)^2}, e^{(\log_2 x)^2}]$,

(ii) every prime in $p - 1$ is below $\log x/(\log_2 x)^2$,

(iii) $p - 1$ is square-free.

By the conjecture,

$$|\mathbf{P}| \gg \psi(e^{(\log_2 x)^2}, \log x/(\log_2 x)^2)/(\log_2 x)^2.$$  

From [3], we thus have

$$|\mathbf{P}| \geq \exp \left\{(\log_2 x)^2 - \log_2 x \left( \log_3 x + \log_4 x - 1 + \frac{\log_4 x - 1}{\log_3 x} + c_9 \left( \frac{\log_4 x}{\log_3 x} \right)^2 \right) \right\}$$

for some constant $c_9$. Let

$$k = \left\lceil \frac{\log x - 2 \log x/(\log_2 x)^2}{(\log_2 x)^2} \right\rceil.$$ 

If $m$ is the product of $k$ primes in $\mathbf{P}$, then

$$x^{1-3(\log_2 x)^{-2}} < m \leq x^{1-2(\log_2 x)^{-2}}$$

for large $x$. Let $\mathbf{S}$ denote the set of all $m\alpha(\phi(m))$, where $m$ runs over the integers composed of $k$ distinct primes in $\mathbf{P}$. Then

$$\alpha(\phi(m)) \leq \prod_{p < \log x/(\log_2 x)^2} p < x^{2(\log_2 x)^{-2}}$$

for large $x$, so that if $n \in \mathbf{S}$, then

$$x^{1-3(\log_2 x)^{-2}} < n < x.$$ 

Write $n \in \mathbf{S}$ in the form $md$, where $m$ is the product of $k$ distinct primes in $\mathbf{P}$ and $d = \alpha(\phi(m))$. Then from (3.3),

$$G(n) \geq \frac{1}{d} f(d,m) = \frac{1}{d} \phi(m) > x^{1-6(\log_2 x)^{-2}}$$
for large $x$ by (4.1) and (4.2). Thus for large $x$,

$$\sum_{n \leq x} \mu^2(n)G(n) > x^{1 - 6(\log_2 x)^{-2}} |S|.$$ 

But

$$|S| = \left( \frac{|P|}{k} \right) > x \cdot \exp \left\{ - \frac{\log x}{\log_2 x} \left( \log_3 x + \log_4 x + \frac{\log_4 - 1}{\log_3 x} + c_{10} \left( \frac{\log_4 x}{\log_3 x} \right)^2 \right) \right\}$$

for any $c_{10} > c_9$ and all large $x$ depending on the choice of $c_{10}$. Thus the theorem is proved for any $c_8 > c_{10}$.

REFERENCES


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