

# Irreducible radical extensions and Euler-function chains

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## Abstract

We discuss the smallest algebraic number field which contains the  $n$ th roots of unity and which may be reached from the rational field  $\mathbb{Q}$  by a sequence of irreducible, radical, Galois extensions. The degree  $M(n)$  of this field over  $\mathbb{Q}$  is  $\phi(m)$ , where  $m$  is the smallest multiple of  $n$  divisible by each prime factor of  $\phi(m)$ . The prime factors of  $m/n$  are precisely the primes not dividing  $n$  but which do divide some number in the “Euler chain”  $\phi(n), \phi(\phi(n)), \dots$ . For each fixed  $k$ , we show that  $M(n) > n^k$  on a set of asymptotic density 1.

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# 1 Introduction

Throughout this paper, all fields which appear are of characteristic zero. Let  $\mathbb{K} \subset \mathbb{L}$  be a field extension. We say  $\mathbb{L}$  is *prime radical* if  $\mathbb{L} = \mathbb{K}[\alpha]$ , where  $\alpha^p \in \mathbb{K}$  for some prime  $p$ , and the polynomial  $f(X) = X^p - \alpha^p \in \mathbb{K}[X]$  is irreducible. Note that for such an extension to also be Galois it is necessary and sufficient that the  $p$ th roots of unity lie in  $\mathbb{K}$ .

The present paper is motivated by the following situation. Not all solvable extensions  $\mathbb{K} \subset \mathbb{L}$  can be decomposed into a chain of prime radical Galois extensions. But perhaps it is possible for such a chain to exist from  $\mathbb{K}$  to a field  $\mathbb{M}$  containing  $\mathbb{L}$ . In fact this is always the case, which we record as follows.

**Theorem 1.** *Let  $\mathbb{K} \subset \mathbb{L}$  be a solvable extension of characteristic zero fields lying in an algebraically closed field  $\mathbb{U}$ . There is a unique minimal extension  $\mathbb{L} \subset \mathbb{M} \subset \mathbb{U}$  such that  $\mathbb{M}$  can be reached from  $\mathbb{K}$  by a sequence of prime radical extensions which are also Galois.*

For example, say  $\mathbb{K} = \mathbb{Q}$  and  $\mathbb{L} = \mathbb{Q}(\zeta_7)$ , where in general we let  $\zeta_n$  denote a primitive  $n$ th root of unity. This extension is not only solvable, it is cyclic. The field  $\mathbb{L}$  has degree 6 over  $\mathbb{Q}$ , and there is the intermediate field  $\mathbb{A} = \mathbb{Q}(\zeta_7 + \zeta_7^2 + \zeta_7^4)$  of degree 2 over  $\mathbb{Q}$ . Clearly every field extension of degree 2 is prime radical and Galois, so there is no problem here. But the degree-3 extension from  $\mathbb{A}$  to  $\mathbb{L}$  is Galois, so cannot be prime radical, since the cube roots of unity are not present. There is no getting around an extension of degree 3 at some point, so we throw in the cube roots of 1, giving us a prime radical degree-2 extension  $\mathbb{B}$  of  $\mathbb{A}$ . It is then possible to show that the degree-3 extension  $\mathbb{B}(\zeta_7)$  over  $\mathbb{B}$  is in fact prime radical, and of course Galois. So

$$\mathbb{M} = \mathbb{Q}(\zeta_7 + \zeta_7^2 + \zeta_7^4)(\zeta_3)(\zeta_7) = \mathbb{Q}(\zeta_{21}),$$

a field of degree 12 over  $\mathbb{Q}$ .

Let us consider more generally the case for  $\mathbb{K} = \mathbb{Q}(\zeta_n)$ . We write  $M(n)$  for the degree of the field  $\mathbb{M}$  determined in Theorem 1. Here, we present a formula for  $M(n)$ . Let  $\phi_k(n)$  be the  $k$ th iterate of the Euler function  $\phi$  at  $n$ . By convention, we have  $\phi_0(n) = n$  and  $\phi_1(n) = \phi(n)$ .

**Theorem 2.** *Let  $F(n)$  be the least common multiple of  $n$  and the largest squarefree divisor of  $\prod_{k \geq 1} \phi_k(n)$ . Then the field  $\mathbb{M}$  determined in Theorem 1*

with  $\mathbb{K} = \mathbb{Q}$  and  $\mathbb{L} = \mathbb{Q}(\zeta_n)$  is  $\mathbb{Q}(\zeta_{F(n)})$ , which has degree  $M(n) = \phi(F(n))$  over  $\mathbb{Q}$ .

These two results were communicated to us by Hendrik Lenstra; we warmly thank him for allowing us to present his argument. Our contribution is in the following result which shows that  $M(n)$ , for most positive integers  $n$ , grows faster than any fixed power of  $n$ .

**Theorem 3.** *For each  $\varepsilon > 0$ , the set of natural numbers  $n$  for which*

$$M(n) > n^{(1-\varepsilon) \log \log n / \log \log \log n}$$

*has asymptotic density 1.*

Throughout this paper, we use  $c_0, c_1, \dots$  to denote computable positive constants and  $x$  to denote a positive real number. We also use the Landau symbols  $O$  and  $o$  and the Vinogradov symbols  $\gg$  and  $\ll$  with their usual meanings. We write  $\log x$  for the maximum of 1 and the natural logarithm of  $x$ . We write  $p$  and  $q$  for prime numbers.

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## 2 The proofs of Theorem 1 and Theorem 2

We prove two lemmas. The first gives a sufficient condition for an extension  $\mathbb{K} \subset \mathbb{L}$  to be reached from  $\mathbb{K}$  by a sequence of prime radical Galois extensions.

**Lemma 4.** *If  $\mathbb{K} \subset \mathbb{L}$  is Galois with a solvable Galois group, and  $\zeta_p \in \mathbb{L}$  for each prime  $p$  dividing  $[\mathbb{L} : \mathbb{K}]$ , then  $\mathbb{L}$  can be reached from  $\mathbb{K}$  by a sequence of prime radical extensions that are Galois.*

*Proof.* The proof relies on the well-known fact from Kummer theory that a cyclic extension of prime degree  $p$  of a field  $\mathbb{K}$  containing a primitive  $p$ th root of 1 is prime radical. We now proceed by induction on  $[\mathbb{L} : \mathbb{K}]$ . If all  $\zeta_p \in \mathbb{K}$  for prime  $p \mid [\mathbb{L} : \mathbb{K}]$ , we then use the solvability of  $\text{Gal}(\mathbb{L}/\mathbb{K})$  to break up

the extension into a tower of cyclic extensions of prime degrees, and apply the above well-known fact to each of them. Otherwise, let  $p$  be minimal with  $\zeta_p \notin \mathbb{K}$ . We now break up the extension  $\mathbb{K} \subset \mathbb{L}$  into  $\mathbb{K} \subset \mathbb{K}(\zeta_p) \subset \mathbb{L}$  and deal with each piece inductively. By  $[\mathbb{K}(\zeta_p) : \mathbb{K}] < p$  and the choice of  $p$ , the above fact applies to the prime degree pieces into which the abelian extension  $\mathbb{K} \subset \mathbb{K}(\zeta_p)$  can be broken up, while the inductive hypothesis applies to  $\mathbb{K}(\zeta_p) \subset \mathbb{L}$ .  $\square$

The second lemma shows that the condition appearing in Lemma 4 is also necessary.

**Lemma 5.** *If  $\mathbb{K} \subset \mathbb{L}$  is Galois and  $\mathbb{L}$  can be reached from  $\mathbb{K}$  by a finite sequence of prime radical extensions (which are not necessarily Galois), then  $\zeta_p \in \mathbb{L}$  for each prime  $p \mid [\mathbb{L} : \mathbb{K}]$ .*

*Proof.* Say the promised sequence of fields is  $\mathbb{K} = \mathbb{K}_0 \subset \mathbb{K}_1 \subset \cdots \subset \mathbb{K}_n = \mathbb{L}$ , and let  $p$  be a prime factor of  $[\mathbb{L} : \mathbb{K}]$ . Then some  $[K_{i+1} : \mathbb{K}_i] = p$ . Since this extension is radical, it must have a polynomial of the form  $X^p - a$  where  $a \in \mathbb{K}_i$ . Thus  $a$  has a  $p$ th root  $b$  in  $\mathbb{L}$ , and since  $\mathbb{L}$  is Galois, all of the conjugates of  $b$  are in  $\mathbb{L}$ . In particular,  $b\zeta_p \in \mathbb{L}$ , so  $\zeta_p \in \mathbb{L}$ .  $\square$

We are now ready to prove Theorems 1 and 2.

*Proof of Theorem 1.* This follows immediately from Lemmas 4 and 5. Indeed, to obtain  $\mathbb{M}$  from  $\mathbb{L}$ , we first adjoin to  $\mathbb{L} = \mathbb{L}_0$  all  $\zeta_p$ , for  $p \mid [\mathbb{L} : \mathbb{K}]$  for which  $\zeta_p \notin \mathbb{L}$ . The resulting field  $\mathbb{L}_1$  is still Galois with a solvable group over  $\mathbb{K}$ . We now adjoin to  $\mathbb{L}_1$  all  $\zeta_p$  for  $p \mid [\mathbb{L}_1 : \mathbb{K}]$  for which  $\zeta_p \notin \mathbb{L}_1$ , and so reach a solvable Galois extension  $\mathbb{L}_2$  of  $\mathbb{K}$ . We continue to iterate the process until we reach a field  $\mathbb{M} = \mathbb{L}_n$  which contains all  $\zeta_p$  for  $p \mid [\mathbb{M} : \mathbb{K}]$ . Note that the iteration does indeed stabilize since the sequence of relative degrees  $[\mathbb{L}_{i+1} : \mathbb{L}_i]$  is strictly decreasing. (If  $[\mathbb{L}_i : \mathbb{L}_{i-1}] = d_i$ , then any  $\zeta_p$  adjoined to  $\mathbb{L}_i$  to form  $\mathbb{L}_{i+1}$  must have  $p \mid d_i$ . Hence this next extension has degree at most  $\prod_{p \mid d_i} (p-1) \leq \phi(d_i)$ . And if  $d_i > 1$ , we have  $\phi(d_i) < d_i$ .)

This construction creates the smallest field extension  $\mathbb{M}$  of  $\mathbb{L}$  which contains each  $\zeta_p$  for  $p$  prime and  $p \mid [\mathbb{M} : \mathbb{K}]$ . It follows from Lemma 4 that  $\mathbb{M}$  may be reached from  $\mathbb{K}$  by a sequence of prime radical Galois extensions. The minimality, and thus uniqueness of  $\mathbb{M}$  follows from Lemma 5.  $\square$

*Proof of Theorem 2.* We apply the algorithm described in the proof of Theorem 1 to  $\mathbb{K} = \mathbb{Q}$  and  $\mathbb{L} = \mathbb{Q}(\zeta_n)$ . We obtain  $\mathbb{M} = \mathbb{Q}(\zeta_m)$ , where  $m$  is the

least multiple of  $n$  that is divisible by all primes dividing  $\phi(m)$ . It is easy to see that

$$m = n \prod_{\substack{p|\phi_k(n) \text{ for some } k \geq 1 \\ p \nmid n}} p,$$

and we immediately recognize that  $m = F(n)$ . Thus,  $M(n) = [\mathbb{Q}[\zeta_m] : \mathbb{Q}] = \phi(m) = \phi(F(n))$ .  $\square$

### 3 The proof of Theorem 3

#### 3.1 Preliminary results

We recall a result from [2]:

**Proposition 6.** *There is an absolute constant  $c_1$  such that for each prime  $p$  and integer  $k \geq 0$ , the number of integers  $n \leq x$  with  $p \mid \phi_k(n)$  is at most  $(x/p)(c_1 \log \log x)^k$ .*

Let

$$F_K(n) = \prod_{0 \leq k \leq K} \phi_k(n).$$

One of our goals will be to establish the following result.

**Proposition 7.** *There is an absolute constant  $c_2$  such that for all sufficiently large numbers  $x$ , all numbers  $y \geq 1$  and all integers  $K \geq 1$ , the number of integers  $n \leq x$  with  $p^2 \mid F_K(n)$  for some prime  $p > y$  is at most  $(x/y)K(c_2 \log \log x)^{2K}$ .*

Let  $\Omega(n)$  denote the number of prime factors of  $n$  counted with multiplicity. We will also prove the following result.

**Proposition 8.** *The number of positive integers  $n \leq x$  with the property that  $\Omega(F_K(n)) > 2(5 \log \log x)^K$  is  $\ll (x/\log x)(c_1 \log \log x)^K$  uniformly in  $K$ , where  $c_1$  is the constant from Proposition 6.*

### 3.2 Proof of Theorem 3

Let  $x$  be a large positive real number and let  $0 < \varepsilon < 1$  be arbitrarily small and fixed. Put

$$K = \lceil (1 - \varepsilon) \log \log x / \log \log \log x \rceil.$$

Assume  $n \leq x$ , and factor  $F_K(n)$  as  $AB$ , where each prime in  $A$  is at most  $(\log x)^3$  and each prime in  $B$  exceeds  $(\log x)^3$ . Since

$$(x / \log x)(c_1 \log \log x)^K = o(x),$$

Proposition 8 implies that but for  $o(x)$  choices of the positive integer  $n \leq x$ , we have

$$A \leq (\log^3 x)^{2(5 \log \log x)^K} \leq \exp(2(5 \log \log x)^{K+1}) = x^{o(1)}.$$

By the minimal order of  $\phi(m)/m$  for  $m \leq x$ , we have that each one of the inequalities  $\phi_{j+1}(n)/\phi_j(n) > 1/(2 \log \log x)$  holds. We also may assume that  $n > x/(2 \log \log x)$ , so that

$$\begin{aligned} F_K(n) &= n^{K+1} \prod_{i=0}^K \frac{\phi_i(n)}{n} = n^{K+1} \prod_{i=0}^K \prod_{j=0}^{i-1} \frac{\phi_{j+1}(n)}{\phi_j(n)} \\ &> n^{K+1} / (2 \log \log x)^{1+2+\dots+K} > x^{K+1} / (2 \log \log x)^{(K+1)(K+2)/2} \\ &> x^{K+1/2} \end{aligned}$$

for  $x$  sufficiently large. Thus, but for  $o(x)$  choices for  $n \leq x$ , we have

$$B > x^{K+1/4}.$$

By Proposition 7, the number of  $n \leq x$  with  $p^2 \mid F_K(n)$  for some prime number  $p > \log^3 x$  is  $O(x / \log x)$ . Thus, for all but  $o(x)$  choices of  $n \leq x$ , the number  $B$  is squarefree. It is clear that  $B \mid F(n)$ , therefore  $\phi(B) \mid M(n)$ . From the minimal order of the Euler function, we have

$$\phi(B) > \frac{B}{2 \log \log B} > \frac{x^{K+1/4}}{2(\log(K+1/4) + \log \log x)} > \frac{x^{K+1/4}}{3 \log \log x} > x^K.$$

Thus,  $M(n) > x^K$  holds for all  $n \leq x$  with  $o(x)$  exceptions, which completes the proof of the theorem.  $\square$

### 3.3 Proofs of the preliminary results

Before we begin the proof of Proposition 7, we establish some helpful notation. For a positive integer  $m$ , let

$$\mathcal{P}_m = \{p \text{ prime} : p \equiv 0 \text{ or } 1 \pmod{m}\}.$$

By the Brun–Titchmarsh inequality and partial summation, we have

$$\sum_{\substack{p \in \mathcal{P}_m \\ p \leq x}} \frac{1}{p} \leq \frac{c_0}{\phi(m)} \log \log x \quad (1)$$

for some absolute constant  $c_0$  (see Lemma 1 in [1] or formula (3.1) in [2]). Note that from Theorem 3.5 in [2], we may (and do) take the constant  $c_1$  from Proposition 6 equal to  $2c_0$ . Let

$$\mathcal{S}_k(x, m) = \{n \leq x : m \mid \phi_k(n)\}, \quad S_k(x, m) = \#\mathcal{S}_k(x, m).$$

**Lemma 9.** *For all sufficiently large values of  $x$ , if  $q_1 \leq q_2$  are primes and  $k$  is any nonnegative integer, then*

$$S_k(x, q_1 q_2) \leq \frac{x}{q_1 q_2} (3c_0 \log \log x)^{2k}.$$

*Proof.* We proceed by induction on  $k$ . The result is clearly true for  $k = 0$ . Assume that the result holds at  $k$ . If  $q_1 q_2 \mid \phi_{k+1}(n)$ , then either  $p \mid \phi_k(n)$  for some  $p \in \mathcal{P}_{q_1 q_2}$ , or  $p_1 p_2 \mid \phi_k(n)$  for some  $p_1 \in \mathcal{P}_{q_1}$  and  $p_2 \in \mathcal{P}_{q_2}$ . Thus,

$$S_{k+1}(x, q_1 q_2) \leq \sum_{p \in \mathcal{P}_{q_1 q_2}} S_k(x, p) + \sum_{p_1 \in \mathcal{P}_{q_1}, p_2 \in \mathcal{P}_{q_2}} S_k(x, p_1 p_2).$$

Thus, by Proposition 6 and the induction hypothesis, we have that

$$S_{k+1}(x, q_1 q_2) \leq \sum_{\substack{p \in \mathcal{P}_{q_1 q_2} \\ p \leq x}} \frac{x}{p} (c_1 \log \log x)^k + \sum_{\substack{p_1 \in \mathcal{P}_{q_1}, p_2 \in \mathcal{P}_{q_2} \\ p_1 \leq x, p_2 \leq x}} \frac{x}{p_1 p_2} (3c_0 \log \log x)^{2k}.$$

We now use (1), and so get

$$\begin{aligned} S_{k+1}(x, q_1 q_2) &\leq \frac{x}{\phi(q_1 q_2)} (c_0 \log \log x) (c_1 \log \log x)^k \\ &\quad + \frac{x}{\phi(q_1) \phi(q_2)} (c_0 \log \log x)^2 (3c_0 \log \log x)^{2k} \\ &\leq \frac{x}{q_1 q_2} (3c_0 \log \log x) (c_1 \log \log x)^k + (2c_0 \log \log x)^2 (3c_0 \log \log x)^{2k}. \end{aligned}$$

Thus, using  $c_1 = 2c_0$ , the inequality at  $k + 1$  follows for all  $x$  beyond some uniform bound. Thus, the lemma has been proved.  $\square$

We introduce the following notation. Let

$$\mathcal{S}_K(x, y) = \bigcup_{\substack{0 \leq k \leq K \\ p > y, p \text{ prime}}} \mathcal{S}_k(x, p^2), \quad S_K(x, y) = \#\mathcal{S}_K(x, y).$$

For nonnegative integers  $k_1$  and  $k_2$  with  $k_1 < k_2$ , and primes  $q_1$  and  $q_2$ , let

$$\mathcal{S}_{k_1, k_2}(x, q_1, q_2) = \{n \leq x : q_1 \mid \phi_{k_1}(n), q_2 \mid \phi_{k_2}(n)\}.$$

**Lemma 10.** *Suppose that  $k_1, k_2$  and  $K$  are integers with  $0 \leq k_1 < k_2 \leq K$  and  $q_1$  and  $q_2$  are primes with  $q_2 > y$  and  $q_2$  not a divisor of  $\phi_{k_2-k_1}(q_1)$ . Then*

$$\#(\mathcal{S}_{k_1, k_2}(x, q_1, q_2) - \mathcal{S}_K(x, y)) \leq \frac{x}{q_1 q_2} (3c_0 \log \log x)^{k_1 + k_2}.$$

*Proof.* We first show that if  $\phi_j(m)$  is not divisible by the square of any prime exceeding  $y$  for  $0 \leq j \leq k - 1$ , then for each prime  $q \mid \phi_k(m)$  with  $q > y$ , there is a prime  $p \mid m$  with  $q \mid \phi_k(p)$ . Indeed take  $k = 1$ . Either there is a prime  $p \mid m$  with  $q \mid \phi(p)$  or  $p^2 \mid m$ . By the hypothesis, the latter case does not occur. Thus, the result is true at  $k = 1$ . Assume that it is true at  $k$  and assume the hypothesis at  $k + 1$ . Then either there is a prime  $p' \mid \phi_k(m)$  with  $q \mid \phi(p')$ , or  $q^2 \mid \phi_k(m)$ . Again, the latter case does not occur, so we have the former case. By the induction hypothesis, there is a prime  $p \mid m$  with  $p' \mid \phi_k(p)$ . Then  $q \mid \phi_{k+1}(p)$ , and the assertion always holds.

Suppose that  $n \in \mathcal{S}_{k_1, k_2}(x, q_1, q_2) - \mathcal{S}_K(x, y)$ , where  $k_1, k_2, K, q_1$  and  $q_2$  are as given in the lemma. By the above with  $m = \phi_{k_1}(n)$ , there is a prime  $p \mid \phi_{k_1}(n)$  with  $q_2 \mid \phi_{k_2-k_1}(p)$ . By the hypothesis of the lemma, we have  $p \neq q_1$ . Thus,  $pq_1 \mid \phi_{k_1}(n)$ . It follows that

$$\begin{aligned} \#(\mathcal{S}_{k_1, k_2}(x, q_1, q_2) - \mathcal{S}_K(x, y)) &\leq \sum_{p: q_2 \mid \phi_{k_2-k_1}(p)} S_{k_1}(x, pq_1) \\ &\leq \sum_{p: q_2 \mid \phi_{k_2-k_1}(p)} \frac{x}{pq_1} (3c_0 \log \log x)^{2k_1}, \end{aligned}$$

by Lemma 9. But from the remark on p. 190 of [2], we have

$$\sum_{p: q_2 \mid \phi_{k_2-k_1}(p)} \frac{1}{p} \leq \frac{1}{q_2} (2c_0 \log \log x)^{k_2-k_1}.$$

Putting this inequality in the prior one gives the lemma.  $\square$

*Proof of Proposition 7.* The count in Proposition 7 is at most

$$S_K(x, y) + \sum_{p>y} \sum_{0 \leq k_1 < k_2 \leq K} \#(\mathcal{S}_{k_1, k_2}(x, p, p) - \mathcal{S}_K(x, y)).$$

By Lemma 9 with  $q_1 = q_2 = p$ , we have

$$S_K(x, y) \leq \sum_{p>y} \sum_{0 \leq k \leq K} \frac{x}{p^2} (3c_0 \log \log x)^{2k} \ll \frac{x}{y} (3c_0 \log \log x)^{2K}.$$

We also take  $q_1 = q_2 = p$  in Lemma 10. Thus,

$$\begin{aligned} \sum_{p>y} \sum_{0 \leq k_1 < k_2 \leq K} \#(\mathcal{S}_{k_1, k_2}(x, p, p) - \mathcal{S}_K(x, y)) &\ll \sum_{p>y} \frac{x}{p^2} K (3c_0 \log \log x)^{2K} \\ &\ll \frac{x}{y} K (3c_0 \log \log x)^{2K}. \end{aligned}$$

Thus, the proposition follows with  $c_2$  any number larger than  $3c_0$ .  $\square$

The next result will be helpful in establishing Proposition 8.

**Lemma 11.** *Uniformly for  $1 < z < 2$ , we have*

$$\sum_{n \leq x} z^{\Omega(n)} \ll \frac{x(\log x)^{z-1}}{2-z}.$$

*Proof.* We follow the suggestion in Exercise 05 in [3]. Let  $g$  be the multiplicative function with  $g(p^a) = z^a - z^{a-1}$  for primes  $p$  and positive integers  $a$ . Then  $z^{\Omega(n)} = \sum_{d|n} g(d)$ . Thus, the sum in the lemma is equal to

$$\begin{aligned} \sum_{m \leq x} g(m) \left\lfloor \frac{x}{m} \right\rfloor &\leq x \sum_{m \leq x} \frac{g(m)}{m} \leq x \prod_{p \leq x} \left( 1 + \frac{z-1}{p} + \frac{z^2-z}{p^2} + \cdots \right) \\ &= x \prod_{p \leq x} \frac{p-1}{p-z} = \frac{x}{2-z} \prod_{3 \leq p \leq x} \frac{p-1}{p-z} \ll \frac{x}{2-z} (\log x)^{z-1}. \end{aligned}$$

This completes the proof of the lemma.  $\square$

**Lemma 12.** *Uniformly for each positive integer  $k$ ,*

$$\sum_{\substack{n \leq x \\ \Omega(n) \geq k}} 1 \ll \frac{k}{2^k} x \log x.$$

*Proof.* This merely involves applying Lemma 11 with  $z = 2 - 1/k$ . Indeed, if  $N$  is the sum in the present lemma, then Lemma 11 implies that

$$N \ll \frac{x(\log x)^{1-1/k}}{(1/k)(2-1/k)^k},$$

and it remains to note that  $(2 - 1/k)^k$  and  $2^k$  are uniformly of the same order.  $\square$

*Proof of Proposition 8.* By Lemma 12, if  $0 < t \leq x$ , the number of primes  $p \leq t$  with  $\Omega(p-1) > 5 \log \log x$  is  $O(t/\log^2 x)$ . This holds since  $5 \log 2 - 1 > 2$ , and indeed the same estimate holds for the number of integers  $n \leq t$  with  $\Omega(n) > 5 \log \log x$ . Thus, by partial summation,

$$\sum_{\substack{p \leq x \\ \Omega(p-1) > 5 \log \log x}} \frac{1}{p} \ll \frac{1}{\log x}. \quad (2)$$

If  $\Omega(n) \leq 5 \log \log x$  and if each prime  $p$  dividing  $F_{K-1}(n)$  has the property that  $\Omega(p-1) \leq 5 \log \log x$ , then for all positive integers  $0 \leq k \leq K$  we have  $\Omega(\phi_k(n)) \leq (5 \log \log x)^k$ , so that  $\Omega(F_K(n)) \leq 2(5 \log \log x)^K$ . We conclude that if  $\Omega(F_K(n)) > 2(5 \log \log x)^K$ , then either  $\Omega(n) > 5 \log \log x$  or there is some prime  $p \mid F_{K-1}(n)$  with  $\Omega(p-1) > 5 \log \log x$ . It follows from Lemma 12, that the number of  $n$  in the first category is  $O(x/\log^2 x)$ , while it follows from (2) and Proposition 6 that the number of  $n$  in the second category is  $O((x/\log x)(c_1 \log \log x)^{K-1})$ . This completes the proof of the proposition.  $\square$

## 4 Thoughts on the normal order of $M(n)$

Let  $k_\phi(n)$  be the least integer  $k$  with  $\phi_k(n) = 1$ . Further, let  $\lambda(n)$  denote Carmichael's function, so that  $\lambda(n)$  is the order of the largest cyclic subgroup of the multiplicative group  $(\mathbb{Z}/n\mathbb{Z})^\times$ . With  $\lambda_k$  as the iterated Carmichael function, let  $k_\lambda(n)$  be the least  $k$  with  $\lambda_k(n) = 1$ . It is easy to see that the prime factors of  $\prod_{k \geq 1} \phi_k(n)$  are the same as the prime factors of  $\prod_{k \geq 1} \lambda_k(n)$ , so that we might have stated Theorem 2 in terms of the iterated  $\lambda$ -function rather than the iterated  $\phi$ -function. Let  $G(n)$  be the product of the prime

factors of  $\prod_{k \geq 1} \lambda_k(n)$  (so that  $F(n)$  in Theorem 2 is equal to the least common multiple of  $n$  and  $G(n)$ ). Thus,

$$M(n) = \phi(F(n)) \leq F(n) \leq nG(n) \leq n^{k_\lambda(n)+1}.$$

It is suggested in [4] that for all  $n$  lying outside a set of asymptotic density 0, the inequality  $k_\lambda(n) \ll \log \log n$  holds. If so, then apart from a factor of order  $\log \log \log n$  in the exponent, Theorem 3 is best possible.

We close by remarking that we have  $k_\lambda(n) \gg \log \log n$  almost always, that is, for all  $n$  outside a set of density 0. Indeed, we have from Theorem 4.5 of [2] that there is a positive constant  $c_3$  such that for almost all  $n$ , there is some iterate  $\phi_j(n)$  divisible by every prime up to  $(\log n)^{c_3}$ . Since every prime that divides some iterate of  $\phi$  at  $n$  also divides some iterate of  $\lambda$  at  $n$  (as remarked above), we have

$$k_\lambda(n) \geq \max_{p \leq (\log n)^{c_3}} k_\lambda(p).$$

Further, by Linnik's theorem, there exists a positive constant  $c_4$  such that for all sufficiently large values of  $x$ , there is a prime  $p \leq x$  with  $2^u \mid p-1$  for some integer  $u$  with  $2^u > x^{c_4}$ . Thus, for this prime  $p$ ,  $k_\lambda(p) > u/2 \gg \log x$ . Applied with  $x = (\log n)^{c_3}$ , we have the assertion.

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