ON PRIMES AND PRACTICAL NUMBERS

CARL POMERANCE AND ANDREAS WEINGARTNER

ABSTRACT. A number n is *practical* if every integer in $[1, n]$ can be expressed as a subset sum of the positive divisors of n. We consider the distribution of practical numbers that are also shifted primes, improving a theorem of Guo and Weingartner. In addition, essentially proving a conjecture of Margenstern, we show that all large odd numbers are the sum of a prime and a practical number. We also consider an analogue of the prime k-tuples conjecture for practical numbers, proving the "correct" upper bound, and for pairs, improving on a lower bound of Melfi.

In memory of Ron Graham (1935–2020)
and Richard Guy (1916–2020)

After Srinivasan [15], we say a positive integer n is *practical* if every integer $m \in [1, n]$ is a subset-sum of the positive divisors of n. The distribution of practical numbers has been of some interest, with work of Margenstern, Melfi, Tenenbaum, Saias, and the second-named author of this paper. In particular, we now know, [21], [22], that there is a constant $c = 1.33607\ldots$ such that the number of practical numbers in $[1, x]$ is $\sim cx/\log x$ as $x \to \infty$. For other problems and results about practical numbers see [4, Sec. B2].

The problem of how frequently a shifted prime $p-h$ can be practical was considered recently in [3]. Since practical numbers larger than 1 are all even, one assumes that the shift h is a fixed odd integer. Under this assumption, it would make sense that the concept of being practical and being a shifted prime are "independent events" and so it is natural to conjecture that the number of primes $p \le x$ with $p-h$ practical is of magnitude $x/\log^2 x$. Towards this conjecture it was shown in [3] that the number of shifted primes up to x that are practical is, for large x

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depending on h, between

$$\frac{x}{(\log x)^{5.7683\ldots}} \text{ and } \frac{x}{(\log x)^{1.0860\ldots}}.$$

Here we make further progress with this problem, proving the conjecture for the upper bound of the count and reducing the lower bound exponent $5.7683\ldots$ to $3.1647\ldots$.

As in [3] we consider a somewhat more general problem. Let θ be an arithmetic function with $\theta(n) \geq 2$ for all n and let $\mathcal{B}_\theta$ be the set of positive integers containing $n = 1$ and all those $n \geq 2$ with canonical prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $p_1 < \ldots < p_k$, $\alpha_1, \ldots, \alpha_k > 0$, which satisfy

$$p_j \leq \theta(p_1^{\alpha_1} \ldots p_{j-1}^{\alpha_{j-1}}) \qquad (1 \leq j \leq k). \tag{1}$$

(It is not necessary that p_i be the i-th prime number.) Stewart [16] and Sierpinski [14] showed that if $\theta(n) = \sigma(n) + 1$, where $\sigma(n)$ is the sum of the positive divisors of n, then the set $\mathcal{B}_\theta$ is precisely the set of practical numbers. Tenenbaum [18] found that if $\theta(n) = yn$, where $y \geq 2$ is a constant, then $\mathcal{B}_\theta$ is the set of integers with y-dense divisors; i.e., the ratios of consecutive divisors are at most y.

Throughout this paper, all constants implied by the big O and $\ll$ notation may depend on the choice of θ. For several of our results we assume that there are constants A, C such that

$$\theta(mn) \leq C m^A \theta(n), \qquad m, n \geq 1. \tag{2}$$

This holds for $\theta(n) = \sigma(n) + 1$ with $A = 2$, $C = 1$, since we trivially have $\sigma(mn) \leq \sigma(m)\sigma(n)$ and $\sigma(m) \leq m^2$.

We write $\log_2 x = \log\log x$ for $x > e^e$ and $\log_2 x = 1$ for $0 < x \leq e^e$, and write $\log_3 x = \log_2 \log x$ for $x > 1$. Let

$$l(x) = \exp\left(\frac{\log x}{\log_2 x \log_3^3 x}\right)$$

and

$$S_h(x) := |\{p \leq x : p \text{ prime}, \ p - h \in \mathcal{B}_\theta\}|.$$

Theorem 1. *Fix a nonzero integer h. Assume (2) and $n \leq \theta(n) \ll nl(n)$ for $n \geq 1$. For x sufficiently large depending on the choice of θ, h, we have*

$$\frac{x}{(\log x)^{3.1648}} < S_h(x) \ll_h \frac{x}{(\log x)^2}, \tag{3}$$

where $h \in \mathbb{Z}$ and h is not divisible by $\prod_{p \leq \theta(1)} p$ in the lower bound.

The exponent in the lower bound can be taken as any number larger than $(e+1)\log(e+1)-e+1$. In the case of practical numbers, where $\theta(n)=\sigma(n)+1$ and $\prod_{p\le\theta(1)} p=2$, Theorem 1 implies the following.

Corollary 1. *For any fixed odd $h\in\mathbb{Z}$, the number of primes $p\le x$ such that $p-h$ is practical satisfies* (3).

It seems likely that the upper bound in (3) is best possible, apart from optimizing the implied constant as a function of the shift parameter h. Our proof shows that this constant is $\ll h/\varphi(h)$.

Margenstern [7, Conjecture 7] conjectured that every natural number other than 1 is the sum of two numbers that are either practical or prime. The case of even numbers was settled by Melfi [9, Theorem 1], who showed that every even number is the sum of two practical numbers. Somewhat weaker versions of the problem for odd numbers were recently stated by Sun [17]. We show that in the case of odd numbers there are at most a finite number of exceptions to Margenstern's conjecture. Tomás Oliveira e Silva has told us that Margenstern's conjecture has no counterexamples to 10^9. Note that it would seem difficult to use our methods to get a numerical bound x_0 for which every odd number $> x_0$ is the sum of a prime and a practical number. However, it would seem quite tractable using our proof if one was prepared to use the extended Riemann Hypothesis in place of the Bombieri–Vinogradov theorem.

Theorem 2. *Assume $\theta(n)\ge n$. Every sufficiently large integer not divisible by $\prod_{p\le\theta(1)} p$ is the sum of a prime and a member of $\mathcal{B}_\theta$.*

Corollary 2. *Every sufficiently large odd integer is the sum of a prime and a practical number.*

Margenstern [7, Theorem 6] showed that for every fixed even number h, there are infinitely many practical numbers n such that $n+h$ is also practical. He conjectured [7, Conjecture 2] that the number of practical pairs $\{n,n+2\}$ up to x is asymptotic to $cx/\log^2 x$ for some positive constant c. Let

$$T_h(x) := |\{n\le x : n\in\mathcal{B}_\theta, n+h\in\mathcal{B}_\theta\}|.$$

Theorem 3. *Fix a nonzero integer h. Assume* (2) *and $\theta(n)\ll nl(n)$ for $n\ge 1$.*
(i) We have

$$T_h(x)\ll_h \frac{x}{\log^2 x}. \tag{4}$$

(ii) Assume further that $\theta(n) \geq n$ for all n, and that $n \in \mathcal{B}_\theta$ and $m \leq 3n/|h|$ imply $mn \in \mathcal{B}_\theta$. Moreover, if $\theta(1) < 3$, assume that

$$\begin{cases} h \in 2\mathbb{Z} & \text{if } \theta(2) \geq 3, \\ h \in 4\mathbb{Z} & \text{if } \theta(2) < 3. \end{cases} \tag{5}$$

Then for sufficiently large x, depending on the choice of h,

$$T_h(x) > \frac{x}{(\log x)^{9.5367}}. \tag{6}$$

When $h \in 2\mathbb{Z}$ and $\theta(n) = \sigma(n) + 1$, all conditions of Theorem 3 are satisfied, since for practical n we have $\sigma(n) + 1 \geq 2n$, by [7, Lemma 2].

Corollary 3. *For every nonzero even integer h, the number of practical n up to x, such that $n + h$ is also practical, satisfies* (4) *and* (6)*.*

Corollary 3 improves on the lower bound by Melfi [10, Thm. 1.1] for twin practical numbers, $T_2(x) \gg x/\exp(k\sqrt{\log x})$ for $k > 2 + \log(3/2)$.

The upper bound in Theorem 3 generalizes as follows to the distribution of practical k-tuples.

Theorem 4. *Fix integers $0 \leq h_1 < h_2 < \ldots < h_k$. Assume* (2) *and $\theta(n) \ll nl(n)$ for $n \geq 1$. We have*

$$\left|\{n \leq x : \{n + h_1, \ldots, n + h_k\} \subset \mathcal{B}_\theta\}\right| \ll_{h_1,\ldots,h_k} \frac{x}{\log^k x}.$$

1. The upper bound of Theorem 1

Lemma 1. *There exists a constant $K > 0$ such that for all $a, b \in \mathbb{Z}\backslash\{0\}$ and all $x > 1$ we have*

$$\#\{m \leq x : m \text{ and } am + b \text{ are both prime}\} \leq K \frac{a|b|}{\varphi(a|b|)} \cdot \frac{x}{\log^2 x}.$$

This result follows immediately from [11, Lemma 5].

Let $P^+(n)$ denote the largest prime factor of $n > 1$ and $P^+(1) = 1$. Define

$$B(x, y, z) = \#\{n \leq x : n \in \mathcal{B}_{z\theta}, P^+(n) \leq y\}.$$

Proposition 1. *Assume $\theta(n) \ll n\,l(n)$. For $x \geq 2$, $y \geq 2$ and $z \geq 1$,*

$$B(x, y, z) \ll \frac{x \log(2z)}{\log x} e^{-u/3},$$

where $u = \log x / \log y$.

Before proving this we establish some consequences.

Corollary 4. *Let $\alpha \in \mathbb{R}$. Assume* (2) *and $\theta(n) \ll n\, l(n)$ for $n \geq 1$. For $x \geq 1$, $y \geq 2$, $z \geq 1$,*

$$\sum_{\substack{n\leq x,\ n\in\mathcal{B}_{z\theta}\\ P^+(n)\leq y}} \left(\frac{\sigma(n)}{n}\right)^{\alpha} \ll_{\alpha} \frac{x\log(2z)}{\log(2x)} \exp\left(-\frac{\log x}{3\log y}\right).$$

Proof. When $\alpha \leq 0$, the result follows from Proposition 1. We will show the result for $\alpha \in \mathbb{N}$ by induction. Note that because of (2) we have that $kd \in \mathcal{B}_\theta$ implies $k \in \mathcal{B}_{\theta_d}$, where $\theta_d(n) = Cd^A\theta(n)$. By Proposition 1 with z replaced by zCd^A,

$$\begin{aligned}
\sum_{\substack{n\leq x,\, n\in\mathcal{B}_{z\theta}\\ P^+(n)\leq y}} \left(\frac{\sigma(n)}{n}\right)^{\alpha} &= \sum_{\substack{n\leq x,\, n\in\mathcal{B}_{z\theta}\\ P^+(n)\leq y}} \left(\frac{\sigma(n)}{n}\right)^{\alpha-1} \sum_{d\mid n}\frac{1}{d}\\
&\leq \sum_{d\leq x} \frac{\sigma(d)^{\alpha-1}}{d^{\alpha}} \sum_{\substack{k\leq x/d,\, k\in\mathcal{B}_{z\theta_d}\\ P^+(k)\leq y}} \left(\frac{\sigma(k)}{k}\right)^{\alpha-1}\\
&\ll_{\alpha} \sum_{d\leq x} \frac{\sigma(d)^{\alpha-1}}{d^{\alpha}} \frac{x\log(2dz)}{d\log(2x/d)} \exp\left(-\frac{\log(x/d)}{3\log y}\right)\\
&\ll x\exp\left(-\frac{\log x}{3\log y}\right) \sum_{d\leq x} \exp\left(\frac{\log d}{3\log y}\right) \frac{(\log_2 d)^{\alpha-1}\log(2dz)}{d^2\log(2x/d)}\\
&\ll_{\alpha} \frac{x\log(2z)}{\log(2x)} \exp\left(-\frac{\log x}{3\log y}\right),
\end{aligned}$$

since $\exp((\log d)/(3\log y)) \leq d^{1/2}$. □

With $y = x$, $z = 1$ and $\alpha = 1$ in Corollary 4, we get

Corollary 5. *Under the assumptions of Corollary 4 we have, for $x > 1$,*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{B}_\theta}} \frac{\sigma(n)}{n} \ll \frac{x}{\log x}.$$

Remark 1. Corollary 5 allows us to replace the relative error term $O(\log_2 x/\log x)$ in [21, Theorem 1.1], the asymptotic for the count of practical numbers up to x, by $O(1/\log x)$. Indeed, in the proof of [21, Theorem 1.1], the estimate $\sigma(n)/n \ll \log_2 n$ leads to the extra factor of $\log_2 x$. Using instead Corollary 5 in the proofs of Lemmas 5.3 and 5.6 of [21], the factor $\log_2 x$ can be avoided.

Proof of the upper bound in Theorem 1. Assume $x \geq 2|h|$. We consider those $n \in \mathcal{B}_\theta$ with $n + h$ prime and $n + h \leq x$. We may assume that $n > x/\log^2 x$. Write $n = mq$, where $q = P^+(n)$. We have $m \in \mathcal{B}_\theta$,

$P^+(m) \le q$ and $q \le \theta(m) \le ml(m)$. So, assuming x is large, we have $m > x^{1/3}$. By Lemma 1,

$$\begin{aligned} S_h(x) &\le \sum_{m\in\mathcal{B}_\theta} |\{q\in\mathbb{P} : mq+h\in\mathbb{P}, q\le (x-h)/m\}| \\ &\ll \sum_{\substack{m\in\mathcal{B}_\theta,\ m>x^{1/3}\\ mP^+(m)\le x-h}} \frac{m|h|}{\varphi(m|h|)} \frac{(x-h)/m}{\log^2(2(x-h)/m)} \\ &\le \frac{2|h|x}{\varphi(|h|)} \sum_{m\in\mathcal{B}_\theta,\ m>x^{1/3}} \frac{1}{\varphi(m)\log^2 P^+(m)}. \end{aligned}$$

We will show that the last sum is $\ll 1/\log^2 x$. With $p = P^+(m)$ and $m = kp$, we have $k \in \mathcal{B}_\theta$ and $k > x^{1/7}$. The last sum is

$$\ll \sum_{p\ge 2} \frac{1}{p\log^2 p} \sum_{\substack{k\in\mathcal{B}_\theta,\ k>x^{1/7}\\ P^+(k)\le p}} \frac{k}{\varphi(k)}\cdot\frac{1}{k}.$$

Since $k/\varphi(k) \ll \sigma(k)/k$, Corollary 4 (with $\alpha = z = 1$) and partial summation applied to the inner sum shows that the last expression is

$$\ll \sum_{p\ge 2} \frac{1}{p\log^2 p}\cdot\frac{\log p}{\log x}\exp\left(-\frac{\log x}{21\log p}\right) \ll \frac{1}{\log^2 x},$$

by the prime number theorem. □

Proof of Proposition 1. We follow the proof of Saias [12, Prop. 1], who established this result in the case when $\theta(n) = yn$ with $y \ge 2$ (integers with y-dense divisors) and in the case when $\theta(n) = \sigma(n) + 1$ (practical numbers) and $z = 1$. Let $f(n)$ be an increasing function with $\theta(n) \le nf(n)$ for all $n \ge 1$ and $f(n) \ll l(n)$. Suppose $n \in \mathcal{B}_{z\theta}$, where $n = p_1p_2\dots p_k$ with $p_1 \le p_2 \le \cdots \le p_k$. Since f is increasing, $p_j \le zp_1\cdots p_{j-1}f(p_1\cdots p_{j-1})$, so $p_j^2 \le znf(n) \le zxf(x)$ for $n \le x$. By sorting the integers counted in $B(x, y, z)$ according to their largest prime factor, we get

$$B(x,y,z) \le 1 + \sum_{p\le\min(y,\sqrt{zxf(x)})} B(x/p, p, z),$$

the analogue of [12, Lemma 8].

Let $\Psi(x, y)$ denote the number of integers $n \le x$ with $P^+(n) \le y$. We write $u = \log x/\log y$ and $\tilde{v} = \log x/\log(2z)$. Let $\tilde{\rho}(u) = \rho(\max\{0, u\})$, where $\rho(u)$ is Dickman's function. Let $\tilde{D}(x, y, z)$ be the function defined in [12, p. 169]. It satisfies

$$\tilde{D}(x,y,2z) \asymp \frac{x}{\tilde{v}}\tilde{\rho}\Big(u\big(1-1/\sqrt{\log y}\big)-1\Big) \qquad (0<u<3(\log x)^{1/3})$$

and

$$\tilde{D}(x,y,2z) = \Psi(x,y) \qquad (u \geq 3(\log x)^{1/3}),$$

Lemma 9 of [12] shows that

$$\tilde{D}(x,y,2z) \geq 1 + \sum_{p \leq \min(y, \sqrt{2zx}l(x))} \tilde{D}(x/p,p,2z),$$

for $z \geq 1$, $y \geq 2$, $\tilde{v} \geq v_0$ and $0 < u \leq 3(\log x)^{1/3}$.

We claim that

$$B(x,y,z) \leq c\tilde{D}(x,y,2z), \tag{7}$$

for some suitable constant c. If $2 \leq x \leq x_0$, we have $\tilde{D}(x,y,2z) \asymp 1$, so we may assume $x \geq x_0$ and hence $\sqrt{f(x)} \leq l(x)$. If $0 < \tilde{v} \leq u < 3(\log x)^{1/3}$, then $2z \geq y$ and $B(x,y,z) = \Psi(x,y) \ll \tilde{D}(x,y,2z)$, where the last estimate is derived in the penultimate display on page 182 of [12]. If $0 < u \leq \tilde{v} \leq v_0$, then $\tilde{D}(x,y,2z) \asymp x$ so (7) holds. If $u \geq 3(\log x)^{1/3}$, then $\tilde{D}(x,y,2z) = \Psi(x,y)$ and (7) holds. Assume that c is such that (7) holds in the domain covered so far. In the remainder we may assume that $u \leq 3(\log x)^{1/3}$ and $\tilde{v} \geq v_0$. We show by induction on k that (7) holds for $y \geq 2, z \geq 1, 2 \leq x \leq 2^k$. We have

$$\begin{aligned} B(x,y,z) &\leq 1 + \sum_{p \leq \min(y,\sqrt{zxf(x)})} B(x/p,p,z) \\ &\leq 1 + c \sum_{p \leq \min(y,\sqrt{zxf(x)})} \tilde{D}(x/p,p,2z) \\ &\leq c\left(1 + \sum_{p \leq \min(y,\sqrt{2zx}l(x))} \tilde{D}(x/p,p,2z)\right) \\ &\leq c\tilde{D}(x,y,2z). \end{aligned}$$

It remains to show that

$$\tilde{D}(x,y,2z) \ll x\frac{\log(2z)}{\log x}e^{-u/3}.$$

We may assume $x \geq x_0$. If $u \leq 3(\log x)^{1/3}$, then $y \geq y_0$ and the result follows from $\rho(u) \ll e^{-u}$. If $u > 3(\log x)^{1/3}$, then

$$\tilde{D}(x,y,2z) = \Psi(x,y) \ll xe^{-u/2} \ll \frac{x}{\log x}e^{-u/3} \ll \frac{x\log(2z)}{\log x}e^{-u/3},$$

where the upper bound for $\Psi(x,y)$ is [19, Thm. III.5.1]. □

2. Some Lemmas

The following observation follows immediately from the definition of the set $\mathcal{B}_\theta$ in (1).

Lemma 2. *Let $\theta(n) \ge n$ for all $n \in \mathbb{N}$. If $n \in \mathcal{B}_\theta$ and $P^+(k) \le n$, then $nk \in \mathcal{B}_\theta$.*

If $\theta(n) = yn$, we write $\mathcal{D}_y$ for $\mathcal{B}_\theta$. For an integer $n > 1$, let $P^-(n)$ denote the least prime dividing n, and let $P^-(1) = +\infty$.

Lemma 3. *There is a number y_0 such that if $x \ge z^4 \ge 1$ and $y \ge \max\{y_0, z + z^{0.535}\}$, we have*

$$|\{n \le x : n \in \mathcal{D}_y,\ P^-(n) > z\}| \asymp \frac{x \log(y/z)}{\log(xy)\log(2z)}.$$

This conclusion continues to hold if $z + 1 \le y \le y_0$ and $(z, y]$ contains at least one prime number.

Proof. When $x \ge y \ge y_0$ and $z \ge 3/2$, then $\log(xy) \asymp \log x$ and the result follows from [13, Thm. 1] and [20, Rem. 2]. When $y > x$, the result follows from $|\{n \le x : P^-(n) > z\}| \asymp x/\log(2z)$. If $1 \le z \le 3/2$, the result follows from [12, Thm. 1]. If $y \le y_0$, the result follows from iterating [13, Lemma 8] a finite number of times. □

Lemma 4. *For $d \in \mathbb{N}$, $x \ge 1$, $z \ge 1$ and $y \ge 2z$, we have*

$$|\{n \le x : n \in \mathcal{D}_y,\ P^-(n) > z,\ d \mid n\}| \ll 1_{d \in \mathcal{D}_y} + \frac{x \log(dy)}{d \log(xy)\log(2z)}.$$

Proof. We first assume that $x/d \ge z^4$. If $d = 1$ the result follows from Lemma 3, so we assume $d > 1$. We have

$$\begin{aligned}|\{dw \le x : dw \in \mathcal{D}_y, P^-(w) > z\}| &\le |\{w \le x/d : w \in \mathcal{D}_{dy}, P^-(w) > z\}| \\ &\ll \frac{x \log(dy)}{d \log(xy)\log(2z)},\end{aligned}$$

by Lemma 3.

If $x/d \le z^4$, then $\log(xy) \le \log(ydz^4) \le 5\log(yd)$, so the result follows from $|\{2 \le w \le x/d : P^-(w) > z\}| \ll x/(d\log(2z))$. □

Lemma 5. *Assume $\theta(n) \ge n$ for all $n \in \mathbb{N}$. For all $h \in \mathbb{N}$ that are not divisible by $\prod_{p \le \theta(1)} p$, we have*

$$|\{x/p_0 < n \le x : n \in \mathcal{B}_\theta,\ \gcd(n,h) = 1\}| \gg \frac{x}{\log x \log(2h)\log_2 h},$$

for $x \ge K\log^5(2h)$, where $p_0 \le \theta(1)$ is the smallest prime not dividing h, and K is some positive constant depending only on θ. Moreover, there exists a constant $\eta > 0$ such that if $L \ge 1$ satisfies

$$\sum_{p|h,\ p>L} \frac{\log p}{p} < \eta$$

then, for $x \ge KL^5$,

$$|\{x/p_0 < n \le x : n \in \mathcal{B}_\theta,\, \gcd(n,h)=1\}| \gg \frac{x}{L\log x \log(2L)}.$$

Proof. Let $p_0 \le \theta(1)$ be the smallest prime with $p_0 \nmid h$. Let $k \in \mathbb{N}$, $L_k = p_0^k/2$, and assume $x \ge 2L_k^5$. Since $\theta(n) \ge n$,

$$|\{x/p_0 < n = p_0^k w \le x : n \in \mathcal{B}_\theta,\ P^-(w) > L_k\}| \\ \ge |\{x/p_0^{k+1} < w \le x/p_0^k : w \in \mathcal{D}_{p_0^k},\ P^-(w) > L_k\}|.$$

We would like to use Lemma 3 to obtain a lower bound for this count, but the fact that w is not free to roam over the entire interval $[1, x/p_0^k]$ is problematic. We note though that Lemma 3 implies there is a set $\mathcal{K} \subset \mathbb{N}$ with bounded gaps such that if $x \ge 2L_k^5$ and $k \in \mathcal{K}$, we have

$$|\{x/p_0^{k+1} < w \le x/p_0^k : w \in \mathcal{D}_{p_0^k},\ P^-(w) > L_k\}| \gg \frac{x\log(p_0^k/L_k)}{p_0^k \log x \log L_k} \\ \asymp \frac{x}{L_k \log x \log L_k}.$$

We have

$$\begin{aligned} &|\{w \le x/p_0^k : w \in \mathcal{D}_{p_0^k},\ P^-(w) > L_k,\ \gcd(h,w) > 1\}| \\ &\quad \le \sum_{\substack{p|h \\ p>L_k}} |\{w \le x/p_0^k : w \in \mathcal{D}_{p_0^k},\ P^-(w) > L_k,\ p \mid w\}| \\ &\quad \ll \sum_{\substack{p|h \\ L_k<p\le 2L_k}} 1 + \sum_{\substack{p|h \\ p>L_k}} \frac{x\log p}{L_k p \log x \log L_k}, \end{aligned}$$

by Lemma 4, since $\log(pp_0^k) \ll \log p$ for $p > L_k$. The sum of 1 is clearly $\le L_k \le (x/2)^{1/5}$. The second statement of the lemma now follows with the smallest $k \in \mathcal{K}$ such that $L_k \ge L$.

Since h has at most $\log h/\log L_k$ prime factors $> L_k$, the last sum above is

$$\ll \frac{\log h}{\log L_k} \cdot \frac{x}{L_k \log L_k \log x} \cdot \frac{\log L_k}{L_k} = \frac{x\log h}{L_k^2 \log L_k \log x}.$$

We need this to be $< x/(CL_k \log x \log L_k)$ for some sufficiently large constant $C > 0$, that is, $L_k \geq C \log(2h)$. The first statement of the lemma now follows with the smallest such $k \in \mathcal{K}$. □

3. The lower bound of Theorem 1

Let h be a fixed integer that is not a multiple of $\prod_{p \leq \theta(1)} p$. Let $\delta = 1/\log_2 x$ and define

$$\mathcal{Q} = \{q \in (x^{1/2-\delta}, x^{1/2}/\log^{10} x] : \ \gcd(q,h) = 1, \ q \in \mathcal{B}_\theta\}.$$

Let $\mathcal{N}_h(x)$ denote the set of pairs (q,m) with $q \in \mathcal{Q}$, $qm + h \leq x$, and $qm + h$ prime, and let $N_h(x) = |\mathcal{N}_h(x)|$. Thus,

$$N_h(x) = \sum_{q \in \mathcal{Q}} \pi(x; q, h).$$

Now, by the Bombieri–Vinogradov theorem, see [19, p. 403], we have

$$\sum_{q \in \mathcal{Q}} \left| \pi(x;q,h) - \frac{\pi(x)}{\varphi(q)} \right| \ll \frac{x}{\log^6 x}.$$

Thus,

$$N_h(x) = \sum_{q \in \mathcal{Q}} \pi(x;q,h) = \sum_{q \in \mathcal{Q}} \frac{\pi(x)}{\varphi(q)} + O\Big(\frac{x}{\log^6 x}\Big).$$

Further, using Lemma 5, we have

$$\sum_{q \in \mathcal{Q}} \frac{1}{\varphi(q)} \geq \sum_{q \in \mathcal{Q}} \frac{1}{q} \gg_h \delta.$$

We conclude that

$$N_h(x) \gg_h \delta x/\log x. \tag{8}$$

Let $\mathcal{N}_{h,1}(x)$ denote the set of those pairs (q,m) in $\mathcal{N}_h(x)$ with $x^\delta < P^+(m) < x^{1/2-\delta}$.

Lemma 6. *We have* $|\mathcal{N}_{h,1}(x)| = |\mathcal{N}_h(x)| + O(\delta^2 x/\log x)$,

Proof. Let $q \in \mathcal{Q}$. The number of integers $m \leq (x-h)/q$ with $P^+(m) \leq x^\delta$ is $\ll (x-h)/(q \log^{10} x)$, see [19, Lem. III.5.19], and so such numbers m are negligible. For $m = rk$, where $r = P^+(m) \geq x^{1/2-\delta}$, we have $k \leq x^{2\delta}$. Thus, the number of such pairs (q, rk) is at most

$$\sum_{q \in \mathcal{Q}} \sum_{k \leq x^{2\delta}} \sum_{\substack{r \leq (x-h)/qk \\ r \text{ prime} \\ qrk+h \text{ prime}}} 1.$$

The inner sum, by Lemma 1, is $\ll_h x/(\varphi(q)\varphi(k)\log^2 x)$. Summing on k gives us $\ll_h \delta x/(\varphi(q)\log x)$, and then summing on q gives us $\ll_h \delta^2 x/\log x$, using $q/\varphi(q) \ll \sigma(q)/q$, Corollary 5, and partial summation. This concludes the proof. □

Corollary 6. *For a pair (q, m) in $\mathcal{N}_{h,1}(x)$ we have $qm \in \mathcal{B}_\theta$.*

Proof. Since $P^+(m) < x^{1/2-\delta} < q$, it follows from Lemma 2 that $qm \in \mathcal{B}_\theta$. □

Let $v_2(n)$ denote the number of factors 2 in the prime factorization of n and let $\Omega(n)$ denote the total number of prime factors of n, counted with multiplicity. Let $\varepsilon > 0$ be arbitrarily small but fixed. Let $\mathcal{N}_{h,2}(x)$ denote the set of pairs $(q, m) \in \mathcal{N}_{h,1}(x)$ with

$$\Omega(m) \le I := \lfloor (1+\varepsilon)\log_2 x \rfloor \text{ and } v_2(m) \le 4\log_3 x.$$

Lemma 7. *We have*

$$|\mathcal{N}_{h,2}(x)| = |\mathcal{N}_h(x)| + O_h(\delta^2 x/\log x).$$

Proof. Assume $(q, m) \in \mathcal{N}_{h,1}(x)$. Let $r = P^+(m)$, so that $r > x^\delta$, and write $m = rk$. If $(q, m) \notin \mathcal{N}_{h,2}(x)$ then either $\Omega(k) > I - 1$ or $v_2(k) > 4\log_3 x$. For a given number k, the number of primes $r \le (x - h)/qk$ with $qrk+h$ prime is, by Lemma 1, $\ll_h x/(\varphi(q)\varphi(k)\log^2(x/qk))$. Summing this expression over k with $v_2(k) > 4\log_3 x$ and $q \in \mathcal{Q}$, it is $\ll_h \delta^2 x/\log x$, since $2^{-4\log_3 x} < \delta^2$. We now wish to consider the case when $\Omega(k) > I - 1$. Following a standard theme (see Exercises 04 and 05 in [5]) we have uniformly for each real number z with $1 < z < 2$ that

$$\sum_{n \le x} \frac{z^{\Omega(n)}}{\varphi(n)} \ll \frac{1}{2-z}(\log x)^z. \tag{9}$$

Applying this with $z = 1 + \varepsilon$, we have

$$\sum_{\substack{k \le x^{1/2} \\ \Omega(k) > I-1}} \frac{1}{\varphi(k)} \le z^{-I+1} \sum_{k \le x^{1/2}} \frac{z^{\Omega(k)}}{\varphi(k)} \ll (\log x)^{1+\varepsilon-(1+\varepsilon)\log(1+\varepsilon)}.$$

This last expression is of the form $(\log x)^{1-\eta}$, where $\eta > 0$ depends on the choice of ε. Thus, the number of pairs (q, m) in this case is $\ll_h \delta x/(\log x)^{1+\eta}$, which is negligible. □

Let $\Omega_3(n) = \Omega(n/v_2(n))$ denote the number of odd prime factors of n counted with multiplicity, and let $\mathcal{N}_{h,3}$ denote the number of pairs $(q, m) \in \mathcal{N}_{h,2}$ with $\Omega_3(q) \le J := \lfloor (e+\varepsilon)\log_2 x \rfloor$.

Lemma 8. *We have $|\mathcal{N}_{h,3}(x)| = |\mathcal{N}_h(x)| + O_h(\delta^2 x/\log x)$.*

Proof. By the same method that gives (9), we have

$$\sum_{n\le x}\frac{z^{\Omega_3(n)}}{\varphi(n)} \ll \frac{1}{3-z}(\log x)^z, \tag{10}$$

uniformly for $1<z<3$. Assuming that ε is small enough that $z=e+\varepsilon<3$, we have

$$\sum_{\substack{q\in\mathcal{Q}\\ \Omega_3(q)>J}}\frac{1}{\varphi(q)} \le \sum_{\substack{q\le x^{1/2}\\ \Omega_3(q)>J}}\frac{1}{\varphi(q)} \le z^{-J}\sum_{q\le x^{1/2}}\frac{z^{\Omega_3(q)}}{\varphi(q)} \ll (\log x)^{z-(e+\varepsilon)\log z}.$$

Since $z-(e+\varepsilon)\log z=-\eta<0$, where η depends on the choice of ε, this calculation shows that those pairs with $\Omega_3(q)>J$ are negligible. □

Let $K=\lfloor 4\log_3 x\rfloor+1$. For a given pair $(q,m)\in\mathcal{N}_{h,3}(x)$, we count the number of pairs $(q',m')\in\mathcal{N}_{h,3}(x)$ with $q'm'=qm$. The pair (q',m') is determined by (q,m) and m', so all we need to do is count the number of divisors d of qm with $\Omega(d)\le I$ and $v_2(d)<K$. This count is at most

$$K\sum_{i\le I}\binom{I+J}{i} \ll K\binom{I+J}{I}.$$

Stirling's formula shows that

$$K\binom{I+J}{I} \ll (\log x)^{\alpha+\eta}\log_3 x,$$

where $\alpha=(e+1)\log(e+1)-e\log e=2.16479...$ and $\eta\to 0$ as $\varepsilon\to 0$. It follows from (8) and Lemma 8 that

$$S_h(x) \gg \frac{\delta x}{\log x}\cdot\frac{1}{(\log x)^{\alpha+\eta}\log_3 x} \gg \frac{x}{(\log x)^{1+\alpha+2\eta}} = \frac{x}{(\log x)^{3.16479...+2\eta}}.$$

Remark 2. The proof of the lower bound of Theorem 1 would be somewhat simpler if instead of the Bombieri–Vinogradov theorem we had used a very new result of Maynard [8]. With the choice of parameters $\delta=0.02$, $\eta=0.001$ in his Corollary 1.2, one has for the set $\mathcal{Q}$ of integers $q\le x^{0.52}$ with a divisor in $(x^{0.041},x^{0.071})$ that

$$\sum_{\substack{q\in\mathcal{Q}\\ \gcd(q,a)=1}}\left|\pi(x;q,a)-\frac{\pi(x)}{\varphi(q)}\right| \ll_{a,A} \frac{x}{\log^A x},$$

for any fixed integer $a\ne 0$ and any positive A. We note that all of the members of $\mathcal{B}_\theta\cap(x^{0.041},x^{0.52}]$ are in $\mathcal{Q}$.

4. Proof of Theorem 2

Let h be an integer in $(x/2, x]$ that is not a multiple of $\prod_{p\le\theta(1)} p$. Define

$$\mathcal{D} = \{q \in \mathcal{B}_\theta \cap (x^{1/2-\delta}, x^{1/2}/\log^{10} x] : \gcd(q,h) = 1\}.$$

By Lemma 5,

$$|\mathcal{D}| \gg \frac{x^{1/2}}{\log^{12} x \log\log x}. \tag{11}$$

For each $q \in \mathcal{D}$, if $p \le x/2 < h$, where p is a prime that satisfies $p \equiv h \bmod q$, then $p = h - qm$ for some $m \in \mathbb{N}$. Let $M_h(x)$ denote the number of pairs (p,q) with p prime, $p \le x/2$, $p \equiv h \bmod q$ and $q \in \mathcal{D}$. As in Section 3, we have

$$M_h(x) = \sum_{q\in\mathcal{D}} \pi(x/2; q, h) = \sum_{q\in\mathcal{D}} \frac{\pi(x/2)}{\varphi(q)} + O\Big(\frac{x}{\log^6 x}\Big).$$

From (11), we have

$$F := \sum_{q\in\mathcal{D}} \frac{1}{\varphi(q)} \ge \sum_{q\in\mathcal{D}} \frac{1}{q} \ge \frac{|\mathcal{D}|}{x^{1/2}/\log^{10} x} \gg \frac{1}{\log^2 x \log\log x}.$$

We conclude that

$$M_h(x) \gg F\frac{x}{\log x} \gg \frac{x}{\log^3 x \log\log x}. \tag{12}$$

We claim that most of the pairs (p,q) counted in $M_h(x)$ are such that $qm = h - p \in \mathcal{B}_\theta$. Since $q > x^{1/2-\delta}$ and $qm < h \le x$, we have $m \le x^{1/2+\delta}$. If $P^+(m) \le x^{1/2-\delta}$, then $P^+(m) < q$ and $mq \in \mathcal{B}_\theta$. If $P^+(m) > x^{1/2-\delta}$, write $r = P^+(m) > x^{1/2-\delta}$ and $m = ra$ with $a < x^{2\delta}$. Given a and q, the number of primes $r < x/(aq)$ with $h - aqr$ prime is

$$\ll \frac{hx}{\varphi(h)\varphi(q)\varphi(a)\log^2 x}, \tag{13}$$

by Lemma 1. We have $h/\varphi(h) \ll \log\log x$ and

$$\sum_{a<x^{2\delta}} \frac{1}{\varphi(a)} \ll \delta \log x.$$

Thus, summing (13) over $q \in \mathcal{D}$ and $a < x^{2\delta}$ amounts to

$$\ll F\frac{x\delta\log\log x}{\log x} = o\left(F\frac{x}{\log x}\right),$$

since $\delta = 1/(\log\log x)^2$. By (12), the number of pairs (p, q) with $h = p + qm$, p prime and $qm \in \mathcal{B}_\theta$ is

$$\gg F\frac{x}{\log x} \gg \frac{x}{\log^3 x \log\log x},$$

which is at least 1 when x is sufficiently large.

5. The upper bound in Theorems 3 and 4

For a natural number n, a divisor d of n is said to be *initial* if $P^+(d) \le P^-(n/d)$. Let $I_y(n)$ be the largest initial divisor of n with $d \le y$. Note that if $n \in \mathcal{B}_\theta$, then $I_y(n) \in \mathcal{B}_\theta$ for all y.

Assume $n \le x$ and $n, n+h \in \mathcal{B}_\theta$. Let $q = I_{x^{1/3}}(n)$, $q' = I_{x^{1/3}}(n+h)$. Since $n, n+h \in \mathcal{B}_\theta$ and $\theta(n) = n^{1+o(1)}$, we may assume that $q, q' \in [x^{1/7}, x^{1/3}]$. Write $n = qm$ and $n + h = q'm'$. We have $q, q' \in \mathcal{B}_\theta$ and $P^-(m) \ge P^+(q) =: r$, $P^-(m') \ge P^+(q') =: r'$. Given $q, q' \in \mathcal{B}_\theta$ with $d = \gcd(q, q')$, we need m, m' such that $q'm' - qm = h$. This equation only has solutions if $d|h$, in which case all solutions have the form

$$m = m_0 + jq'/d, \quad m' = m_0' + jq/d, \quad j \in \mathbb{Z}.$$

If m_0, m_0' are the smallest positive solutions to $q'm' - qm = h$, then $1 \le n = mq \le x$ implies $0 \le j \le dx/qq' \le hx/qq'$. Let

$$\mathcal{A} = \{(m_0 + jq'/d)(m_0' + jq/d) : 0 \le j \le hx/qq'\},$$

and let $S(\mathcal{A})$ be the number of elements of $\mathcal{A}$ remaining after removing all products mm', where either m is a multiple of a prime $p < r$, $p \nmid hqq'$, or m' is a multiple of a prime $p < r'$, $p \nmid hqq'$. For each prime $p \nmid hqq'$, each of the conditions $p|m$ and $p|m'$ is equivalent to j belonging to a unique residue class modulo p (because $p \nmid qq'$), and those two residue classes are distinct (because $p \nmid h$). Selberg's sieve [2, Prop. 7.3 and Thm. 7.14] shows that

$$S(\mathcal{A}) \ll \frac{hx/qq'}{\log r \log r'} \left(\frac{hqq'}{\varphi(hqq')}\right)^2 \ll_h \frac{xqq'}{\varphi(q)^2\varphi(q')^2 \log P^+(q) \log P^+(q')}.$$

Summing this estimate over $q, q' \in [x^{1/7}, x^{1/3}] \cap \mathcal{B}_\theta$, the upper bound in Theorem 3 follows from Lemma 9 with $\alpha = 2$.

This argument generalizes naturally to yield Theorem 4: For $1 \le i \le k$, let $n + h_i = m_i q_i \in \mathcal{B}_\theta$, where $q_i = I_{x^{1/(k+1)}}(n + h_i)$, so that $q_i \in \mathcal{B}_\theta \cap [x^{1/(2k+3)}, x^{1/(k+1)}]$. One finds that if $\gcd(q_i, q_l)|(h_l - h_i)$, for $1 \le i < l \le k$, then

$$m_i = m_{i,0} + j \operatorname{lcm}(q_1, \ldots, q_k)/q_i \qquad (1 \le i \le k),$$

where $0 \leq j \leq x/\mathrm{lcm}(q_1, \ldots, q_k) \leq \frac{x}{q_1 \ldots q_k} \prod_{1 \leq i < l \leq k}(h_l - h_i)$. Eliminating values of j for which $p | m_i$, where $p < P^+(q_i)$, $p \nmid \prod_{i \leq k} q_i$ and $p \nmid \prod_{1 \leq i < l \leq k}(h_l - h_i)$, we find that

$$S(\mathcal{A}) \ll_{h_1,\ldots,h_k} x \prod_{i=1}^{k} \frac{q_i^{k-1}}{\varphi(q_i)^k \log P^+(q_i)}.$$

Theorem 4 now follows from Lemma 9 with $\alpha = k$.

Lemma 9. *Let $\alpha \in \mathbb{R}$. Assume (2) and $\theta(n) \ll n\, l(n)$ for $n \geq 1$. We have*

$$\sum_{q \geq x,\ q \in \mathcal{B}_\theta} \frac{q^{\alpha-1}}{\varphi(q)^\alpha \log P^+(q)} \ll_\alpha \frac{1}{\log x}.$$

Proof. It suffices to estimate the sum restricted to $q \in I := [x, x^{4/3}]$. We write $q = mr$, where $r = P^+(q)$. Note that $q \in \mathcal{B}_\theta \cap I$ and $\theta(n) < n^{1+o(1)}$ implies that $r \leq x^{3/4}$. We have

$$\sum_{q \in \mathcal{B}_\theta \cap I} \frac{q^{\alpha-1}}{\varphi(q)^\alpha \log P^+(q)} \ll \sum_{r \leq x^{3/4}} \frac{1}{r \log r} \sum_{\substack{m \in \mathcal{B}_\theta \cap (I/r) \\ P^+(m) \leq r}} \left(\frac{m}{\varphi(m)}\right)^\alpha \frac{1}{m}.$$

Since $m/\varphi(m) \ll \sigma(m)/m$, partial summation and Corollary 4 applied to the inner sum shows that the last expression is

$$\ll_\alpha \sum_{r \leq x^{3/4}} \frac{1}{r \log r} \cdot \frac{\log r}{\log x} \exp\left(-\frac{\log x}{3 \log r}\right) \ll \frac{1}{\log x},$$

by the prime number theorem. □

6. The lower bound in Theorem 3

Lemma 10. *Assume (2) and $\theta(n) \ll n\, l(n)$ for $n \geq 1$. For $L \geq 1$ and $x \geq 1$, we have*

$$\sum_{\substack{n \in \mathcal{B}_\theta \\ n \leq x}} \sum_{\substack{p | n \\ p > L}} \frac{\log p}{p} \ll \frac{x \log(2L)}{L \log(2x)}.$$

Proof. As in the proof of Corollary 4,

$$\sum_{\substack{n \in \mathcal{B}_\theta \\ n \leq x}} \sum_{\substack{p | n \\ p > L}} \frac{\log p}{p} = \sum_{L < p < x^{2/3}} \frac{\log p}{p} \sum_{\substack{mp \in \mathcal{B}_\theta \\ m \leq x/p}} 1 \leq \sum_{L < p < x^{2/3}} \frac{\log p}{p} \sum_{\substack{m \in \mathcal{B}_{\theta_p} \\ m \leq x/p}} 1$$

$$\ll \sum_{L < p < x^{2/3}} \frac{\log p}{p} \cdot \frac{x \log p}{p \log(2x)} \ll \frac{x \log(2L)}{L \log(2x)},$$

by Proposition 1 and the prime number theorem. □

Say a pair $n_1, n_2 \in \mathcal{B}_\theta$ is *h-ε-special* if $\gcd(n_1, n_2) = h$ and $\Omega_3(n_i) \le (e+\varepsilon)\log_2 n_i$ for $i = 1, 2$.

Lemma 11. *Assume* (2) *and $n \le \theta(n) \ll n\, l(n)$ for $n \ge 1$. For $h \ge 1$ satisfying* (5) *and $0 < \varepsilon < 1$, the number of h-ε-special pairs $n_1, n_2 \in \mathcal{B}_\theta$ with $N/3 < n_1, n_2 < N$ and $v_2(n_1), v_2(n_2) \le C$, where C is some number depending only on h, is $\gg_{h,\varepsilon} N^2/\log^2 N$.*

Proof. Write $h = 2^a 3^b h'$, where $P^-(h') > 3$, $a, b \ge 0$, but assume that $a \ge 1$ or $a \ge 2$, according to the two cases in (5). We consider $n_1 \in \mathcal{B}_\theta$ of the form

$$n_1 = 2^{a+k} 3^b h' n_1' = 2^k h n_1'$$

where $P^-(n_1') > \max\{3, P^+(h)\} =: p$ and $2^k > 2p$. Since $\theta(n) \ge n$, the number of such n_1 with $N/2 < n_1 \le N$ is at least

$$\left| \left\{ \frac{N}{h2^{k+1}} < n_1' \le \frac{N}{h2^k} : n_1' \in \mathcal{D}_{h2^k}, P^-(n_1') > p \right\} \right| \asymp_h \frac{N}{\log N}, \tag{14}$$

by Lemma 3, for a suitable k with $2^k > 2p > 2^{k+O(1)}$. In particular, $v_2(n_1) \ll_h 1$.

As in the proof of the lower bound of Theorem 1, we can remove those n_1 with $\Omega_3(n_1) > (e+\varepsilon)\log_2 n_1$ without affecting (14). This follows from an estimate analogous to (10):

$$\sum_{n \le x} z^{\Omega_3(n)} \ll \frac{x}{3-z} \log^{z-1} x$$

uniformly for $1 < z < 3$ (cf. [19, Exercise 217(b)]).

Let $\eta > 0$ be an arbitrary constant. Lemma 10 shows that we can choose a sufficiently large constant $L = L(\eta)$ such that removing those n_1 for which

$$\sum_{\substack{p \mid n_1 \\ p > L}} \frac{\log p}{p} > \eta$$

will not affect (14). For each of the $\asymp_{h,\varepsilon} N/\log N$ values of n_1 that remain, consider $n_2 \in \mathcal{B}_\theta$ of the form

$$n_2 = 2^a 3^{b+j} h' n_2' = 3^j h n_2',$$

where $\gcd(n_2', 2n_1') = 1$, and j is the smallest integer with $3^j > p$. Given n_1, the number of such $n_2 \le N$ is at least

$$\sum_{\substack{N/h3^{j+1} < n_2' \le N/h3^j \\ n_2' \in \mathcal{D}_{h3^j} \\ \gcd(n_2', 2n_1') = 1}} 1 \gg_h \frac{N}{\log(NL)\log(2L)} \gg \frac{N}{\log N},$$

by Lemma 5 with $p_0 = 3$. As with n_1, this estimate is unchanged if we remove those n_2 with $\Omega(n_2) > (e+\varepsilon)\log_2 n_2$. Further, $v_2(n_2) = v_2(h) \ll_h 1$. □

Let $N = \sqrt{xh}$. Suppose $a, a' \in \mathcal{B}_\theta \cap (N/3, N]$ is an h-ε-special pair, with $v_2(a), v_2(a') \le C$, where $C = C(h)$ is as in Lemma 11. For each such pair $\{a, a'\}$, there is a unique pair $\{b, b'\}$ such that $ab - a'b' = h$ and $1 \le b \le a'/h$, $1 \le b' \le a/h$. We have $ab, a'b' \le aa'/h \le x$. Now $b, b' \le \sqrt{x/h} < 3a/h, 3a'/h$, so $ab, a'b' \in \mathcal{B}_\theta$ by the assumption on θ. By Lemma 11, it would seem we have created $\gg_{h,\varepsilon} x/\log^2 x$ pairs $\{ab, a'b'\} \subset \mathcal{B}_\theta \cap [1, x]$ with $ab - a'b' = h$, but we have to check for possible multiple representations.

Note that in a graph of average degree $\ge d$, there is an induced subgraph of minimum degree $\ge d/2$. This folklore result can be proved by induction on d, see [1]. (Also see [6, Prop. 3] for a somewhat sharper version.) We apply this to the graph on members of $\mathcal{B}_\theta \cap (N/3, N]$, where two integers are connected by an edge if they form an h-ε-special pair. From Lemma 11 the average degree in this graph is $\gg N/\log N$, so there is a subgraph G of minimum degree $\gg N/\log N$.

We use this to say something about $\Omega_3(b), \Omega_3(b')$. For edges (a, a') in G, note that for any residue class mod a' there are at most 2 choices for a, and similarly for any residue class mod a there are at most 2 choices for a'. For (a, a') with corresponding pair (b, b') as above, let $f(a, a') = b$ and $g(a, a') = b'$. For each fixed a' the function f is at most two-to-one in the variable a, since $(a/h)b \equiv 1 \pmod{a'/h}$ and $b \le a'/h$. Similarly, for each fixed a, the function $g(a, a') = b'$ is at most two-to-one in the variable a'. Thus, for each fixed a' there are $\gg N/\log N$ distinct values of b and for each fixed a there are $\gg N/\log N$ distinct values of b'. Now $b, b' \le N$ and as we have seen, the number of integers $n \le N$ with $\Omega_3(n) > (e+\varepsilon)\log_2 x$ is $o(N/\log N)$. So, by possibly discarding $o(x/\log^2 x)$ pairs (a, a'), we may assume that the corresponding pair (b, b') satisfies $\Omega_3(b), \Omega_3(b') \le (e+\varepsilon)\log_2 x$.

The numbers ab and $a'b'$ might arise from many different pairs (a, a'). However, we have $\Omega_3(ab), \Omega_3(a'b') \le 2(e+\varepsilon)\log_2 x$, so the number of odd divisor pairs of $ab, a'b'$ is

$$\le 2^{4(e+\varepsilon)\log_2 x} = (\log x)^{4(e+\varepsilon)\log 2}.$$

Since $v_2(a), v_2(a') \ll_h 1$, there are $\gg_{h,\varepsilon} x/(\log x)^{2+4(e+\varepsilon)\log 2}$ pairs $n, n+h \in \mathcal{B}_\theta$ with $n \le x$. This completes the proof of the theorem.

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Mathematics Department, Dartmouth College, Hanover, NH 03784
E-mail address: `carl.pomerance@dartmouth.edu`

Department of Mathematics, Southern Utah University, Cedar City, UT 84720
E-mail address: `weingartner@suu.edu`