# REMARKS ON THE MIDDLE BINOMIAL COEFFICIENT

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**Abstract**

We show that $\binom{m+k}{k} \mid \binom{2m}{m}$ for all $k \le \exp(.8\sqrt{\log m})$ on a set of numbers $m$ of asymptotic density 1. We also show that $(m+1)(m+2)\dots(m+k)$ divides $\binom{2m}{m}$ on a set of asymptotic density 1 for $k$ as large as $.7\log m$.

## 1. Introduction

In [4] I proved some elementary results about the middle binomial coefficient $\binom{2m}{m}$. In particular, Theorem 2 of that paper shows that for each fixed positive integer $k$, we have $m+k \mid \binom{2m}{m}$ for a set of integers $m$ of asymptotic density 1. Following the proof it is left as an exercise to show that the product $(m+1)(m+2)\dots(m+k)$ divides $\binom{2m}{m}$ on a set of asymptotic density 1. We prove the following strengthening.

**Theorem 1.** *For any fixed $\eta < 1/\log 4 = .721\dots$, we have for a set of integers $m$ of asymptotic density 1 that*

$$\frac{(m+k)!}{m!} \;\Big|\; \binom{2m}{m} \tag{1.1}$$

*for all positive integers $k \le \eta \log m$.*

Replacing the product with $\binom{m+k}{k}$, we prove the following theorem.

**Theorem 2.** *For $m$ in a set of asymptotic density 1, we have*

$$\binom{m+k}{k} \;\Big|\; \binom{2m}{m} \tag{1.2}$$

*for all positive integers $k \le \exp(.8\sqrt{\log m})$.*

These theorems may be of interest in the context of recent work towards settling a problem of Erdős using AI.[1] One might compare this note with the write-up of Sothanaphan [6].

[1] See T. Tao, https://mathstodon.xyz/@tao/115855840223258103.

The paper [4] spawned some other work as well. Since $m+1$ always divides $\binom{2m}{m}$ and $m+k \mid \binom{2m}{m}$ almost always when $k > 1$, what about $m \mid \binom{2m}{m}$? It was shown that the upper density of this set is $\le 1 - \log 2$ and conjectured that the density exists and is positive. Sanna [6] showed that the upper density of the set is smaller than $1/4$, while Ford and Konyagin [3] proved the conjecture, showing the density is slightly larger than $1/11$. In fact, they showed that for each fixed positive integer $\ell$, the density of the set of $m$ with $m^\ell \mid \binom{2m}{m}$ exists and is positive.

## 2. Proof of Theorem 1

We let $p$ denote a prime variable, and we let $v_p$ be the function which returns the exponent on $p$ in the prime factorization of its argument.

**Lemma 1.** *Let*

$$\alpha_p(m) = \frac{v_p\left(\binom{2m}{m}\right)}{\log m/\log p}.$$

*For a set of integers $m$ of asymptotic density 1 we have $\alpha_2(m) = 1/2 + o(1)$, $\alpha_3(m) \ge 34/81 + o(1)$, and $\alpha_p(m) \ge .39$ uniformly for all $3 < p < 2\log m$.*

*Proof.* Note that $m$ has $\lfloor \log m/\log p \rfloor + 1$ base-$p$ digits and we expect roughly half of these to be $\ge p/2$. More precisely, for $p = 2$, we expect half to be $\ge p/2$ and for $p$ odd we expect $(p-1)/2p$ of them to be $\ge p/2$. The probability of getting at most $.4N$ heads when flipping a fair coin $N$ times is

$$2^{-N} \sum_{k \le .4N} \binom{N}{k} = e^{(-.4\log .4 - .6 \log .6 + o(1))N} 2^{-N} < e^{-.02N}$$

for $N$ large. For $3 < p < 2\log x$ a calculation using this probability shows that the number of $m \le x$ with fewer than $.39 \log m/\log p$ base-$p$ digits at least $p/2$ is $O(x^{1-c/\log p})$. Here $c$ is a small positive constant. Summing this for $p < 2\log x$ we get an expression that is $o(x)$ as $x \to \infty$. We can add to this exceptional set those $m$ with $\alpha_2(m) \le 1/2 - \epsilon$ for any fixed positive $\epsilon$, with $c$ now depending on $\epsilon$. For $p = 3$ we consider the base-27 expansion of $m$, finding that the average number of base-3 carries in doubling a base-27 digit is $34/27$, so our assertion about $\alpha_3(m)$ follows as well. □

If $p > 2k$, then $v_p((m+k)!/m!) = \max\{v_p(m+i) : 1 \le i \le k\}$. As in [4], if this max is $j$, occurring at $m + i_0$, then the $j$ least significant base-$p$ digits of $m + i_0$ are 0, so the $j$ least significant base-$p$ digits of $m$ are $p - i_0 \ge p - k > p/2$, and so $v_p\left(\binom{2m}{m}\right) \ge j = v_p((m+k)!/m!)$. So assume that $p \le 2k$.

**Lemma 2.** *For a set of integers $m$ of asymptotic density 1 we have*

$$\max\{v_p(m+i) : 1 \le i \le k\} \le 3\frac{\log k}{\log p}$$

*for all $k$ with $\frac{1}{2}\log m < k < \log m$ and for all $p \le 2k$.*

*Proof.* We may assume that $x/\log x < m \le x$. If $1 \le i \le k$, the number of $m \le x$ with $v_p(m+i) > 3\log k/\log p$ is $\le x/k^3$. Summing this for $i \le k$ and $p \le 2k$, the count is $\ll x/k = o(x)$. Thus, but for these exceptional values of $m$ we have the inequality in the lemma. □

**Lemma 3.** *For all integers $m, k > 0$ and primes $p$, we have*

$$v_p\left(\frac{(m+k)!}{m!}\right) \le v_p(k!) + \max\{v_p(m+i) : 1 \le i \le k\}.$$

*Proof.* We may assume that the max in the lemma is positive, say it first occurs at $m+i_0$. If $p^j \mid m+i_0$, then the number of multiples of $p^j$ in $\{m+1, \ldots, m+k\} \backslash \{m+i_0\}$ is $\le \lceil k/p^j \rceil - 1 \le \lfloor k/p^j \rfloor$. Summing on $j$ we have

$$v_p\left(\frac{(m+k)!}{m!}\right) \le v_p(m+i_0) + \sum_{p^j \,\mid\, m+i_0} \lfloor k/p^j \rfloor \le v_p(m+i_0) + v_p(k!),$$

which was to be proved. □

As a corollary, we have for all integers $m, k > 0$ and primes $p$,

$$v_p\left(\binom{m+k}{k}\right) \le \max\{v_p(m+i) : 1 \le i \le k\}. \tag{2.1}$$

Using Lemmas 2, 3 we have on a set of integers $m$ of density 1 and for $p \le 2k$, $\frac{1}{2}\log m < k < \log m$ that

$$v_p((m+k)!/m!) \le v_p(k!) + \max\{v_p(m+i) : 1 \le i \le k\} < \frac{k}{p-1} + 3\frac{\log k}{\log p}.$$

For a fixed $\epsilon > 0$, let $\beta_2 = 1/2 - \epsilon$, $\beta_3 = 34/81 - \epsilon$, and $\beta_p = .39$ for $p > 3$. It remains for us to find, in light of Lemma 1, how large we may take $k$ so that

$$\frac{k}{p-1} + 3\frac{\log k}{\log p} \le \beta_p \frac{\log m}{\log p}$$

holds for all $p \le 2k$. The most difficult value of $p$ to accommodate is $p = 2$ and we find that the inequality holds for $k$ as large as $(1/\log 4 - 2\epsilon)\log m$.

**Remark**. It should be possible to show in Lemma 1 that $\alpha_p \sim 1/2$ for all $p \le 2\log x$. A result in this direction is the theorem at the bottom of page 89 of [2]. The constant $1/\log 4$ is optimal in that if $k$ is slightly larger, the set of $m$ where the divisibility holds does not have density 1. In fact, a somewhat larger constant times $\log m$ for $k$ would eliminate all examples, cf. [1].

## 3. Proof of Theorem 2

Let $x$ be large, $D = D_p = 1 + \lfloor \log x / \log p \rfloor$, and $K = \lfloor \exp(.8\sqrt{\log x}) \rfloor$.

**Lemma 4.** *The number of integers $m \le x$ with $v_p(\binom{2m}{m}) \le D/\log D$ for some prime $p \le 2K$ is $o(x)$.*

*Proof.* We follow the proof of Lemma 2 in [4], with some improvements. Let $B = D/\log D$ and fix a prime $p \le 2K$. The number of $m \le x$ with fewer than $B$ base-$p$ digits $\ge p/2$ is smaller than

$$\lceil p/2 \rceil^D \sum_{j<B} \binom{D}{j}.$$

Since $B$ is small compared to $D$, the sum here is $\ll \binom{D}{\lfloor B \rfloor}$. A short calculation with Stirling's formula shows that this expression is $\le \exp(O(D \log\log D/\log D))$, so $\binom{D}{\lfloor B \rfloor} \le x^{.01/\log p}$ for $x$ larger than an absolute constant. Replacing $\lfloor p/2 \rfloor$ with $p/2$ creates an error smaller than $x^{.01/\log p}$, so our count is then at most

$$p^D 2^{-D} x^{.02/\log p} \le x^{1-(2/3)/\log p}$$

for $x$ large. Summing this for $p \le 2K$ we get $\le K x^{1-(2/3)/\log(2K)}$ choices of $m \le x$ with $v_p(\binom{2m}{m}) \le B$. Since this bound is $o(x)$ as $x \to \infty$, the lemma is proved. □

**Lemma 5.** *The number of $m \le x$ such that for some $p < 2K$, $\max\{v_p(m+i) : 1 \le i \le K\} > D/\log D$ is $o(x)$ as $x \to \infty$.*

*Proof.* The count in question is at most

$$\sum_{p<2K} \sum_{i \le K} x/p^{D/\log D} \le K \sum_{p<2K} x/p^{D/\log D} < K \sum_{p<2K} x^{1-1/\log D},$$

since $p^D > x$. As in [4], the summand here is $< K x^{1-1/(1+\log\log x)}$, so the count is at most $K^2 x^{1-1/(1+\log\log x)}$, which by our choice of $K$ is $o(x)$. □

To prove the theorem we need to show that for most integers $m$

$$v_p\left(\binom{m+k}{k}\right) \le v_p\left(\binom{2m}{m}\right) \tag{3.1}$$

for all primes $p$ and for all $k \le K$. First assume that $p > 2k$. Then $v_p\left(\binom{m+k}{k}\right) = v_p((m+k)!/m!)$, so as before (3.1) holds. Thus, it suffices to consider the case that $p \le 2k$. From Lemma 4 we may assume that $v_p\left(\binom{2m}{m}\right) > D/\log D$ for all $p \le 2k$. Using (2.1) and Lemma 5, we may assume that $v_p\left(\binom{m+k}{k}\right) \le D/\log D$, so the desired inequality follows. This completes the proof of Theorem 2.

**Remark**. The constant .8 in Theorem 2 may be replaced with any number $< \sqrt{\log 2}$, but beyond that we do not know of any improvements. But at least one can see that if (1.2) holds for asymptotically all $m$, then for any fixed $\epsilon > 0$ we have $k \ll m^\epsilon$. To see this, take a large prime $p$ and consider integers $m = p^d + c_{d-1}p^{d-1} + \cdots + c_0$, where each $c_i < p/2$. Then $p \nmid \binom{2m}{m}$. Now take $k = p$ so that $p \mid (m+k)!/m!$. The number of choices for $m$ is $> p^d/2^d$, and each $m \le 2p^d$. The proportion of such numbers $m \le 2p^d$ is $\le 2^{1-d}$

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