REMARKS ON THE
MIDDLE BINOMIAL COEFFICIENT

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ABSTRACT. We show that $\binom{m+k}{k} \mid \binom{2m}{m}$ for all $k \leq \exp(.8\sqrt{\log m})$ on a set of numbers m of asymptotic density 1. We also show that $(m+1)(m+2)\ldots(m+k)$ divides $\binom{2m}{m}$ on a set of asymptotic density 1 for k as large as $.7 \log m$.

1. INTRODUCTION

In a recent paper [3] I proved some elementary results about the middle binomial coefficient $\binom{2m}{m}$. In particular, Theorem 2 of that paper shows that for each fixed positive integer k, $m + k \mid \binom{2m}{m}$ for a set of integers m of asymptotic density 1. Following the proof it is left as an exercise to show that the product $(m+1)(m+2)\ldots(m+k)$ divides $\binom{2m}{m}$ on a set of asymptotic density 1. We prove the following strengthening.

Theorem 1.1. *For any fixed $\eta < 1/\log 4 = .721\ldots$, we have for a set of integers m of asymptotic density 1 that*

$$(1.1) \qquad \frac{(m+k)!}{m!} \,\bigg|\, \binom{2m}{m}$$

for all positive integers $k \leq \eta \log m$.

Replacing the product with $\binom{m+k}{k}$, we prove the following theorem.

Theorem 1.2. *For m in a set of asymptotic density 1, we have*

$$(1.2) \qquad \binom{m+k}{k} \,\bigg|\, \binom{2m}{m}$$

for all positive integers $k \leq \exp(.8\sqrt{\log m})$.

These theorems may be of interest in the context of recent work towards settling a problem of Erdős using AI (see [5]). One might compare this note with the write-up of Sothanaphan [4].

2. Proof of Theorem 1.1

We let p denote a prime variable, and we let v_p be the function which returns the exponent on p in the prime factorization of its argument.

Lemma 2.1. *Let*

$$\alpha_p(m) = \frac{v_p\left(\binom{2m}{m}\right)}{\log m / \log p}.$$

For a set of integers m of asymptotic density 1 we have $\alpha_2(m) = 1/2 + o(1)$, $\alpha_3(m) \geq 34/81 + o(1)$, and $\alpha_p(m) \geq .39$ for all $3 < p < 2\log m$.

Proof. Note that m has $\lfloor \log m / \log p \rfloor + 1$ base-p digits and we expect roughly half of these to be $\geq p/2$. More precisely, for $p = 2$, we expect half to be $\geq p/2$ and for p odd we expect $(p-1)/2p$ of them to be $\geq p/2$. For $3 < p < 2\log x$ a calculation using the binomial distribution shows that the number of $m \leq x$ with fewer than $.39\log m / \log p$ base-p digits at least $p/2$ is $O(x^{1-c/\log p})$. Here c is a small positive constant. Summing this for $p < 2\log x$ we get an expression that is $o(x)$ as $x \to \infty$. We can add to this exceptional set those m with $\alpha_2(m) \leq 1/2 - \epsilon$ for any fixed positive ϵ, with c now depending on ϵ. For $p = 3$ we consider the base-27 expansion of m, finding that the average number of base-3 carries doubling such a digit engenders is $34/27$, so our assertion about $\alpha_3(m)$ follows as well. $\square$

If $p > 2k$, $v_p((m+k)!/m!) = \max\{v_p(m+i) : 1 \leq i \leq k\}$. As in [3], if this max is j, occurring at $m + i_0$, then the j least significant base-p digits of $m + i_0$ are 0, so the j least significant base-p digits of m are $p - i_0 \geq p - k > p/2$, and so $v_p(\binom{2m}{m}) \geq j = v_p((m+k)!/m!)$. So assume that $p \leq 2k$.

Lemma 2.2. *For a set of integers m of asymptotic density 1 we have*

$$\max\{v_p(m+i) : 1 \leq i \leq k\} \leq 3\frac{\log k}{\log p}$$

for all k with $\frac{1}{2}\log m < k < \log m$ and for all $p \leq 2k$.

Proof. We may assume that $x/\log x < m \leq x$. If $1 \leq i \leq k$, the number of $m \leq x$ with $v_p(m+i) > 3\log k / \log p$ is $\leq x/k^3$. Summing this for $i \leq k$ and $p \leq 2k$, the count is $\ll x/k = o(x)$. Thus, but for these exceptional values of m we have the inequality in the lemma. $\square$

Lemma 2.3. *For all integers $m, k > 0$ and primes p, we have*

$$v_p\left(\frac{(m+k)!}{m!}\right) \leq v_p(k!) + \max\{v_p(m+i) : 1 \leq i \leq k\}.$$

Proof. We may assume that the max in the lemma is positive, say it first occurs at $m + i_0$. If $p^j \mid m + i_0$, then the number of multiples of p^j in $\{m + 1, \ldots, m + k\} \setminus \{m + i_0\}$ is $\leq \lceil k/p^j \rceil - 1 \leq \lfloor k/p^j \rfloor$. Summing on j we have

$$v_p\Big(\frac{(m+k)!}{m!}\Big) \leq v_p(m + i_0) + \sum_{p^j \mid m + i_0} \lfloor k/p^j \rfloor \leq v_p(m + i_0) + v_p(k!),$$

which was to be proved. □

As a corollary, we have for all integers $m, k > 0$ and primes p,

$$(2.1) \qquad v_p\Big(\binom{m+k}{k}\Big) \leq \max\{v_p(m + i) : 1 \leq i \leq k\}.$$

Using Lemmas 2.2, 2.3 we have on a set of integers m of density 1 and for $p \leq 2k$, $\frac{1}{2}\log m < k < \log m$ that

$$v_p((m+k)!/m!) \leq v_p(k!) + \max\{v_p(m+i) : 1 \leq i \leq k\} < \frac{k}{p-1} + 3\frac{\log k}{\log p}.$$

For a fixed $\epsilon > 0$, let $\beta_2 = 1/2 - \epsilon$, $\beta_3 = 34/81 - \epsilon$, and $\beta_p = .39$ for $p > 3$. It remains for us to find, in light of Lemma 2.1, how large we may take k so that

$$\frac{k}{p-1} + 3\frac{\log k}{\log p} \leq \beta_p \frac{\log m}{\log p}$$

holds for all $p \leq 2k$. The most difficult value of p to accommodate is $p = 2$ and we find that the inequality holds for k as large as $(1/\log 4 - 2\epsilon)\log m$.

Remark. It should be possible to show in Lemma 2.1 that $\alpha_p \sim 1/2$ for all $p \leq 2\log x$. A result in this direction is the theorem at the bottom of page 89 of [2]. The constant $1/\log 4$ is optimal in that if k is slightly larger, the set of m where the divisibility holds does not have density 1. In fact, a somewhat larger constant times $\log m$ for k would eliminate all examples, cf. [1].

3. Proof of Theorem 1.2

Let x be large, $D = D_p = 1 + \lfloor \log x/\log p \rfloor$, and $K = \lfloor \exp(.8\sqrt{\log x}) \rfloor$.

Lemma 3.1. *The number of integers $m \leq x$ with $v_p(\binom{2m}{m}) \leq D/\log D$ for some prime $p \leq 2K$ is $o(x)$.*

Proof. We follow the proof of [3, Lemma 2] with some improvements. Let $B = D/\log D$ and fix a prime $p \le 2K$. The number of $m \le x$ with fewer than B base-p digits $\ge p/2$ is smaller than

$$\lceil p/2 \rceil^D \sum_{j < B} \binom{D}{j}.$$

Since B is small compared to D, the sum here is $\ll \binom{D}{\lfloor B \rfloor}$. A short calculation with Stirling's formula shows that this expression is $\le \exp(O(D \log \log D / \log D))$, so $\binom{D}{\lfloor B \rfloor} \le x^{.01/\log p}$ for x larger than an absolute constant. Replacing $\lfloor p/2 \rfloor$ with $p/2$ creates an error smaller than $x^{.01/\log p}$, so our count is then at most

$$p^D 2^{-D} x^{.02/\log p} \le x^{1-(2/3)/\log p}$$

for x large. Summing this for $p \le 2K$ we get $\le Kx^{1-(2/3)/\log(2K)}$ choices of $m \le x$ with $v_p(\binom{2m}{m}) \le B$. Since this bound is $o(x)$ as $x \to \infty$, the lemma is proved. $\qquad\square$

Lemma 3.2. *The number of $m \le x$ such that for some $p < 2K$, $\max\{v_p(m+i) : 1 \le i \le K\} > D/\log D$ is $o(x)$ as $x \to \infty$.*

Proof. The count in question is at most

$$\sum_{p < 2K} \sum_{i \le K} x/p^{D/\log D} \le K \sum_{p < 2K} x/p^{D/\log D} < K \sum_{p < 2K} x^{1-1/\log D},$$

since $p^D > x$. As in [3], the summand here is $< Kx^{1-1/(1+\log\log x)}$, so the count is at most $K^2 x^{1-1/(1+\log\log x)}$, which by our choice of K is $o(x)$. $\qquad\square$

To prove the theorem we need to show that for most integers m

$$(3.1) \qquad v_p\left(\binom{m+k}{k}\right) \le v_p\left(\binom{2m}{m}\right)$$

for all primes p and for all $k \le K$. First assume that $p > 2k$. Then $v_p(\binom{m+k}{k}) = v_p((m+k)!/m!)$, so as before (3.1) holds. Thus, it suffices to consider the case that $p \le 2k$. From Lemma 3.1 we may assume that $v_p(\binom{2m}{m}) > D/\log D$ for all $p \le 2k$. Using (2.1) and Lemma 3.2, we may assume that $v_p(\binom{m+k}{k}) \le D/\log D$, so the desired inequality follows. This completes the proof of Theorem 1.2.

Remark. The constant .8 in Theorem 1.2 may be replaced with any number $< \sqrt{\log 2}$, but beyond that we do not know of any improvements. But at least one can see that if (1.2) holds for asymptotically all m, then for any fixed $\epsilon > 0$ we have $k \ll m^\epsilon$. To see this, take a large prime p and consider integers $m = p^d + c_{d-1}p^{d-1} + \cdots + c_0$, where each

$c_i < p/2$. Then $p \nmid \binom{2m}{m}$. Now take $k = p$ so that $p \mid (m+k)!/m!$. The number of choices for m is $> p^d/2^d$, and each $m \le 2p^d$. The proportion of such numbers $m \le 2p^d$ is $\le 2^{1-d}$

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