THE ERDÖS CONJECTURE FOR PRIMITIVE SETS

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ABSTRACT. A subset of the integers larger than 1 is primitive if no member divides another. Erdős proved in 1935 that the sum of $1/(a \log a)$ for $a$ running over a primitive set $A$ is universally bounded over all choices for $A$. In 1988 he asked if this universal bound is attained for the set of prime numbers. In this paper we make some progress on several fronts and show a connection to certain prime number "races" such as the race between $\pi(x)$ and $\text{li}(x)$.

1. Introduction

A set of positive integers $>1$ is called primitive if no element divides any other (for convenience, we exclude the singleton set \{1\}). There are a number of interesting and sometimes unexpected theorems about primitive sets. After Besicovitch [4], we know that the upper asymptotic density of a primitive set can be arbitrarily close to 1/2, whereas the lower asymptotic density is always 0. Using the fact that if a primitive set has a finite reciprocal sum, then the set of multiples of members of the set has an asymptotic density, Erdős gave an elementary proof that the set of nondeficient numbers (i.e., $\sigma(n)/n \geq 2$, where $\sigma$ is the sum-of-divisors function) has an asymptotic density. Though the reciprocal sum of a primitive set can possibly diverge, Erdős [9] showed that for a primitive set $A$,

$$\sum_{a \in A} \frac{1}{a \log a} < \infty.$$ 

In fact, the proof shows that these sums are uniformly bounded as $A$ varies over primitive sets.

Some years later in a 1988 seminar in Limoges, Erdős suggested that in fact we always have

$$(1.1) \quad f(A) := \sum_{a \in A} \frac{1}{a \log a} \leq \sum_{p \in \mathcal{P}} \frac{1}{p \log p},$$

where $\mathcal{P}$ is the set of prime numbers. The assertion (1.1) is now known as the Erdős conjecture for primitive sets.

In 1991, Zhang [19] proved the Erdős conjecture for primitive sets $A$ with no member having more than 4 prime factors (counted with multiplicity).
After Cohen [6], we have
\[ C := \sum_{p \in \mathcal{P}} \frac{1}{p \log p} = 1.63661632336 \ldots , \]
the sum over primes in \((\ref{eq:1.1})\). Using the original Erdős argument in [9], Erdős and Zhang showed that \(f(A) < 2.89\) for a primitive set \(A\), which was later improved by Robin to 2.77. These unpublished estimates are reported in [11] by Erdős–Zhang, who used another method to show that \(f(A) < 1.84\). Shortly after, Clark [5] claimed that \(f(A) \leq e^\gamma = 1.781072\ldots\). However, his brief argument appears to be incomplete.

Our principal results are the following.

**Theorem 1.1.** For any primitive set \(A\) we have \(f(A) < e^\gamma\).

**Theorem 1.2.** For any primitive set \(A\) with no element divisible by 8, we have \(f(A) < C + 2.37 \times 10^{-7}\).

Say a prime \(p\) is **Erdős strong** if for any primitive set \(A\) with the property that each element of \(A\) has the same least prime factor \(p\), we have \(f(A) \leq 1/(p \log p)\). We conjecture that every prime is Erdős strong. Note that the Erdős conjecture \((\ref{eq:1.1})\) would immediately follow, though it is not clear that the Erdős conjecture implies our conjecture. Just proving our conjecture for the case of \(p = 2\) would give the inequality in Theorem 1.2 for all primitive sets \(A\). Currently the best we can do for a primitive set \(A\) of even numbers is that \(f(A) < e^\gamma/2\); see Proposition 2.1 below.

For part of the next result, we assume the Riemann Hypothesis (RH) and the Linear Independence Hypothesis (LI), which asserts that the sequence of numbers \(\gamma_n > 0\) such that \(\zeta(1/2 + i\gamma_n) = 0\) is linearly independent over \(\mathbb{Q}\).

**Theorem 1.3.** Unconditionally, all of the odd primes among the first \(10^8\) primes are Erdős strong. Assuming RH and LI, the Erdős strong primes have relative lower logarithmic density > 0.995.

The proof depends strongly on a recent result of Lamzouri [13], who was interested in the “Mertens race” between \(\prod_{p \leq x} (1 - 1/p)\) and \(1/(e^\gamma \log x)\).

For a primitive set \(A\), let \(\mathcal{P}(A)\) denote the support of \(A\), i.e., the set of prime numbers that divide some member of \(A\). It is clear that the Erdős conjecture \((\ref{eq:1.1})\) is equivalent to the same assertion where the prime sum is over \(\mathcal{P}(A)\).

**Theorem 1.4.** If \(A\) is a primitive set with \(\mathcal{P}(A) \subset [3, \exp(10^6)]\), then
\[ f(A) \leq \sum_{p \in \mathcal{P}(A)} \frac{1}{p \log p}. \]

If some primitive set \(A\) of odd numbers exists with \(f(A) > \sum_{p \in \mathcal{P}(A)} 1/(p \log p)\), Theorem 1.4 suggests that it will be very difficult indeed to give a concrete example!

For a positive integer \(n\), let \(\Omega(n)\) denote the number of prime factors of \(n\) counted with multiplicity. Let \(\mathbb{N}_k\) denote the set of integers \(n\) with \(\Omega(n) = k\). Zhang [20] proved a result that implies \(f(\mathbb{N}_k) < f(\mathbb{N}_1)\) for each \(k \geq 2\), so that the Erdős conjecture holds for the primitive sets \(\mathbb{N}_k\). More recently, Banks and Martin [2] conjectured that \(f(\mathbb{N}_1) > f(\mathbb{N}_2) > f(\mathbb{N}_3) > \cdots\). The inequality \(f(\mathbb{N}_2) > f(\mathbb{N}_3)\) was just established by Bayless, Kinlaw, and Klyve [3]. We prove the following result.

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Theorem 1.5. There is a positive constant $c$ such that $f(\mathbb{N}_k) \geq c$ for all $k$.

We let the letters $p, q, r$ represent primes. In addition, we let $p_n$ represent the $n$th prime. For an integer $a > 1$, we let $P(a)$ and $p(a)$ denote the largest and smallest prime factors of $a$. Modifying the notation introduced in [11], for a primitive set $A$ let

\[
A_p = \{a \in A : p(a) \geq p\}, \\
A'_p = \{a \in A : p(a) = p\}, \\
A''_p = \{a/p : a \in A'_p\}.
\]

We let $f(a) = 1/(a \log a)$, and so $f(A) = \sum_{a \in A} f(a)$. In this language, Zhang’s full result [20] states that $f((\mathbb{N}_k)'_p) \leq f(p)$ for all primes $p$, $k \geq 1$. We also let

\[
g(a) = \frac{1}{a} \prod_{p < P(a)} \left(1 - \frac{1}{p}\right), \quad h(a) = \frac{1}{a \log P(a)},
\]

with $g(A) = \sum_{a \in A} g(a)$ and $h(A) = \sum_{a \in A} h(a)$.

2. The Erdős approach

In this section we will prove Theorem 1.1. We begin with an argument inspired by the original 1935 paper of Erdős [9].

Proposition 2.1. For any primitive set $A$, if $q \notin A$, then

\[
f(A'_q) < e^\gamma g(q) = \frac{e^\gamma}{q} \prod_{p < q} \left(1 - \frac{1}{p}\right).\]

Proof. For each $a \in A'_q$, let $S_a = \{ba : p(b) \geq P(a)\}$. Note that $S_a$ has asymptotic density $g(a)$. Since $A'_q$ is primitive, we see that the sets $S_a$ are pairwise disjoint. Further, the union of the sets $S_a$ is contained in the set of all natural numbers $m$ with $p(m) = q$, which has asymptotic density $g(q)$. Thus, the sum of densities for each $S_a$ is dominated by $g(q)$, that is,

(2.1) \[ g(A'_q) = \sum_{a \in A'_q} g(a) \leq g(q). \]

By Theorem 7 in [17], we have for $x \geq 285$,

(2.2) \[ \prod_{p \leq x} \left(1 - \frac{1}{p}\right) > \frac{1}{e^\gamma \log(2x)}, \]

which may be extended to all $x \geq 1$ by a calculation. Thus, since each $a \in A'_q$ is composite,

\[
g(a) = \frac{1}{a} \prod_{p < P(a)} \left(1 - \frac{1}{p}\right) > \frac{e^{-\gamma}}{a \log (2P(a))} \geq \frac{e^{-\gamma}}{a \log a} = e^{-\gamma} f(a).
\]

Hence by (2.1),

\[
f(A'_q)/e^\gamma < g(A'_q) \leq g(q).
\]

\[\square\]
Remark 2.2. Let \( \sigma \) denote the sum-of-divisors function, and let \( A \) be the set of \( n \) with \( \sigma(n)/n \geq 2 \) and \( \sigma(d)/d < 2 \) for all proper divisors \( d \) of \( n \), the set of primitive nondeficient numbers. Then an appropriate analog of \( g(A) \) gives the density of nondeficient numbers recently shown in [12] to lie in the tight interval \((0.2476171, 0.2476475)\). In [14], an analog of Proposition 2.1 is a key ingredient for sharp bounds on the reciprocal sum of the primitive nondeficient numbers.

Remark 2.3. We have \( g(\mathcal{P}) = 1 \). Indeed, it is easy to see by induction over primes \( r \) that

\[
\sum_{p \leq r} g(p) = \sum_{p \leq r} \frac{1}{p} \prod_{q < p} \left( 1 - \frac{1}{q} \right) = 1 - \prod_{p \leq r} \left( 1 - \frac{1}{p} \right).
\]

Letting \( r \to \infty \) we get that \( g(\mathcal{P}) = 1 \). There is also a holistic way of seeing this. Since \( g(p) \) is the density of the set of integers with least prime factor \( p \), it would make sense that \( g(\mathcal{P}) \) is the density of the set of integers which have a least prime factor, and this density is 1. To make this rigorous, one notes that the density of the set of integers whose least prime factor is \( > y \) tends to 0 as \( y \to \infty \). As a consequence of \( g(\mathcal{P}) = 1 \), we have

\[
(2.3) \quad \sum_{p > 2} g(p) = \frac{1}{2},
\]

an identity we will find to be useful.

For a primitive set \( A \), let

\[
A^k = \{ a \in A : 2^k \parallel a \}, \quad B^k = \{ a/2^k : a \in A^k \}.
\]

The next result will help us prove Theorem 1.1.

Lemma 2.4. For a primitive set \( A \), let \( k \geq 1 \) be such that \( 2^k/ \in A \). Then we have

\[
f(A^k) < e^\gamma \sum_{p \in A, p > 2} g(p).
\]

Proof. The hypothesis \( 2^k \notin A \) implies that \( 1 \notin B^k \), so that \( B^k \) is a primitive set. If \( 2^k p \notin A \) for a prime \( p > 2 \), then \( (B^k)'_p \) is a primitive set of odd composite numbers, so by Proposition 2.1

\[
f((B^k)'_p) < e^\gamma g(p).
\]

Now if \( 2^k p \in A \) for some odd prime \( p \), then \( (B^k)'_p = \{ p \} \), and note that \( p \notin A \) by primitivity. We have \( f(2^k p) < 2^{-k} e^\gamma g(p) \) since

\[
\frac{1}{2^k p \log(2^k p)} \leq \frac{1}{2^k p \log(2p)} < \frac{e^\gamma}{2^k} g(p),
\]

which follows from (2.2). Since \( (B^k)'_p \neq \emptyset \) implies \( p \notin A \), we have

\[
f(A^k) = \sum_{p \notin A, p > 2} f(2^k \cdot (B^k)'_p) \leq \sum_{p \in B^k, p \notin A, p > 2} f(2^k p) + 2^{-k} \sum_{p \notin B^k, p \notin A, p > 2} f((B^k)'_p)
\]

\[
< \frac{e^\gamma}{2^k} \sum_{p \notin A, p > 2} g(p).
\]

With Lemma 2.4 in hand, we prove \( f(A) < e^\gamma \).
Proof of Theorem 2.1. From Erdős–Zhang [11], we have that $f(A_3) < 0.92$. If $2 \in A$, then $A'_2 = \{2\}$, so that $f(A) = f(A_3) + f(A'_2) < 0.92 + 1/(2 \log 2) < e^{-\gamma}$. Hence we may assume that $2 \notin A$. If $A$ contains every odd prime, then $A'_2$ consists of at most one power of 2, and the calculation just concluded shows we may assume this is not the case. Hence there is at least one odd prime $p_0 \notin A$. By Proposition 2.1, we have

$$f(A) = \sum_{p \in A} f(A'_p) = \sum_{p \in A} f(p) + \sum_{\rho \notin A} f(A'_\rho) < \sum_{p \in A} f(p) + e^{\gamma} \sum_{p \notin A, p > 2} g(p) + f(A'_2).$$

First suppose $A$ contains no powers of 2. Then by Lemma 2.4,

$$f(A'_2) = \sum_{k \geq 1} f(A^k) < \sum_{k \geq 1} \frac{e^{\gamma}}{2^k} \sum_{p \notin A, p > 2} g(p) = e^{\gamma} \sum_{p \notin A, p > 2} g(p).$$

Substituting into (2.4), we conclude, using (2.3), that

$$f(A) < \sum_{p \in A} f(p) + 2e^{\gamma} \sum_{p \notin A, p > 2} g(p) \leq 2e^{\gamma} \sum_{p > 2} g(p) = e^{-\gamma}.$$

For the last inequality in (2.5) we used that for every prime $p$,

$$\frac{f(p)}{e^{\gamma} g(p)} < 1.082,$$

which follows after a short calculation using [17, Theorem 7].

Now if $2^K \in A$ for some positive integer $K$, then $K$ is unique and $K \geq 2$. Also $A^K = \{2^K\}$ and $A^k = \emptyset$ for all $k > K$, so again by Lemma 2.4

$$f(A'_2) = \sum_{k=1}^K f(A^k) < f(2^K) + \sum_{k=1}^{K-1} \frac{e^{\gamma}}{2^k} \sum_{p \notin A, p > 2} g(p) = f(2^K) + (1 - 2^{1-K}) e^{\gamma} \sum_{p \notin A, p > 2} g(p).$$

Substituting into (2.4) gives

$$f(A) < \sum_{p \in A} f(p) + f(2^K) + (2 - 2^{1-K}) e^{\gamma} \sum_{p \notin A, p > 2} g(p) \leq f(2^K) + (2 - 2^{1-K}) e^{\gamma} \sum_{p > 2} g(p)$$

(2.7)

\[ \leq f(2^K) + (1 - 2^{-K}) e^{\gamma} < e^{\gamma}, \]

using identity (2.3), inequality (2.6), and $f(2^K) < 2^{-K} e^{\gamma}$. This completes the proof.

\[ \square \]

3. MERTENS PRIMES

In this section we will prove Theorems 1.3 and 1.4. Note that by Mertens’ theorem,

$$\prod_{p < x} \left(1 - \frac{1}{p}\right) \sim \frac{1}{e^{\gamma} \log x}, \quad x \to \infty,$$

where $\gamma$ is Euler’s constant. We say a prime $q$ is Mertens if

$$e^{\gamma} \prod_{p < q} \left(1 - \frac{1}{p}\right) \leq \frac{1}{\log q}$$

(3.1)
and let $\mathcal{P}^{\text{Mert}}$ denote the set of Mertens primes. We are interested in Mertens primes because of the following consequence of Proposition 2.1, which shows that every Mertens prime is Erdős strong.

**Corollary 3.1.** Let $A$ be a primitive set. If $q \in \mathcal{P}^{\text{Mert}}$, then $f(A_q') \leq f(q)$. Hence if $A_q' \subset \{q\}$ for all $q \notin \mathcal{P}^{\text{Mert}}$, then $A$ satisfies the Erdős conjecture.

**Proof.** By Proposition 2.1 we have

$$f(A_q') \leq \max\{e^\gamma g(q), f(q)\}.$$  

If $q \in \mathcal{P}^{\text{Mert}}$, then

$$e^\gamma g(q) = e^\gamma \prod_{p < q} \left(1 - \frac{1}{p}\right) \leq \frac{1}{q \log q} = f(q),$$

so $f(A_q') \leq f(q)$. □

Now, one would hope that the Mertens inequality (3.1) holds for all primes $q$. However, (3.1) fails for $q = 2$ since $e^\gamma > 1/\log 2$. Nevertheless, we have computed that $q$ is indeed a Mertens prime for all $2 < q \leq p_{10^8} = 2,038,074,743$, thus proving the unconditional part of Theorem 1.3.

### 3.1. Proof of Theorem 1.3

To complete the proof, we use a result of Lamzouri [13] relating the Mertens inequality to the race between $\pi(x)$ and $\text{li}(x)$, studied by Rubinstein and Sarnak [18]. Under the assumption of RH and LI, he proved that the set $\mathcal{N}$ of real numbers $x$ satisfying

$$e^\gamma \prod_{p \leq x} \left(1 - \frac{1}{p}\right) > \frac{1}{\log x}$$

has logarithmic density $\delta(\mathcal{N})$ equal to the logarithmic density of numbers $x$ with $\pi(x) > \text{li}(x)$, and in particular

$$\delta(\mathcal{N}) = \lim_{x \to \infty} \frac{1}{\log x} \int_{t \in \mathcal{N} \cap [2,x]} \frac{dt}{t} = 0.00000026 \ldots .$$

We note that if a prime $p = p_n \in \mathcal{N}$, then for $p' = p_{n+1}$ we have $[p,p') \subset \mathcal{N}$ because the prime product on the left-hand side is constant on $[p,p')$, while $1/\log x$ is decreasing for $x \in [p,p')$.

The set of primes $Q$ in $\mathcal{N}$ is precisely the set of non-Mertens primes, so $Q = \mathcal{P} \setminus \mathcal{P}^{\text{Mert}}$. From the above observation, we may leverage knowledge of the continuous logarithmic density $\delta(\mathcal{N})$ to obtain an upper bound on the relative (upper) logarithmic density of non-Mertens primes

$$\delta(Q) := \limsup_{x \to \infty} \frac{1}{\log x} \sum_{p \leq x} \log p \frac{\log p}{p},$$  

From the above observation, we have

$$\delta(\mathcal{N}) \geq \limsup_{x \to \infty} \frac{1}{\log x} \sum_{p \leq x} \int_p^{p_0} \frac{dt}{t} = \limsup_{x \to \infty} \frac{1}{\log x} \sum_{p \leq x} \log(p'/p).$$

Then letting $d_p = p' - p$ be the gap between consecutive primes, we have

$$\delta(\mathcal{N}) \geq \limsup_{x \to \infty} \frac{1}{\log x} \sum_{p \leq x} \frac{d_p}{p},$$
since \( \sum \log(p'/p) = \sum d_p/p + O(1) \). The average gap is roughly \( \log p \), so we may consider the primes for which \( d_p < \epsilon \log p \) for a small positive constant \( \epsilon \) to be determined.

We claim that
\[
\limsup_{x \to \infty} \frac{1}{\log x} \sum_{p \leq x, d_p < \epsilon \log p} \frac{\log p}{p} \leq 16\epsilon,
\]
from which it follows that
\[
\bar{\delta}(Q) = \limsup_{x \to \infty} \frac{1}{\log x} \sum_{\substack{p \leq x, \ \bar{p} \in Q \ \text{prime}}} \frac{\log p}{p} \leq \limsup_{x \to \infty} \frac{1}{\log x} \left( \sum_{\substack{p \leq x, \ \bar{p} \in Q \ \text{prime}}} \frac{d_p/\epsilon}{p} + \sum_{d_p \geq \epsilon \log p} \frac{\log p}{p} \right)
\]
\[
\leq \bar{\delta}(N)/\epsilon + 16\epsilon.
\]
Hence to prove Theorem 1.3 it suffices to prove (3.4), since taking \( \epsilon = \sqrt{\bar{\delta}(N)}/4 \) gives
\[
\bar{\delta}(Q) < 8\sqrt{\bar{\delta}(N)} < 4.2 \times 10^{-3}.
\]

By Riesel-Vaughan [16, Lemma 5], the number of primes \( p \) up to \( x \) with \( p + d \) also prime is at most
\[
\sum_{\substack{p \leq x, \ \bar{p} \in Q \ \text{prime}}} 1 \leq \frac{8c_2 x}{\log^2 x} \prod_{p > 2} \frac{p - 1}{p - 2},
\]
where \( c_2 \) is for the twin-prime constant \( 2 \prod_{p \geq 2} \frac{p(p - 2)}{(p - 1)^2} = 1.3203 \ldots \). Denote the prime product by \( M_n \), and consider the multiplicative function \( H(d) = \sum_{u \mid d} H(u)F(d/u) \). We have \( H(2^k) = 0 \) for all \( k \geq 1 \), and for \( p > 2 \) we have \( H(p) = F(p) - 1 \) and \( H(p^k) = 0 \) if \( k \geq 2 \). Thus,
\[
\sum_{d \leq y} F(d) = \sum_{d \leq y} \sum_{u \mid d} H(u) = \sum_{u \leq y} H(u) \sum_{d \leq y/u} 1 \leq y \sum_{u \leq y} \frac{H(u)}{u} \leq y \prod_{p > 2} \left( 1 + \frac{H(p)}{p} \right)
\]
\[
= y \prod_{p > 2} \left( 1 + \frac{(p - 1)/(p - 2) - 1}{p} \right) = y \prod_{p > 2} \left( 1 + \frac{1}{p(p - 2)} \right) = \frac{2y}{e^2}.
\]
Using (3.6), we have
\[
\sum_{\substack{p \leq x, \ \bar{p} \in Q \ \text{prime}}} 1 \leq \sum_{d \leq \epsilon \log x} \sum_{\substack{p \leq x, \ \bar{p} \in Q \ \text{prime}}} 1 \leq \frac{8c_2 x}{\log^2 x} \sum_{d \leq \epsilon \log x} F(d) \leq \epsilon \frac{8c_2 e^2 x}{\log x} \leq \epsilon \frac{16x}{\log x}.
\]
Thus, (3.4) now follows by partial summation, and the proof is complete.

Remark 3.2. The concept of relative upper logarithmic density of the set of non-Mertens primes in (3.3) can be replaced in the theorem with
\[
\delta_0(Q) := \limsup_{x \to \infty} \frac{1}{\log \log x} \sum_{\substack{p \leq x, \ \bar{p} \in Q \ \text{prime}}} \frac{1}{p}.
\]
Indeed, \( \tilde{\delta}_0(Q) \leq \tilde{\delta}(Q) \) follows from the identity

\[
\sum_{\substack{p \leq x \atop p \in \mathcal{Q}}} \frac{1}{p} = \frac{1}{\log x} \sum_{p \leq x \atop p \in \mathcal{Q}} \log p \quad + \quad \int_2^x \frac{1}{t \log t} \sum_{p \leq t \atop p \in \mathcal{Q}} \log p \, dt.
\]

**Remark 3.3.** Greg Martin has indicated to us that one may be able to prove (under RH and LI) that the relative logarithmic density of \( \mathcal{Q} \) exists and is equal to the logarithmic density of \( \mathcal{N} \). This topic will be addressed in a future paper.

### 3.2. Proof of Theorem 1.4.

We now use some numerical estimates of Dusart [8] to prove Theorem 1.4.

We say a pair of primes \( p \leq q \) is a Mertens pair if

\[
\prod_{p \leq r < q} \left( 1 - \frac{1}{r} \right) > \frac{\log p}{\log pq}.
\]

We claim that every pair of primes \( p, q \) with \( 2 < p \leq q < e^{10^6} \) is a Mertens pair. Assume this and let \( A \) be a primitive set supported on the odd primes up to \( e^{10^6} \). By (2.1), if \( p \not\in A \), we have

\[
\frac{1}{p} \geq \sum_{a \in A_p'} \frac{1}{a} \prod_{p \leq r < P(a)} \left( 1 - \frac{1}{r} \right) > \sum_{a \in A_p'} \frac{\log p}{a \log P(a)}
\]

\[
\geq \sum_{a \in A_p'} \frac{\log p}{a \log a} = f(A_p') \log p.
\]

Dividing by \( \log p \) we obtain \( f(A_p') \leq f(p) \), which also holds if \( p \in A \). Thus, the claim about Mertens pairs implies the theorem.

To prove the claim, first note that if \( p \) is a Mertens prime, then \( p, q \) is a Mertens pair for all primes \( q \geq p \). Indeed, we have

\[
\prod_{p \leq r < q} \left( 1 - \frac{1}{r} \right) = \prod_{r < p} \left( 1 - \frac{1}{r} \right)^{-1} \prod_{r < q} \left( 1 - \frac{1}{r} \right) > e^{\gamma} \log p \prod_{r < q} \left( 1 - \frac{1}{r} \right).
\]

By (2.2), this last product exceeds \( e^{-\gamma} / \log(2q) > e^{-\gamma} / \log(pq) \), and using this in the above display shows that \( p, q \) is indeed a Mertens pair. Since all of the odd primes up to \( p_{10^8} \) are Mertens, to complete the proof of our assertion, it suffices to consider the case when \( p > p_{10^8} \). Define \( E_p \) via the equation

\[
\prod_{r < p} \left( 1 - \frac{1}{r} \right) = \frac{1 + E_p}{e^{\gamma} \log p}.
\]

Using [8, Theorem 5.9], we have for \( p > 2,278,382 \),

\[
|E_p| \leq 2/(\log p)^3.
\]

A routine calculation shows that if \( p \leq q < e^{4.999(\log p)^4} \), then

\[
\prod_{p \leq r < q} \left( 1 - \frac{1}{r} \right) = \left( \frac{\log p}{\log q} \right) \cdot \frac{1 + E_q}{1 + E_p} > \frac{\log p}{\log pq}.
\]

It remains to note that \( 4.999(\log p_{10^8})^4 > 1,055,356 \).

It seems interesting to record the principle that we used in the proof.
Corollary 3.4. If \( A \) is a primitive set such that \( p(a), P(a) \) is a Mertens pair for each \( a \in A \), then \( f(A) \leq f(P(A)) \).

Remark 3.5. Kevin Ford has noted to us the remarkable similarity between the concept of Mertens primes in this paper and the numbers

\[
\gamma_n = \left( \gamma + \sum_{k \leq n} \frac{\log p_k}{p_k - 1} \right) \prod_{k \leq n} \left( 1 - \frac{1}{p_k} \right)
\]
discussed in Diamond–Ford [7]. In particular, while it may not be obvious from the definition, the analysis in [7] on whether the sequence \( \gamma_1, \gamma_2, \ldots \) is monotone is quite similar to the analysis in [13] on the Mertens inequality. Though the numerical evidence seems to indicate we always have \( \gamma_{n+1} < \gamma_n \), this is disproved in [7], and it is indicated there that the first time this fails may be near \( 1.9 \cdot 10^{215} \). This may also be near where the first odd non-Mertens prime exists. If this is the case and under assumption of RH, it may be that every pair of primes \( p > q \) is a Mertens pair when \( p > 2 \) and \( q < \exp(3 \cdot 10^{11}) \).

4. Odd primitive sets

We say a primitive set is odd if every member of the set is an odd number. In this section we prove Theorem 1.2 and establish a curious result on parity for primitive sets.

Let

\[
\epsilon_0 = \sum_{p > 2 \atop p \notin P^{Mert}} (e^\gamma g(p) - f(p)).
\]

Lemma 4.1. We have \( 0 \leq \epsilon_0 < 2.37 \times 10^{-7} \).

Proof. By the definition of \( P^{Mert} \), the summands in the definition of \( \epsilon_0 \) are non-negative, so that \( \epsilon_0 \geq 0 \). If \( p > 2 \) is not Mertens, then \( p > p_{10^8} > 2 \times 10^9 \), so that [8,7] shows that

\[
e^\gamma g(p) - f(p) < \frac{1}{5p(\log p)^4}.
\]

By [8, Proposition 5.16], we have

\[
p_n > n(\log n + \log \log n - 1 + (\log \log n - 2.1)/\log n), \quad n \geq 2.
\]

Using this we find that

\[
\sum_{n > 10^8} \frac{1}{5p_n(\log p_n)^4} < 2.37 \times 10^{-7},
\]

which with (4.1) completes the proof.

Remark 4.2. Clearly, a smaller bound for \( \epsilon_0 \) would follow by raising the search limit for Mertens primes. Another small improvement could be made using the estimate in [4] for \( p_n \). It follows from the ideas in Remark 3.3 that \( \epsilon_0 > 0 \). Further, it may be provable from the ideas in Remark 3.5 that \( \epsilon_0 < 10^{-12} \) if the Riemann Hypothesis holds.
We have the following result.

**Theorem 4.3.** For any odd primitive set $A$, we have

\[ f(A) \leq f(\mathcal{P}(A)) + \varepsilon_0. \]

*Proof.* Assume that $A$ is an odd primitive set. If $p \in \mathcal{P}(A)$ is Mertens, Corollary 3.1 implies that $f(A'_p) \leq f(p)$, while if $p \in \mathcal{P}(A)$ is not Mertens we have by Proposition 2.1 that $f(A'_p) \leq \max\{f(p), e^g(p)\} = e^g(p)$. Thus,

\[ f(A) = \sum_{p \in \mathcal{P}(A)} f(A'_p) \leq \sum_{p \in \mathcal{P}(A) \cap \mathcal{P}^\text{Mert}} f(p) + \sum_{p \in \mathcal{P}(A) \setminus \mathcal{P}^\text{Mert}} e^g(p) \leq \varepsilon_0 + \sum_{p \in \mathcal{P}(A)} f(p) \]

by the definition of $\varepsilon_0$. This completes the proof. \qed

This theorem yields the following corollary.

**Corollary 4.4.** If $A$ is a primitive set containing no multiple of 8, then (4.2) holds.

*Proof.* We have seen the corollary in the case that $A$ is odd. Next, suppose that $A$ contains an even number but no multiple of 4. If $2 \in A$, the result follows by applying Theorem 4.3 to $A \setminus \{2\}$, so assume $2 \notin A$. Then $A'_2$ is an odd primitive set and $f(A'_2) \leq f(A'_2)/2$. We have by the odd case that

\[ f(A) = f(A_3) + f(A'_2) < f(\mathcal{P}(A_3)) + \varepsilon_0 + \frac{1}{2} (f(\mathcal{P}(A'_2)) + \varepsilon_0). \]

Since

\[ \frac{1}{2} f(\mathcal{P}(A'_2)) \leq \frac{1}{2} f(\mathcal{P} \setminus \{2\}) < 0.4577 \]

and $f(2) = 0.7213 \ldots$, (4.3) and Lemma 4.1 imply that $f(A) < f(\mathcal{P}(A))$, which is stronger than required. The case when $A$ contains a multiple of 4 but no multiple of 8 follows in a similar fashion. \qed

Since a cube-free number cannot be divisible by 8, (4.2) holds for all primitive sets $A$ of cube-free numbers. Also, the proof of Corollary 4.4 can be adapted to show that (4.2) holds for all primitive sets $A$ containing no number that is $4 \mod 8$.

We close out this section with a curious result about those primitive sets $A$ where (4.2) does not hold. Namely, the Erdős conjecture must then hold for the set of odd members of $A$. Put another way, (4.2) holds for any primitive set $A$ for which the Erdős conjecture for the odd members of $A$ fails.

**Theorem 4.5.** If $A$ is a primitive set with $f(A) > f(\mathcal{P}(A)) + \varepsilon_0$, then $f(A_3) < f(\mathcal{P}(A_3))$.

*Sketch of proof.* Without loss of generality, we may include in $A$ all primes not in $\mathcal{P}(A)$ and so assume that $\mathcal{P}(A) = \mathcal{P}$ and $f(A) > C + \varepsilon_0$. By Theorem 4.3, we may assume that $A$ is not odd, and by Corollary 4.4 we may assume that $2 \notin A$. By the proof of Theorem 1.1 (see (2.5) and (2.7)), if $3 \in A$, we have

\[ f(A) < f(3) + \frac{2}{3} e^\gamma < C, \]

a contradiction, so we may assume that $3 \notin A$. We now apply the method of proof of Theorem 1.1 to $A_3$, where powers of 3 replace powers of 2. This leads to

\[ f(A_3) < \frac{1}{2} e^\gamma < C - f(2) = f(\mathcal{P}(A_3)). \]

This completes the argument. \qed
5. **Zhang primes and the Banks–Martin conjecture**

Note that

\[
\sum_{p \geq x} \frac{1}{p \log p} \sim \frac{1}{\log x}, \quad x \to \infty.
\]

In Erdős–Zhang [11] and in Zhang [20], numerical approximations to this asymptotic relation are exploited. Say a prime \( q \) is Zhang if

\[
\sum_{p \geq q} \frac{1}{p \log p} \leq \frac{1}{\log q}.
\]

Let \( \mathcal{P}^{Zh} \) denote the set of Zhang primes. We are interested in Zhang primes because of the following result.

**Theorem 5.1.** If \( \mathcal{P}(A'_p) \subset \mathcal{P}^{Zh} \), then \( f(A'_p) \leq f(p) \). Hence the Erdős conjecture holds for all primitive sets \( A \) supported on \( \mathcal{P}^{zh} \).

**Proof.** As in [11] it suffices to prove the theorem in the case that \( A \) is a finite set. By \( d^0(A) \) we mean the maximal value of \( \Omega(a) \) for \( a \in A \). We proceed by induction on \( d^0(A'_p) \). If \( d^0(A'_p) \leq 1 \), then \( f(A'_p) \leq f(p) \). If \( d^0(A'_p) > 1 \), then \( f(A'_p) \leq f(A''_p)/p \).

The primitive set \( B := A''_p \) satisfies \( f(B) = f(B_p) = \sum_{q \geq p} f(B'_q) \). Since \( d^0(B'_q) \leq d^0(B) < d^0(A'_p) \), by induction we have \( f(B'_q) \leq f(q) \). Thus, since \( p \) is Zhang,

\[
f(A''_p) = f(B) = \sum_{q \geq p} f(B'_q) \leq \sum_{q \geq p} \frac{1}{q \log q} \leq \frac{1}{\log p},
\]

from which we obtain \( f(A'_p) \leq f(A''_p)/p \leq 1/(p \log p) \). This completes the proof. \( \square \)

From this one might hope that all primes are Zhang. However, the prime 2 is not Zhang since \( C > 1/\log 2 \), and the prime 3 is not Zhang since \( C - 1/(2 \log 2) > 1/\log 3 \). Nevertheless, as with Mertens primes, it is true that the remaining primes up to \( p_{10^8} \) are Zhang. Indeed, starting from [12], we computed that

\[
(5.1) \quad \sum_{p \geq q} \frac{1}{p \log p} = C - \sum_{p < q} \frac{1}{p \log p} \leq \frac{1}{\log q} \quad \text{for all } 3 < q \leq p_{10^8}.
\]

The computation stopped at \( 10^8 \) for convenience, and one could likely extend this further with some patience. It seems likely that there is also a “race” between \( \sum_{p \geq q} 1/(p \log p) \) and \( 1/\log q \), as with Mertens primes, and that a large logarithmic density of primes \( q \) are Zhang, with a small logarithmic density of primes failing to be Zhang.

A related conjecture due to Banks and Martin [2] is the chain of inequalities

\[
\sum_{p} \frac{1}{p \log p} > \sum_{p \leq q} \frac{1}{pq \log pq} > \sum_{p \leq q \leq r} \frac{1}{pqr \log pqr} > \cdots,
\]

succinctly written as \( f(N_k) > f(N_{k+1}) \) for all \( k \geq 1 \), where \( N_k = \{ n : \Omega(n) = k \} \). As mentioned in the introduction, we know only that \( f(N_1) > f(N_k) \) for all \( k \geq 2 \) and \( f(N_2) > f(N_3) \). More generally, for a subset \( Q \) of primes, let \( N_k(Q) \) denote the subset of \( N_k \) supported on \( Q \). A result of Zhang [20] implies that \( f(N_1(Q)) > f(N_k(Q)) \) for all \( k > 1 \), while Banks and Martin showed that \( f(N_k(Q)) > f(N_{k+1}(Q)) \) if \( \sum_{p \in Q} 1/p \) is not too large. We prove a similar result in the case where \( Q \) is a subset of the Zhang primes and we replace \( f(N_k(Q)) \) with \( h(N_k(Q)) \). Recall that \( h(A) = \sum_{a \in A} 1/(a \log P(a)) \).
Proposition 5.2. For all $k \geq 1$ and $Q \subset \mathcal{P}^{zh}$, we have $h(N_k(Q)) \geq h(N_{k+1}(Q))$.

Proof. Since the primes in $Q$ are Zhang primes, we have

$$h(N_{k+1}(Q)) = \sum_{q_1 \leq \cdots \leq q_{k+1}} \frac{1}{q_1 \cdots q_k q_{k+1} \log q_{k+1}}$$

$$= \sum_{q_1 \leq \cdots \leq q_k} \frac{1}{q_1 \cdots q_k} \sum_{q_{k+1} \geq q_k} \frac{1}{q_{k+1} \log q_{k+1}}$$

$$\leq \sum_{q_1 \leq \cdots \leq q_k} \frac{1}{q_1 \cdots q_k \log q_k} = h(N_k(Q)).$$

This completes the proof. \qed

It is interesting that if we do not in some way restrict the primes used, the analog of the Banks–Martin conjecture for the function $h$ fails. In particular, we have

$$h(N_2) > \sum_{m \leq 10^4} \frac{1}{p_m} \sum_{n \geq m} \frac{1}{p_n \log p_n} = \sum_{m \leq 10^4} \frac{1}{p_m} \left( C - \sum_{k < m} \frac{1}{p_k \log p_k} \right) > 1.638,$$

while $h(N_1) = C < 1.637$.

It is also interesting that the analog of the Banks–Martin conjecture for the function $g$ is false since

$$1 = g(N_1) = g(N_2) = g(N_3) = \cdots.$$

We have already shown in (2.1) that $g(A^*_q) \leq g(q)$ for any primitive set $A$ and prime $q$, so the analog for $g$ of the strong Erdős conjecture holds.

5.1. Proof of Theorem 1.5 We now return to the function $f$ and prove Theorem 1.5.

We may assume that $k$ is large. Let $m = \lfloor \sqrt{k} \rfloor$ and let $B(n) = e^{c''}$. We have

$$f(N_k) = \sum_{\Omega(a) = k} \frac{1}{a \log a} > \sum_{\Omega(a) = k} \frac{1}{a \log a} \sum_{e^k < a \leq e^{k+m}} e^{c''} > \sum_{j \leq m} \frac{1}{a \log a} \sum_{B(k+j-1) < a \leq B(k+j)} \frac{1}{a}.$$}

Thus it suffices to show that there is a positive constant $c$ such that for $j \leq m$ we have

$$(5.2) \sum_{\Omega(a) = k} \frac{1}{a} \geq c \frac{\log B(k+j)}{m} = c \frac{e^{k+j}}{m},$$

since the proposition will follow.

Let $N_k(x)$ denote the number of members of $N_k$ in $[1, x]$. We use the Sathe–Selberg theorem (see [15, Theorem 7.19]), from which we have that uniformly for $B(k) < x \leq B(k+m)$, as $k \to \infty$,

$$N_k(x) \sim \frac{x}{k!} \left( \frac{\log \log x}{\log x} \right)^k.$$
This result also follows from Erdős [10].

We have

$$\sum_{\Omega(a)=k \atop B(k+j-1)<a\leq B(k+j)} \frac{1}{a} > \int_{B(k+j-1)}^{B(k+j)} \frac{N_k(x) - N_k(B(k+j-1))}{x^2} \, dx$$

$$\gg \int_{2B(k+j-1)}^{B(k+j)} \frac{N_k(x)}{x^2} \, dx.$$ 

Thus,

$$\sum_{\Omega(a)=k \atop B(k+j-1)<a\leq B(k+j)} \frac{1}{a} \gg \left( \frac{\log B(k+j-1)}{k!} \right)^k \int_{2B(k+j-1)}^{B(k+j)} \frac{dx}{x \log x}$$

$$\gg \left( \frac{k+j-1}{k!} \right)^k$$

the last estimate following from Stirling’s formula. This proves (5.2) and so the theorem.

The sets $N_k$ and Theorem 1.5 give us the following result.

**Corollary 5.3.** We have that

$$\limsup_{x \to \infty} \{ f(A) : A \subset [x, \infty), A \text{ primitive} \} > 0.$$ 

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