A NOTE ON CARMICHAEL NUMBERS
IN RESIDUE CLASSES

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Abstract. Improving on some recent results of Matomäki and of Wright, we show that the number of Carmichael numbers to \( X \) in a coprime residue class exceeds \( X^{1/(6 \log \log \log X)} \) for all sufficiently large \( X \) depending on the modulus of the residue class.

In memory of Ron Graham (1935–2020)
and Richard Guy (1916–2020)

1. Introduction

The “little theorem” of Fermat asserts that when \( p \) is a prime number, we have \( b^p \equiv b \pmod{p} \) for all integers \( b \). Given two integers \( b, p \) with \( p > b > 0 \), it is computationally easy to check this congruence, taking \( O(\log p) \) arithmetic operations in \( \mathbb{Z}/p\mathbb{Z} \). So, if the congruence is checked and we find that \( b^p \not\equiv b \pmod{p} \) we immediately deduce that \( p \) is composite. Unfortunately there are easily found examples where \( n \) is composite and the Fermat congruence holds for a particular \( b \). For example it always holds when \( b = 1 \). It holds when \( b = 2 \) and \( n = 341 \), and another example is \( b = 3, n = 91 \). There are even composite numbers \( n \) where \( b^n \equiv b \pmod{n} \) holds for all \( b \), the least example being \( n = 561 \). These are the Carmichael numbers, named after R. D. Carmichael who published the first few examples in 1910, see [4]. (Interestingly, Šimerka published the first few examples 25 years earlier, see [8].)

We now know that there are infinitely many Carmichael numbers, see [1], the number of them at most \( X \) exceeding \( X^c \) for a fixed \( c > 0 \) and \( X \) sufficiently large.

A natural question is if a given residue class contains infinitely many Carmichael numbers. After work of Matomäki [7] and Wright [9], we now know there are infinitely many in a coprime residue class. More
precisely, we have the following two theorems. Let
\[ C_{a,M}(X) = \# \{ n \leq X : n \text{ is a Carmichael number, } n \equiv a \pmod{M} \}. \]

**Theorem M** (Matomäki). Suppose that \( a, M \) are positive coprime integers and that \( a \) is a quadratic residue \( \text{mod} \ M \). Then \( C_{a,M} \geq X^{1/5} \) for \( X \) sufficiently large depending on the choice of \( M \).

**Theorem W** (Wright). Suppose that \( a, M \) are positive coprime integers. There are positive numbers \( K_M, X_M \) depending on the choice of \( M \) such that \( C_{a,M}(X) \geq X^{K_M/\left(\log \log \log X\right)^2} \) for all \( X \geq X_M \).

Thus, Wright was able to remove the quadratic residue condition in Matomäki’s theorem but at the cost of lowering the count to an expression that is of the form \( X^{o(1)} \). The main contribution of this note is to somewhat strengthen Wright’s bound.

**Theorem 1.** Suppose that \( a, M \) are positive coprime integers. Then \( C_{a,M}(X) \geq X^{1/\left(6 \log \log \log X\right)} \) for all sufficiently large \( X \) depending on the choice of \( M \).

That is, we reduce the power of \( \log \log \log X \) to the first power and we remove the dependence on \( M \) in the bound, though there still remains the condition that \( X \) must be sufficiently large depending on \( M \). (It’s clear though that such a condition is necessary since if \( M > X \) and \( a = 1 \), then there are no Carmichael numbers \( n \leq X \) in the residue class \( a \mod M \).)

Our proof largely follows Wright’s proof of Theorem W, but with a few differences.

Unlike with primes, it is conceivable that a non-coprime residue class contains infinitely many Carmichael numbers, e.g., there may be infinitely many that are divisible by 3. This is unknown, but seems likely. Let \( \lambda(n) \) denote the universal exponent of the group \( \mathbb{Z}/n\mathbb{Z}^* \) (so that a composite number \( n \) is a Carmichael number if and only if \( \lambda(n) \mid n-1 \)). For a residue class \( a \pmod{M} \), let \( g = \gcd(a, M) \) and let \( h = \gcd(\lambda(2g), M) \). A necessary condition that there is a Carmichael number \( n \equiv a \pmod{M} \) is that \( h \mid a-1 \). I conjecture that if this condition holds then there are infinitely many Carmichael numbers \( n \equiv a \pmod{M} \). (This modifies a similar conjecture in [3].) Though we don’t know this for any example with \( g > 1 \), the old heuristic of Erdős [5] suggests that \( C_{a,M}(X) \geq X^{1-o(1)} \) as \( X \to \infty \).

2. **Proof of Theorem 1**

There is an elementary and easily-proved criterion for Carmichael numbers: a composite number \( n \) is one if and only if it is squarefree
and $p - 1 \mid n - 1$ for each prime $p$ dividing $n$. This is due to Korselt, and perhaps others, and is over a century old. In our construction we will have a number $L$ composed of many primes, a number $k$ coprime to $L$ that is not much larger than $L$, and primes $p$ of the form $dk + 1$ where $d \mid L$. We will show there are many $n \equiv a \pmod{M}$ that are squarefree products of the $p$'s and are $1 \pmod{kL}$. Such $n$, if they involve more than a single $p$, will satisfy Korselt’s criterion and so are therefore Carmichael numbers.

We may assume that $M \geq 2$. Let $\mu = \varphi(4M)$, so that $4 \mid \mu$. Let $y$ be an independent variable; our other quantities will depend on it. For a positive integer $n$ let $P(n)$ denote the largest prime factor of $n$ (with $P(1) = 1$), and let $\omega(n)$ denote the number of distinct prime factors of $n$.

Let $Q_0 = \{q \text{ prime} : y < q \leq y \log^2 y, q \equiv -1 \pmod{\mu}, P(q - 1) \leq y\}$. If $q \leq y \log^2 y$ and $P(q - 1) > y$, then $q$ is of the form $mr + 1$, where $m < \log^2 y$ and $r$ is prime. By Brun’s sieve (see [6, (6.1)]), the number of such primes $q$ is at most

$$\sum_{m < \log^2 y} \sum_{\substack{r \text{ prime} \atop rm \leq y \log^2 y \atop r \equiv -1 \pmod{\mu}}} 1 \ll \sum_{m < \log^2 y} \frac{y \log^2 y}{\varphi(m) \log^2 y} \ll y \log \log y.$$

Also, the number of primes $q \leq y \log^2 y$ with $q \equiv -1 \pmod{\mu}$ is $\sim \frac{1}{\varphi(\mu)} y \log y$ as $y \to \infty$ by the prime number theorem for residue classes.

We conclude that

(1) $$\#Q_0 \sim \frac{1}{\varphi(\mu)} y \log y \quad \text{and} \quad \prod_{q \in Q_0} q = \exp\left(\frac{1 + o(1)}{\varphi(\mu)} y \log^2 y\right), \quad y \to \infty.$$ 

We also record that

(2) $$\sum_{q \in Q_0} \frac{1}{q} = o(1), \quad y \to \infty,$$

since this holds for all of the primes in the interval $(y, y \log^2 y)$.

Fix $0 < B < 5/12$; we shall choose a numerical value for $B$ near to $5/12$ at the end of the argument. Let

(3) $$x = M^{1/B} \prod_{q \in Q_0} q^{1/B}.$$ 

It follows from [1, (0.3)] that there is an absolute constant $D$ and a set $\mathcal{D}(x)$ of at most $D$ integers greater than $\log x$, such that if $n \leq x^B$, $n$ is
not divisible by any member of \(\mathcal{D}(x)\), \(b\) is coprime to \(n\), and \(z \geq nx^{1-B}\), then the number of primes \(p \leq z\) with \(p \equiv b \pmod{n}\) is \(> \frac{1}{2} \pi(z)/\varphi(n)\).

For each number in \(\mathcal{D}(x)\) we choose a prime factor and remove this prime from \(Q_0\) if it happens to be there. Let \(L\) be the product of the primes in the remaining set \(Q\), so that \(L\) is not divisible by any member of \(\mathcal{D}(x)\), and \(Q\) satisfies (1) and (2). In particular,

\[
L = \exp\left(\frac{1+o(1)}{\varphi(\mu)} y \log^2 y\right), \quad \omega(L) \sim \frac{1}{\varphi(\mu)} y \log y, \\
\text{and } \sum_{q \mid L} \frac{1}{q} = o(1) \text{ as } y \to \infty.
\]

In addition, we have \(ML \leq x^B\).

For each \(d \mid L\) and each quadratic residue \(b \pmod{L/d}\) we consider the primes

- \(p \leq dx^{1-B}\),
- \(p \equiv a \pmod{M}\),
- \(p \equiv 1 \pmod{d}\),
- \(p \equiv b \pmod{L/d}\).

Since \(M\) is coprime to \(L\), the congruences may be glued to a single congruence modulo \(ML\), and the number of such primes \(p\) is

\[
> \frac{\pi(dx^{1-B})}{2\varphi(ML)} > \frac{dx^{1-B}}{3\varphi(ML) \log x}
\]

for \(y\) sufficiently large.

We add these inequalities over the various choices of \(b\), the number of which is \(\varphi(L/d)/2^{\omega(L/d)}\), so the number of primes \(p\) corresponding to \(d \mid L\) is

\[
> \frac{dx^{1-B}2^{\omega(d)}}{3 \cdot 2^{\omega(L)} \varphi(Md) \log x}.
\]

We wish to impose an additional restriction on these primes \(p\), namely that \(\gcd((p-1)/d, L) = 1\). For a given prime \(q \mid L\) the number of primes \(p\) just counted and for which \(q \mid (p-1)/d\) is, via the Brun–Titchmarsh inequality,

\[
\ll \frac{dx^{1-B}2^{\omega(d)}}{2^{\omega(L)}q \varphi(Md) \log(x/(qML))} \ll \frac{dx^{1-B}2^{\omega(d)}}{2^{\omega(L)}q \varphi(Md) \log x}.
\]

Summing this over all \(q \mid L\) and using that \(\sum_{q \mid L} 1/q = o(1)\), these primes \(p\) are seen to be negligible. It follows that for \(y\) sufficiently
large, there are
\[ \frac{dx^{1-B}2^\omega(d)}{2^\omega(L)+2\varphi(Md)\log x} > \frac{x^{1-B}2^\omega(d)}{2^\omega(L)+2\varphi(M)\log x} \]
primes \( p \leq dx^{1-B} \) with \( p \equiv 1 \pmod{d} \), \( \gcd((p-1)/d,L) = 1 \), \( p \equiv a \pmod{M} \), and \( p \) is a quadratic residue \( (\pmod{L}) \) (noting that \( 1 \pmod{d} \) is a quadratic residue \( (\pmod{d}) \)).

For each pair \( p,d \) as above, we map it to \( (p-1)/d \) which is an integer \( \leq x^{1-B} \) coprime to \( L \). The number of pairs \( p,d \) is
\[ > \sum_{d \mid L} x^{1-B}2^\omega(d) = x^{1-B}3^\omega(L) \]
We conclude that there is a number \( k \leq x^{1-B} \) coprime to \( L \) which has \( > (3/2)^\omega(L)/(4\varphi(M)\log x) \) representations as \( (p-1)/d \). Let \( P \) be the set of primes \( p = dk + 1 \) that arise in this way. Then
\[ (5) \quad \#P > \frac{(3/2)^\omega(L)}{4\varphi(M)\log x} \]

For a finite abelian group \( G \), let \( n(G) \) denote Davenport’s constant, the least number such that in any sequence of group elements of length \( n(G) \) there is a non-empty subsequence with product the group identity. It is easy to see that \( n(G) \geq \lambda(G) \) (the universal exponent for \( G \), and in general it is not much larger: \( n(G) \leq \lambda(G)(1 + \log(\#G)) \). This result is essentially due to van Emde Boas–Kruyswijk and Meshulam, see [1].

Let \( G \) be the subgroup of \( (\mathbb{Z}/kML\mathbb{Z})^* \) of residues \( \equiv 1 \pmod{k} \). We have \( \#G \leq ML \). Also, \( \lambda(G) \leq M\lambda(L) \). (Note that, as usual, we denote \( \lambda((\mathbb{Z}/L\mathbb{Z})^*) \) by \( \lambda(L) \). It is the lcm of \( q - 1 \) for primes \( q \mid L \), using that \( L \) is squarefree.) Each prime dividing \( \lambda(L) \) is at most \( y \) and each prime power dividing \( \lambda(L) \) is at most \( y \log^2 y \), so that
\[ \lambda(L) \leq (y \log^2 y)^\pi(y) \]
Thus, for large \( y \), using (4),
\[ (6) \quad n(G) \leq M(y \log^2 y)^\pi(y) \log(ML) \leq e^{2y} \]

For a sequence \( A \) of elements in a finite abelian group \( G \), let \( A^* \) denote the set of nonempty subsequence products of \( A \). In Baker–Schmidt [2, Proposition 1] it is shown that there is a number \( s(G) \) such that if \( \#A \geq s(G) \), then \( G \) has a nontrivial subgroup \( H \) such that \( (A \cap H)^* = H \). Further,
\[ s(G) \leq 5\lambda(G)^2\Omega(\#G) \log(3\lambda(G)\Omega(\#G)) \]
where $\Omega(m)$ is the number of prime factors of $m$ counted with multiplicity. Thus, with $G$ the group considered above, we have

$$s(G) \leq e^{3y}$$

for $y$ sufficiently large.

It is this theorem that Matomäki and Wright use in their papers on Carmichael numbers. The role of the sequence $A$ is played by $\mathcal{P}$, the set of primes constructed above of the form $dk + 1$ where $d \mid L$. So, if $\# \mathcal{P} > s(G)$ we are guaranteed that every member of a nontrivial subgroup $H$ of $G$ is represented by a subset product of $\mathcal{P} \cap H$.

We don’t know precisely what this subgroup $H$ is, but we do know that it is nontrivial and that it is generated by members of $\mathcal{P}$. Well, suppose $p_0$ is in $\mathcal{P} \cap H$. Then $p_0^n \in H$ for every integer $m$. Note that by construction, $\gcd(\lambda(L)/2, \varphi(M)) = 1$, so there is an integer $m \equiv 1 \pmod{\varphi(M)}$ and $m \equiv 0 \pmod{\lambda(L)/2}$. Further, since $p_0$ is a quadratic residue (mod $L$), it follows that $p_0^{\lambda(L)/2} \equiv 1 \pmod{L}$. Thus, $p_0^n \equiv 1 \pmod{L}$ and $p_0^n a \pmod{M}$ (since $m \equiv 1 \pmod{\varphi(M)}$).

Thus, there is a subsequence product $n$ of $\mathcal{P}$ that is $1 \pmod{kL}$ and $a \pmod{M}$. (Note that every member of $G$ is $1 \pmod{k}$.) Further, $n$ is squarefree and for each prime factor $p$ of $n$ we have $p - 1 \mid kL$. Since $n \equiv 1 \pmod{kL}$ we have $p - 1 \mid n - 1$. Thus, $n \equiv a \pmod{M}$ is either a prime or a Carmichael number.

We actually have many subsequence products $n$ of $\mathcal{P}$ that satisfy these conditions, and $\mathcal{P}$ has at most one element that is $1 \pmod{L}$, so we do not need to worry about the case that $n$ is prime. We let $t = \lceil e^{3y} \rceil$, so that $t \geq s(G)$. As shown in [7], [9], the Baker–Schmidt result implies that $\mathcal{P}$ has at least

$$N := \left( \frac{\# \mathcal{P} - n(G)}{t - n(G)} \right)^{t - n(G)} \left( \frac{\# \mathcal{P} - n(G)}{n(G)} \right)$$

subsequence products $n$ of length at most $t$ which are Carmichael numbers in the residue class $a \pmod{M}$. Thus,

$$N > \left( \frac{\# \mathcal{P} - n(G)}{t - n(G)} \right)^{t - n(G)} \left( \frac{\# \mathcal{P} - n(G)}{n(G)} \right)\left( \frac{\# \mathcal{P}}{t} \right)^{t - n(G)} \left( \frac{\# \mathcal{P}}{t} \right)^{-n(G)} > \left( \frac{\# \mathcal{P}}{t} \right)^{t - 2n(G)} t^{-t}.$$

Let $X = x^t$. Since each $p \in \mathcal{P}$ has $p \leq x$, it follows that all of the Carmichael numbers constructed above are at most $X$. Using (1), (3),
and (6), we have
\[ X = \exp \left( \frac{1}{B} + o(1) \frac{\log y}{\varphi(\mu)} \right), \]
and using (5) and (4) gives
\[ N \geq \exp \left( \frac{\log(3/2) + o(1)}{\varphi(\mu)} \frac{t y \log y - t \log t}{t y \log y} \right). \]
Thus, \( N \geq X^{(B \log(3/2) + o(1))/\log y}. \) Now,
\[ \log X \sim \frac{1}{B \varphi(\mu)} t y \log^2 y, \]
\[ \log \log X \sim 3 y + O(\log y), \quad \log \log \log X \sim \log y + O(1). \]
We thus have \( N \geq X^{(B \log(3/2) + o(1))/\log \log X}. \) The number \( B < 5/12 \) can be chosen arbitrarily close to 5/12 and since \((5/12) \log(3/2) > 1/6, \) the theorem is proved.

References

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