A NOTE ON CARMICHAEL NUMBERS IN RESIDUE CLASSES

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Abstract. Improving on some recent results of Matomäki and of Wright, we show that the number of Carmichael numbers to X in a coprime residue class exceeds $X^{1/(6 \log\log\log X)}$ for all sufficiently large X depending on the modulus of the residue class.

In memory of Ron Graham (1935–2020)
and Richard Guy (1916–2020)

1. Introduction

The "little theorem" of Fermat asserts that when p is a prime number, we have $b^p \equiv b \pmod{p}$ for all integers b. Given two integers b, p with $p > b > 0$, it is computationally easy to check this congruence, taking $O(\log p)$ arithmetic operations in $\mathbb{Z}/p\mathbb{Z}$. So, if the congruence is checked and we find that $b^p \not\equiv b \pmod{p}$ we immediately deduce that p is composite. Unfortunately there are easily found examples where n is composite and the Fermat congruence holds for a particular b. For example it always holds when $b = 1$. It holds when $b = 2$ and $n = 341$, and another example is $b = 3$, $n = 91$. There are even composite numbers n where $b^n \equiv b \pmod{n}$ holds for all b, the least example being $n = 561$. These are the *Carmichael numbers*, named after R. D. Carmichael who published the first few examples in 1910, see [4]. (Interestingly, Šimerka published the first few examples 25 years earlier, see [8].)

We now know that there are infinitely many Carmichael numbers, see [1], the number of them at most X exceeding X^c for a fixed $c > 0$ and X sufficiently large.

A natural question is if a given residue class contains infinitely many Carmichael numbers. After work of Matomäki [7] and Wright [9], we now know there are infinitely many in a coprime residue class. More

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precisely, we have the following two theorems. Let

$$C_{a,M}(X) = \#\{n \le X : n \text{ is a Carmichael number, } n \equiv a \pmod{M}\}.$$

Theorem M (Matomäki). *Suppose that a, M are positive coprime integers and that a is a quadratic residue* mod M. *Then $C_{a,M} \ge X^{1/5}$ for X sufficiently large depending on the choice of M.*

Theorem W (Wright). *Suppose that a, M are positive coprime integers. There are positive numbers K_M, X_M depending on the choice of M such that $C_{a,M}(X) \ge X^{K_M/(\log\log\log X)^2}$ for all $X \ge X_M$.*

Thus, Wright was able to remove the quadratic residue condition in Matomäki's theorem but at the cost of lowering the count to an expression that is of the form $X^{o(1)}$. The main contribution of this note is to somewhat strengthen Wright's bound.

Theorem 1. *Suppose that a, M are positive coprime integers. Then $C_{a,M}(X) \ge X^{1/(6\log\log\log X)}$ for all sufficiently large X depending on the choice of M.*

That is, we reduce the power of $\log\log\log X$ to the first power and we remove the dependence on M in the bound, though there still remains the condition that X must be sufficiently large depending on M. (It's clear though that such a condition is necessary since if $M > X$ and $a = 1$, then there are no Carmichael numbers $n \le X$ in the residue class a mod M.)

Our proof largely follows Wright's proof of Theorem W, but with a few differences.

Unlike with primes, it is conceivable that a non-coprime residue class contains infinitely many Carmichael numbers, e.g., there may be infinitely many that are divisible by 3. This is unknown, but seems likely. Let $\lambda(n)$ denote the universal exponent of the group $(\mathbb{Z}/n\mathbb{Z})^*$ (so that a composite number n is a Carmichael number if and only if $\lambda(n) \mid n-1$). For a residue class $a \pmod{M}$, let $g = \gcd(a, M)$ and let $h = \gcd(\lambda(2g), M)$. A necessary condition that there is a Carmichael number $n \equiv a \pmod{M}$ is that $h \mid a-1$. I conjecture that if this condition holds then there are infnitely many Carmichael numbers $n \equiv a \pmod{M}$. (This modifies a similar conjecture in [3].) Though we don't know this for any example with $g > 1$, the old heuristic of Erdős [5] suggests that $C_{a,M}(X) \ge X^{1-o(1)}$ as $X \to \infty$.

2. Proof of Theorem 1

There is an elementary and easily-proved criterion for Carmichael numbers: a composite number n is one if and only if it is squarefree and $p-1 \mid n-1$ for each prime p dividing n. This is due to Korselt, and perhaps others, and is over a century old. In our construction we will have a number L composed of many primes, a number k coprime to L that is not much larger than L, and primes p of the form $dk+1$ where $d \mid L$. We will show there are many $n \equiv a \pmod M$ that are squarefree products of the p's and are $1 \pmod{kL}$. Such n, if they involve more than a single p, will satisfy Korselt's criterion and so are therefore Carmichael numbers.

We may assume that $M \geq 2$. Let $\mu = \varphi(4M)$, so that $4 \mid \mu$. Let y be an independent variable; our other quantities will depend on it. For a positive integer n let $P(n)$ denote the largest prime factor of n (with $P(1)=1$), and let $\omega(n)$ denote the number of distinct prime factors of n.

Let

$$\mathcal{Q}_0 = \{q \text{ prime} : y < q \leq y\log^2 y,\ q \equiv -1 \pmod{\mu},\ P(q-1) \leq y\}.$$

If $q \leq y\log^2 y$ and $P(q-1) > y$, then q is of the form $mr+1$, where $m < \log^2 y$ and r is prime. By Brun's sieve (see [6, (6.1)]), the number of such primes q is at most

$$\sum_{m<\log^2 y} \sum_{\substack{r \text{ prime} \\ mr \leq y\log^2 y \\ rm+1 \text{ prime}}} 1 \ll \sum_{m<\log^2 y} \frac{y\log^2 y}{\varphi(m)\log^2 y} \ll y\log\log y.$$

Also, the number of primes $q \leq y\log^2 y$ with $q \equiv -1 \pmod{\mu}$ is $\sim \frac{1}{\varphi(\mu)} y\log y$ as $y \to \infty$ by the prime number theorem for residue classes. We conclude that

(1)
$$\#\mathcal{Q}_0 \sim \frac{1}{\varphi(\mu)} y\log y \quad \text{and} \quad \prod_{q\in\mathcal{Q}_0} q = \exp\Big(\frac{1+o(1)}{\varphi(\mu)} y\log^2 y\Big), \quad y \to \infty.$$

We also record that

$$\sum_{q\in\mathcal{Q}_0} \frac{1}{q} = o(1), \quad y \to \infty, \tag{2}$$

since this holds for all of the primes in the interval $(y, y\log^2 y]$.

Fix $0 < B < 5/12$; we shall choose a numerical value for B near to $5/12$ at the end of the argument. Let

$$x = M^{1/B} \prod_{q\in\mathcal{Q}_0} q^{1/B}. \tag{3}$$

It follows from [1, (0.3)] that there is an absolute constant D and a set $\mathcal{D}(x)$ of at most D integers greater than $\log x$, such that if $n \leq x^B$, n is not divisible by any member of $\mathcal{D}(x)$, b is coprime to n, and $z \ge nx^{1-B}$, then the number of primes $p \le z$ with $p \equiv b \pmod{n}$ is $> \frac{1}{2}\pi(z)/\varphi(n)$.

For each number in $\mathcal{D}(x)$ we choose a prime factor and remove this prime from $\mathcal{Q}_0$ if it happens to be there. Let L be the product of the primes in the remaining set $\mathcal{Q}$, so that L is not divisible by any member of $\mathcal{D}(x)$, and $\mathcal{Q}$ satisfies (1) and (2). In particular,

$$\begin{aligned} L = \exp\Big(\frac{1+o(1)}{\varphi(\mu)} y \log^2 y\Big), \quad \omega(L) \sim \frac{1}{\varphi(\mu)} y \log y, \\ \text{and} \quad \sum_{q \mid L} \frac{1}{q} = o(1) \quad \text{as} \quad y \to \infty. \end{aligned} \tag{4}$$

In addition, we have $ML \le x^B$.

For each $d \mid L$ and each quadratic residue $b \pmod{L/d}$ we consider the primes

- $p \le dx^{1-B}$,
- $p \equiv a \pmod{M}$,
- $p \equiv 1 \pmod{d}$,
- $p \equiv b \pmod{L/d}$.

Since M is coprime to L, the congruences may be glued to a single congruence modulo ML, and the number of such primes p is

$$> \frac{\pi(dx^{1-B})}{2\varphi(ML)} > \frac{dx^{1-B}}{3\varphi(ML)\log x}$$

for y sufficiently large.

We add these inequalities over the various choices of b, the number of which is $\varphi(L/d)/2^{\omega(L/d)}$, so the number of primes p corresponding to $d \mid L$ is

$$> \frac{dx^{1-B}2^{\omega(d)}}{3 \cdot 2^{\omega(L)}\varphi(Md)\log x}.$$

We wish to impose an additional restriction on these primes p, namely that $\gcd((p-1)/d, L) = 1$. For a given prime $q \mid L$ the number of primes p just counted and for which $q \mid (p-1)/d$ is, via the Brun–Titchmarsh inequality,

$$\ll \frac{dx^{1-B}2^{\omega(d)}}{2^{\omega(L)}q\varphi(Md)\log(x/(qML))} \ll \frac{dx^{1-B}2^{\omega(d)}}{2^{\omega(L)}q\varphi(Md)\log x}.$$

Summing this over all $q \mid L$ and using that $\sum_{q|L} 1/q = o(1)$, these primes p are seen to be negligible. It follows that for y sufficiently large, there are

$$> \frac{dx^{1-B}2^{\omega(d)}}{2^{\omega(L)+2}\varphi(Md)\log x} > \frac{x^{1-B}2^{\omega(d)}}{2^{\omega(L)+2}\varphi(M)\log x}$$

primes $p \le dx^{1-B}$ with $p \equiv 1 \pmod d$, $\gcd((p-1)/d, L) = 1$, $p \equiv a \pmod M$, and p is a quadratic residue $\pmod L$ (noting that $1 \pmod d$ is a quadratic residue $\pmod d$)).

For each pair p, d as above, we map it to $(p-1)/d$ which is an integer $\le x^{1-B}$ coprime to L. The number of pairs p, d is

$$> \frac{x^{1-B}}{2^{\omega(L)+2}\varphi(M)\log x}\sum_{d\mid L} 2^{\omega(d)} = \frac{x^{1-B}3^{\omega(L)}}{2^{\omega(L)+2}\varphi(M)\log x}.$$

We conclude that there is a number $k \le x^{1-B}$ coprime to L which has $> (3/2)^{\omega(L)}/(4\varphi(M)\log x)$ representations as $(p-1)/d$. Let $\mathcal{P}$ be the set of primes $p = dk + 1$ that arise in this way. Then

$$\#\mathcal{P} > \frac{(3/2)^{\omega(L)}}{4\varphi(M)\log x}. \tag{5}$$

For a finite abelian group G, let $n(G)$ denote Davenport's constant, the least number such that in any sequence of group elements of length $n(G)$ there is a non-empty subsequence with product the group identity. It is easy to see that $n(G) \ge \lambda(G)$ (the universal exponent for G), and in general it is not much larger: $n(G) \le \lambda(G)(1 + \log(\#G))$. This result is essentially due to van Emde Boas–Kruyswijk and Meshulam, see [1].

Let G be the subgroup of $(\mathbb{Z}/kML\mathbb{Z})^*$ of residues $\equiv 1 \pmod k$. We have $\#G \le ML$. Also, $\lambda(G) \le M\lambda(L)$. (Note that, as usual, we denote $\lambda((\mathbb{Z}/L\mathbb{Z})^*)$ by $\lambda(L)$. It is the lcm of $q-1$ for primes $q \mid L$, using that L is squarefree.) Each prime dividing $\lambda(L)$ is at most y and each prime power dividing $\lambda(L)$ is at most $y\log^2 y$, so that

$$\lambda(L) \le (y\log^2 y)^{\pi(y)}.$$

Thus, for large y, using (4),

$$n(G) \le M(y\log^2 y)^{\pi(y)}\log(ML) \le e^{2y}. \tag{6}$$

For a sequence A of elements in a finite abelian group G, let A^* denote the set of nonempty subsequence products of A. In Baker–Schmidt [2, Proposition 1] it is shown that there is a number $s(G)$ such that if $\#A \ge s(G)$, then G has a nontrivial subgroup H such that $(A \cap H)^* = H$. Further,

$$s(G) \le 5\lambda(G)^2\Omega(\#G)\log(3\lambda(G)\Omega(\#G)),$$

where $\Omega(m)$ is the number of prime factors of m counted with multiplicity. Thus, with G the group considered above, we have

$$s(G) \le e^{3y}$$

for y sufficiently large.

It is this theorem that Matomäki and Wright use in their papers on Carmichael numbers. The role of the sequence A is played by $\mathcal{P}$, the set of primes constructed above of the form $dk+1$ where $d \mid L$. So, if $\#\mathcal{P} > s(G)$ we are guaranteed that every member of a nontrivial subgroup H of G is represented by a subset product of $\mathcal{P} \cap H$.

We don't know precisely what this subgroup H is, but we do know that it is nontrivial and that it is generated by members of $\mathcal{P}$. Well, suppose p_0 is in $\mathcal{P} \cap H$. Then $p_0^m \in H$ for every integer m. Note that by construction, $\gcd(\lambda(L)/2, \varphi(M)) = 1$, so there is an integer $m \equiv 1 \pmod{\varphi(M)}$ and $m \equiv 0 \pmod{\lambda(L)/2}$. Further, since p_0 is a quadratic residue (mod L), it follows that $p_0^{\lambda(L)/2} \equiv 1 \pmod{L}$. Thus, $p_0^m \equiv 1 \pmod{L}$ and $p_0^m \equiv a \pmod{M}$ (since $m \equiv 1 \pmod{\varphi(M)}$).

Thus, there is a subsequence product n of $\mathcal{P}$ that is 1 (mod kL) and a (mod M). (Note that every member of G is 1 (mod k.) Further, n is squarefree and for each prime factor p of n we have $p-1 \mid kL$. Since $n \equiv 1 \pmod{kL}$ we have $p-1 \mid n-1$. Thus, $n \equiv a \pmod{M}$ is either a prime or a Carmichael number.

We actually have many subsequence products n of $\mathcal{P}$ that satisfy these conditions, and $\mathcal{P}$ has at most one element that is 1 (mod L), so we do not need to worry about the case that n is prime. We let $t = \lceil e^{3y} \rceil$, so that $t \ge s(G)$. As shown in [7], [9], the Baker–Schmidt result implies that $\mathcal{P}$ has at least

$$N := \binom{\#\mathcal{P} - n(G)}{t - n(G)} \Big/ \binom{\#\mathcal{P} - n(G)}{n(G)}$$

subsequence products n of length at most t which are Carmichael numbers in the residue class $a \pmod{M}$. Thus,

$$\begin{aligned} N &> \left(\frac{\#\mathcal{P} - n(G)}{t - n(G)}\right)^{t-n(G)} (\#\mathcal{P})^{-n(G)} \\ &> \left(\frac{\#\mathcal{P}}{t}\right)^{t-n(G)} (\#\mathcal{P})^{-n(G)} > (\#\mathcal{P})^{t-2n(G)} t^{-t}. \end{aligned}$$

Let $X = x^t$. Since each $p \in \mathcal{P}$ has $p \le x$, it follows that all of the Carmichael numbers constructed above are at most X. Using (1), (3), and (6), we have

$$X = \exp\Big(\frac{1/B + o(1)}{\varphi(\mu)} ty \log^2 y\Big),$$

and using (5) and (4) gives

$$N \ge \exp\Big(\frac{\log(3/2) + o(1)}{\varphi(\mu)} ty \log y - t \log t\Big)$$
$$= \exp\Big(\frac{\log(3/2) + o(1)}{\varphi(\mu)} ty \log y\Big).$$

Thus, $N \ge X^{(B\log(3/2)+o(1))/\log y}$. Now,

$$\log X \sim \frac{1}{B\varphi(\mu)} ty \log^2 y,$$

G so that using $t = \lceil e^{3y} \rceil$,

$$\log\log X = 3y + O(\log y), \quad \log\log\log X = \log y + O(1).$$

We thus have $N \ge X^{(B\log(3/2)+o(1))/\log\log\log X}$. The number $B < 5/12$ can be chosen arbitrarily close to $5/12$ and since $(5/12)\log(3/2) > 1/6$, the theorem is proved.

References

[1] W. R. Alford, A. Granville, and C. Pomerance, There are infinitely many Carmichael numbers, *Ann. of Math.* (2) **139** (1994), 703–722.

[2] R. C. Baker and W. M. Schmidt, Diophantine problems in variables restricted to the values 0 and 1, *J. Number Theory* **12** (1980), 460–486.

[3] W. D. Banks and C. Pomerance, On Carmichael numbers in arithmetic progressions, *J. Australian Math. Soc.* **28** (2010), 313–321.

[4] R. D. Carmichael, A new number-theoretic function, *Bull. Amer. Math. Soc.* **16** (1910), 232–238.

[5] P. Erdős, On pseudoprimes and Carmichael numbers, *Publ. Math. Debrecen* **4** (1956), 201–206.

[6] H. Halberstam and H.-E. Richert, Sieve Methods, London Mathematical Society Monographs, No. 4. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], London – New York, 1974.

[7] K. Matomäki, Carmichael numbers in arithmetic progressions, *J. Australian Math. Soc.* **94** (2013), 268–275.

[8] V. Šimerka, Zbytky z arithmetické posloupnosti. (Czech) [On reminders from arithmetical sequence]. Časopis pro pěstování mathematiky a fysiky, **14** (1885), 221–225.

[9] T. Wright, Infinitely many Carmichael numbers in arithmetic progressions, *Bull. London Math. Soc.* **45** (2013), 943–952.

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