# Divisors of the middle binomial coefficient 

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#### Abstract

We study some old and new problems involving divisors of the middle binomial coefficient $\binom{2 n}{n}$.


1. INTRODUCTION. In the center of the $2 n$th row of Pascal's triangle we meet the maximal entry, $\binom{2 n}{n}$. These middle binomial coefficients have a rich history. For example, the fact that $\binom{2 n}{n}$ is divisible by the product of the primes in the interval $(n, 2 n)$ was exploited by Chebyshev in 1850 to obtain upper bounds and lower bounds for the distribution of primes. These bounds were so good that it seemed a promising path to the prime number theorem, but eventually that goal was reached by other methods.

The central binomial coefficient $\binom{2 n}{n}$ also figures prominently in the definition of the Catalan numbers:

$$
\begin{equation*}
C(n)=\frac{1}{n+1}\binom{2 n}{n} \tag{1}
\end{equation*}
$$

They too have a rich history with many combinatorial applications (see Stanley [11]). Note that $C(n)$ is an integer; that is, $n+1$ divides $\binom{2 n}{n}$.

This paper originated with the naive question: Is there some number $k$ other than 1 such that $n+k$ always divides $\binom{2 n}{n}$ ?

It turns out that the short answer is "no". For each $k \neq 1$, there are infinitely many $n$ with $n+k$ not dividing $\binom{2 n}{n}$. However, if $k \geq 2, n+k$ "usually" divides $\binom{2 n}{n}$ (in a way that we will make precise). On the other hand, we show that if $k \leq 0, n+k$ divides $\binom{2 n}{n}$ less frequently than not, leaving unresolved the precise nature of this frequency. While not particularly deep, these results appear to be new. The proofs stem from a number-theoretic (as opposed to combinatorial) proof that $C(n)$ is integral.

Along the way we shall meet the notorious problem of Ron Graham (which has a cash prize attached) on whether there are infinitely many numbers $n$ with $\binom{2 n}{n}$ relatively prime to 105 .

But let us begin at the very beginning, taking nothing for granted.
2. WHY ARE THE BINOMIAL COEFFICIENTS INTEGERS? The binomial coefficient $\binom{m}{k}$, defined as

$$
\frac{m!}{k!(m-k)!},
$$

is integral. Really? Perhaps it is not so obvious that the denominator divides the numerator. There are numerous proofs of course, for example one can use induction and Pascal's rule:

$$
\binom{m+1}{k}=\binom{m}{k}+\binom{m}{k-1} .
$$

Or one can argue combinatorially that $\binom{m}{k}$ counts the number of $k$-element subsets of an $m$-element set.

Instead of looking at the shortest proof, let's instead consider a more complicated proof, one that allows us to introduce some useful notation and also prove other results.

For a prime $p$ and a positive integer $m$, let $v_{p}(m)$ denote the number of factors of $p$ in the prime factorization of $m$. For example, $v_{2}(10)=1, v_{2}(11)=0$, and $v_{2}(12)=$ 2. This function can be extended to positive rational numbers via $v_{p}(a / b)=v_{p}(a)-$ $v_{p}(b)$. Thus, $a / b$ is integral if and only if $v_{p}(a / b) \geq 0$ for all primes $p$.

The power of $p$ in $m!$ is given by

$$
v_{p}(m!)=\sum_{k=1}^{m} v_{p}(k)=\sum_{k=1}^{m} \sum_{j=1}^{v_{p}(k)} 1=\sum_{j \geq 1} \sum_{\substack{k=1 \\ p^{j} \mid k}}^{m} 1=\sum_{j \geq 1}\left\lfloor\frac{m}{p^{j}}\right\rfloor
$$

the last step using that $\left\lfloor m / p^{j}\right\rfloor$ is the number of multiples of $p^{j}$ in $\{1,2, \ldots, m\}$. This result, sometimes referred to as Legendre's formula, is well known in elementary number theory. We also have the almost trivial inequality

$$
\begin{equation*}
\lfloor x+y\rfloor \geq\lfloor x\rfloor+\lfloor y\rfloor \tag{2}
\end{equation*}
$$

for all real numbers $x, y$. From these two results, it is immediate that $\binom{m}{k}$ is an integer. Indeed, for each prime $p$ we have

$$
\begin{equation*}
v_{p}\left(\binom{m}{k}\right)=\sum_{j \geq 1}\left(\left\lfloor\frac{m}{p^{j}}\right\rfloor-\left\lfloor\frac{k}{p^{j}}\right\rfloor-\left\lfloor\frac{m-k}{p^{j}}\right\rfloor\right) \geq \sum_{j \geq 1} 0=0 \tag{3}
\end{equation*}
$$

3. KUMMER'S THEOREM AND CATALAN NUMBERS. There is another consequence of this line of thinking. For a real number $x$ let $\{x\}=x-\lfloor x\rfloor$, the fractional part of $x$. The inequality (2) can be improved to an equation:

$$
\lfloor x+y\rfloor-\lfloor x\rfloor-\lfloor y\rfloor=\{x\}+\{y\}-\{x+y\}= \begin{cases}1, & \text { if }\{x\}+\{y\} \geq 1 \\ 0, & \text { if }\{x\}+\{y\}<1\end{cases}
$$

So by (3), $v_{p}\left(\binom{m}{k}\right)$ is the number of values of $j$ such that $\left\{k / p^{j}\right\}+\left\{(m-k) / p^{j}\right\}$ is at least 1 .

Let's write $k$ and $m-k$ in the base $p$, so that

$$
k=a_{0}+a_{1} p+\ldots, \quad m-k=b_{0}+b_{1} p+\ldots
$$

where the "digits" $a_{i}, b_{i}$ are integers in the range 0 to $p-1$. For $j \geq 1$,

$$
\left\{\frac{k}{p^{j}}\right\}=\frac{a_{0}+a_{1} p+\cdots+a_{j-1} p^{j-1}}{p^{j}}, \quad\left\{\frac{m-k}{p^{j}}\right\}=\frac{b_{0}+b_{1} p+\cdots+b_{j-1} p^{j-1}}{p^{j}}
$$

and so we see that $\left\{k / p^{j}\right\}+\left\{(m-k) / p^{j}\right\} \geq 1$ if and only if in the addition of $k$ and $m-k$ in the base $p$ there is a carry into place $j$ caused by the earlier digits.

This implies the remarkable result of Kummer from 1852: For each prime $p$ and integers $0 \leq k \leq m$, $v_{p}\left(\binom{m}{k}\right)$ is the number of carries in the addition $k+(m-k)=$ $m$ when done in the base $p$.

Recall the definition (1) of the Catalan number $C(n)$. Any of the numerous combinatorial applications of the Catalan numbers lead to a proof that $C(n)$ is an integer, being the solution to a counting problem. From a number-theoretic perspective, one
can ask for a direct proof of the integrality of $C(n)$. This is not difficult, perhaps the easiest way to see it is via the identity

$$
C(n)=\binom{2 n}{n}-\binom{2 n}{n-1}
$$

It also follows from Kummer's theorem. Indeed, say $p$ is a prime and $v_{p}(n+1)=$ $j$. Then the least significant $j$ digits in base $p$ of $n+1$ are all 0 , so the least significant $j$ digits in base $p$ of $n$ are all $p-1$. Thus, in the addition $n+n=2 n$ performed in the base $p$, we have $j$ carries from the least significant $j$ digits, and perhaps some other carries as well. So we have $\left.v_{p}\binom{2 n}{n}\right) \geq j$. Since this is true for all primes $p$, we have $n+1 \left\lvert\,\binom{ 2 n}{n}\right.$ and so $C(n)=\binom{2 n}{n} /(n+1)$ is an integer.

It may seem that the path of this paper is to give difficult proofs of easy theorems! However we shall see that this proof of the integrality of $C(n)$ "has legs" and can be used to also prove some perhaps surprising new results. But first we take a notunrelated detour to view the notorious 105 problem.
4. WHEN IS $\binom{2 n}{n}$ RELATIVELY PRIME TO 105? The numbers $n=1,10$, and 756 have $\binom{2 n}{n}$ relatively prime to 105 . Are there infinitely many others? This problem is due to Ron Graham, and according to [2, 4], Graham offers a prize of $\$ 1,000$ to settle it.

What's the deal with 105 ? It is $3 \times 5 \times 7$, the product of the first three odd primes. More generally, one can ask for any fixed number $m$ divisible by at least three distinct odd primes, if $\binom{2 n}{n}$ is relatively prime to $m$ for infinitely many $n$. In the case when $m=p q$, the product of just two odd primes, we do know that $\binom{2 n}{n}$ is relatively prime to $m$ for infinitely many $n$, a result of Erdős, Graham, Ruzsa, and Straus, see [3].

These problems and results stem from the point of view taken in Kummer's theorem, discussed in the previous section. We can use Kummer's theorem to show that $\binom{2 n}{n}$ is usually divisible by all small primes $p$. This is obvious for $p=2$ since in the base 2 , the number $2 n$ has one more digit than $n$ so there is at least one carry in the addition $n+n$; that is, $\binom{2 n}{n}$ is always even.

For an odd prime $p$, let

$$
R_{p}=\left\{0,1, \ldots, \frac{1}{2}(p-1)\right\}, \quad r_{p}=\# R_{p}=\frac{1}{2}(p+1), \quad \theta_{p}=\frac{\log r_{p}}{\log p} .
$$

We see that $p$ does not divide $\binom{2 n}{n}$ precisely when all of the base- $p$ digits of $n$ come from $R_{p}$. We show this is an unusual event.
Lemma 1. For each odd prime $p$ and all real numbers $x \geq 2$, the number of integers $1 \leq n \leq x$ with $p \nmid\binom{2 n}{n}$ is at most $p x^{\theta_{p}}$.
Proof. If $p \nmid\binom{2 n}{n}$, then by Kummer's theorem, every base- $p$ digit of $n$ is in $R_{p}$. Let $D=\lfloor 1+\log x / \log p\rfloor$, so that if $1 \leq n \leq x$ is an integer, then $n$ has at most $D$ base- $p$ digits. If we restrict these digits so that they are in $R_{p}$, we would have at most $r_{p}$ choices in each place. Thus, the number of choices for $n$ is at most $r_{p}^{D}$. It remains to note that

$$
r_{p}^{D}<p r_{p}^{\log x / \log p}=p x^{\theta_{p}},
$$

so concluding the proof.

For example, there are at most $3 x^{\theta_{3}}$ integers $n \leq x$ with $\binom{2 n}{n}$ not divisible by 3 . Since the exponent $\theta_{3}=\log 2 / \log 3=0.6309 \ldots$ is smaller than 1 , we see that the vast majority of integers $n$ up to $x$ have $\binom{2 n}{n}$ divisible by 3 .

Here is a heuristic argument for why there are infinitely many $n$ with $\binom{2 n}{n}$ relatively prime to 105 . The idea is to view Lemma 1 as an assertion about the probability that $p$ does not divide $\binom{2 n}{n}$ when $n$ is randomly chosen in $[1, x]$. When $x$ is an integer, this probability is at most $p x^{\theta_{p}-1}$. The exponents $\theta_{p}-1$ for $p=3,5,7$ are greater than $-0.37,-0.32,-0.29$, respectively. If these events are independent, as would seem only just (why would the base- $p$ expansion of $n$ have anything to do with the base- $q$ expansion when $p$ and $q$ are different primes?), then the probability that $\binom{2 n}{n}$ is relatively prime to 105 , where $n \leq x$, exceeds $x^{-0.98}$ when $x$ is large. Thus, we expect at least $x^{0.02}$ examples, and this expression tends to infinity when $x \rightarrow \infty$, albeit fairly slowly.

It is interesting to note that if one redoes this heuristic for the four primes $3,5,7,11$, then it suggests that there are at most finitely many numbers, such as $n=3160$, where $\binom{2 n}{n}$ is relatively prime to 1155 . No example larger than 3160 is known, though they have been searched for up to $10^{10^{4}}$, see [7].
5. HOW FREQUENTLY DOES $n+k$ DIVIDE $\binom{2 n}{n}$ ? We have seen that $n+1$ divides $\binom{2 n}{n}$ for all $n$. We now ask what happens with $n+k$ when $k \neq 1$. Is there a value of $k$ where $n+k$ divides $\binom{2 n}{n}$ for all $n$ or for all sufficiently large $n$ ?

We prove the following results which show an important cleavage between the cases when $k \geq 2$ and the cases when $k \leq 0$. However, we first state a universal result.

Theorem 1. For each integer $k \neq 1$ there are infinitely many positive integers $n$ with $n+k \nmid\binom{2 n}{n}$.

Thus, the case $k=1$ of Catalan numbers is indeed special. However, the set of numbers $n$ satisfying the condition of Theorem 1 when $k \geq 2$ is rather sparse. To measure how dense or sparse a set $S$ of positive integers is, let $S(x)$ denote the number of members of $S$ in $[1, x]$. Then the "asymptotic density" of $S$ is $\lim _{x \rightarrow \infty} S(x) / x$ if this limit exists. In general, the limsup gives the upper asymptotic density of $S$ and the liminf the lower asymptotic density. For example, the set of odd numbers has asymptotic density $\frac{1}{2}$, the set of prime numbers has asymptotic density 0 , and the set of numbers which have an even number of decimal digits has upper asymptotic density $10 / 11$ and lower asymptotic density $1 / 11$.

Theorem 2. For each positive integer $k$, the set of positive integers $n$ with

$$
n+k \left\lvert\,\binom{ 2 n}{n}\right.
$$

has asymptotic density 1 .
So it is common for $n+k$ to divide $\binom{2 n}{n}$, but what about $n-k$ ?
Theorem 3. For each integer $k \geq 0$ the set of integers $n>k$ with $n-k \left\lvert\,\binom{ 2 n}{n}\right.$ is infinite, but has upper asymptotic density smaller than $\frac{1}{3}$.

A remark: That there are infinitely many integers $n$ with $n \left\lvert\,\binom{ 2 n}{n}\right.$ and also infinitely many with $n \nmid\binom{2 n}{n}$ follow from Theorems 3.2 and 3.4 in the recent paper of Ulas [13].
6. THE PROOFS OF THEOREMS 1, 2, 3. There is a quick proof of Theorem 1. First assume that $k \geq 2$. Let $p$ be a prime factor of $k$ and let $n=p^{j}-k$ where $j$ is large enough so that $n>0$. In base $p, n$ has at most $j$ digits, with the least significant digit being 0 . Hence, there are at most $j-1$ carries when adding $n$ to $n$ and hence $n+k=p^{j} \nmid\binom{2 n}{n}$. For $k \leq 0$, let $p>2|k|$ be an odd prime number. For $n=p+|k|$, we have that there are no carries when $n$ is added to itself in the base $p$, so that $p \nmid\binom{2 n}{n}$. But $p=n+k$, so we are done.

The proof of Theorem 2 is a bit more difficult. We begin with a lemma that extends Lemma 1 to prime powers.
Lemma 2. Let $p$ be an arbitrary prime, let $x \geq p$ be a real number, and let $D=$ $\lfloor 1+\log x / \log p\rfloor$. The number of integers $1 \leq n \leq x$ with $v_{p}\left(\begin{array}{c}\left.\binom{2 n}{n}\right) \leq D /(5 \log D)\end{array}\right.$ is at most $3 p x^{1-1 /(5 \log p)}$.

Proof. The calculation here is similar to the one in probability where you compute the chance that a coin flipped $D$ times lands heads fairly frequently. We consider the number of assignments of $D$ base- $p$ digits where all but at most $B:=\lfloor D /(5 \log D)\rfloor$ of them are smaller than $p / 2$. Since there are $\lceil p / 2\rceil$ integers in $[0, p / 2)$, the number of assignments is at most

$$
\begin{aligned}
\sum_{j=0}^{B}\binom{D}{j}(\lceil p / 2\rceil)^{D-j}(p-\lceil p / 2\rceil)^{j} & =\sum_{j=0}^{B}\binom{D}{J}(\lceil p / 2\rceil)^{D}\left(\frac{p-\lceil p / 2\rceil}{\lceil p / 2\rceil}\right)^{j} \\
& \leq(\lceil p / 2\rceil)^{D} \sum_{j=0}^{B}\binom{D}{j} .
\end{aligned}
$$

A crude estimation using $D \geq 2$ gets us

$$
\sum_{j=0}^{B}\binom{D}{j} \leq \sum_{j=0}^{B} D^{j}<2 D^{B}
$$

so the number of $n \leq x$ with $\left.v_{p}\binom{2 n}{n}\right) \leq B$ is at most $2(\lceil p / 2\rceil)^{D} D^{B}$. Now

$$
2 D^{B}=2 e^{B \log D} \leq 2 e^{D / 5} \leq 2 e^{1 / 5} x^{1 /(5 \log p)}
$$

$\lceil p / 2\rceil \leq \frac{2}{3} p,\left(\frac{2}{3}\right)^{D} \leq x^{-\log (3 / 2) / \log p}$, and $p^{D} \leq p x$. We conclude that the number of $n \leq x$ with $v_{p}\left(\binom{2 n}{n}\right) \leq B$ is at most

$$
2(\lceil p / 2\rceil)^{D} D^{B} \leq 2 D^{B}\left(\frac{2}{3} p\right)^{D} \leq 2 e^{1 / 5} p x^{1-\log (3 / 2) / \log p+1 /(5 \log p)} .
$$

Since $2 e^{1 / 5}<3$ and $\log (3 / 2)>2 / 5$, the lemma follows at once.
We are now ready to prove Theorem 2. Fix a value of $k \geq 1$. First we claim that for $p \geq 2 k$,

$$
\begin{equation*}
v_{p}\left(\binom{2 n}{n}\right) \geq v_{p}(n+k) \tag{4}
\end{equation*}
$$

Indeed, if $v_{p}(n+k)=j>0$, then the $j$ least significant digits of $n+k$ in base $p$ are 0 , so the $j$ least significant digits of $n$ are at least $p-k \geq p / 2$. By Kummer's
theorem, (4) holds. (Note that this argument is essentially a reprise of the proof given in Section 3 that Catalan numbers are integers.)

It remains to show that for "most" integers $n$, (4) holds for all primes $p<2 k$. Let $x \geq 2 k+1$ be a real number and assume that we are considering values of $n \leq x$. With $D$ as in Lemma 2, we consider two cases: those $n$ with $v_{p}(n+k) \leq$ $D /(5 \log D)$ and those $n$ with $v_{p}(n+k)>D /(5 \log D)$. In the first case, if (4) fails, we would have $v_{p}\left(\binom{2 n}{n}\right)<D /(5 \log D)$, so by Lemma 2, the number of $n$ being considered is at most $3 p x^{1-1 /(5 \log p)}$. Summing this expression for primes $p<2 k$ gives a quantity smaller than $6 k^{2} x^{1-1 /(5 \log (2 k))}$. Divided by $x$, this expression tends to 0 as $x \rightarrow \infty$, so we are left with the second case: those $n$ with $v_{p}(n+k)>$ $D /(5 \log D)$. We shall show in this case that there are very few values of $n \leq x$ to consider, regardless if (4) holds. In fact, the number of choices for $n$ is at most $(x+k) / p^{\lceil D /(5 \log D)\rceil}<2 x / p^{D /(5 \log D)}$. Since $p^{D}>p^{\log x / \log p}=x$, we thus have that the number of choices for $n$ in the second case corresponding to the prime $p<2 k$ is at most $2 x^{1-1 /(5 \log D)}$. We have $\log D \leq \log (1+\log x / \log 2)<1+\log \log x$ (using $\log (a+b)<a / b+\log b$ when $0<a<b$ ), so that if we sum for $p<2 k$, we get that the number of choices for $n$ is at most $2 k x^{1-1 /(5+5 \log \log x)}$. When divided by $x$, this too goes to 0 as $x \rightarrow \infty$, which completes the proof of Theorem 2.

It is not difficult to amend the proof to show that for each fixed positive integer $k$, the set of integers $n$ with

$$
\begin{equation*}
(n+1)(n+2) \ldots(n+k) \left\lvert\,\binom{ 2 n}{n}\right. \tag{5}
\end{equation*}
$$

has asymptotic density 1 . In addition, the proof allows for $k$ to tend to infinity, provided it does not do so too quickly in comparison with $x$. The result (5) might be compared with Harborth [6] where it is shown that for any fixed positive integer $k$, "almost all" entries $\binom{m}{j}$ in Pascal's triangle are divisible by $m(m-1) \ldots(m-k+1)$. Also see http://oeis.org, sequences A065344-9.

We now proceed to the proof of Theorem 3, starting with the second assertion. Suppose that $k \geq 0, n>2 k^{2}$, and $m=n-k$ has a prime factor $p>\sqrt{2 n}$. Then $p>2 k$ and writing $m=c p$, we have

$$
c<\frac{m}{\sqrt{2 n}} \leq \frac{n}{\sqrt{2 n}}=\frac{1}{2} \sqrt{2 n}<\frac{1}{2} p
$$

We see that both base- $p$ digits of $n=c p+k$, namely $c$ and $k$, are smaller than $\frac{1}{2} p$, and so there are no carries when adding $n$ to itself in base $p$. By Kummer's theorem, we have $p \nmid\binom{2 n}{n}$, so that $n-k \nmid\binom{2 n}{n}$.

Suppose that $x>2 k^{4}$. To show the second assertion in the theorem, it will suffice to show that when $x$ is sufficiently large, at least $\frac{2}{3} x$ integers $n \in\left(2 k^{2}, x\right]$ have $m=n-$ $k$ divisible by a prime $p>\sqrt{2 x}$. For each prime $p$ satisfying this inequality, we count numbers $n$ in $\left(2 k^{2}, x\right]$ with $p \mid n-k$, and this is at least $\left\lfloor\left(x-2 k^{2}\right) / p\right\rfloor>x / p-2$. No choice of $n$ corresponds to two different values of $p$, since their product would be too large to have $n \leq x$. Thus, the number of choices for $n$ is at least

$$
\sum_{\sqrt{2 x}<p \leq x}\left(\frac{x}{p}-2\right)=x \sum_{\sqrt{2 x}<p \leq x} \frac{1}{p}-2 \pi(x)
$$

where $\pi(x)$ denotes the number of primes in $[1, x]$. Euler proved long ago in 1737
that the sum of the reciprocals of the primes diverges to infinity like the double-log function, and using a finer estimation (such as the theorem of Mertens from 1874), we have

$$
\sum_{\sqrt{2 x}<p \leq x} \frac{1}{p}=\log \log x-\log \log (\sqrt{2 x})+E(x)
$$

where $E(x) \rightarrow 0$ as $x \rightarrow \infty$, see $[\mathbf{8}, 9]$. The difference of double logs simplifies to

$$
\log 2+\log \left(\frac{\log x}{\log (2 x)}\right)
$$

which tends to $\log 2$ as $x \rightarrow \infty$. As mentioned before, the set of primes has asymptotic density 0 , in fact $\pi(x) / x$ goes to 0 like $1 / \log x$ by Chebyshev's estimates or the prime number theorem. Putting these thoughts together, it follows that for each $\epsilon>0$ and $x$ sufficiently large, there are more than $(\log 2-\epsilon) x$ values of $n \in\left(2 k^{2}, x\right]$ with $n-k$ divisible by a prime $p>\sqrt{2 x}$. We have seen that for each such number $n$, we have $n-k \nmid\binom{2 n}{n}$. Since $\log 2>0.6931>\frac{2}{3}$, the second part of Theorem 3 follows.

To complete the proof, we wish to show there are infinitely many values of $n$ where $n-k \left\lvert\,\binom{ 2 n}{n}\right.$. We leave the few details for the reader, but using Kummer's theorem, we have that if $n=p q+k$ where $p, q$ are primes with $k<p$ and $\frac{3}{2} p<q<2 p$, then $n-k \left\lvert\,\binom{ 2 n}{n}\right.$. The number of such numbers $n \leq x$ is greater than a positive constant times $x /(\log x)^{2}$.

## 7. THE "GOVERNOR SET". For each integer $k$, let

$$
D_{k}=\left\{n: n+k \left\lvert\,\binom{ 2 n}{n}\right.\right\} .
$$

We call $D_{0}$ the governor set for a reason that will soon be clear.
Say two sets $A, B$ of positive integers are asymptotically equivalent if the symmetric difference $(A \cup B) \backslash(A \cap B)$ has asymptotic density 0 . In this case, we write $A \simeq B$. For example, if $A$ is the set of all positive integers exceeding a googol ( $10^{100}$ ) and $B$ is the set of all composite positive integers, then $A \simeq B$. In particular, any two sets of asymptotic density 1 are asymptotically equivalent, as are any two sets of asymptotic density 0 .

If $A$ is a set of positive integers and $n$ is a positive integer, we let

$$
A+n:=\{a+n: a \in A\} .
$$

Theorem 4. For each positive integer $k$ we have $D_{0}+k \simeq D_{-k}$.
We sketch the proof. Suppose that $p$ is a prime with $p \mid n$ and $p>2 k$. Then, as in the proof of Theorem 2, $v_{p}\left(\binom{2 n}{n}\right)=v_{p}\left(\binom{2(n+k)}{n+k}\right)$. And for primes $p$ at most $2 k$, again following the argument for Theorem 2, for most numbers $n$, the power of $p$ in both $\binom{2 n}{n}$ and in $\binom{2(n+k)}{n+k}$ is higher than the power of $p$ in $n$. Thus, most of the time, the condition $n \in D_{0}$ (that is, $n \left\lvert\,\binom{ 2 n}{n}\right.$ ) is equivalent to the condition $n+k \in D_{-k}$ (that is, $\left.n \left\lvert\, \begin{array}{c}2(n+k) \\ n+k\end{array}\right.\right)$ ).

It is not clear if the governor set $D_{0}$ has positive lower asymptotic density, though I conjecture this the case. In Theorem 3 we essentially learned that the upper asymptotic
density is at most $1-\log 2$, and perhaps some improvement can be made. It also would be highly interesting to investigate the problem numerically. The numbers $n$ which divide $\binom{2 n}{n}$ are $1,2,6,15,20,28,42,45,66,77,88,91, \ldots$. What are these numbers trying to tell us? A census to higher levels may be illuminating. Note that $D_{0}$ may also be described as the set of $n$ with $n \mid C(n)$, and in this guise there is some information to be found at http://oeis.org, sequence A014847.

To cement the role of the governor set $D_{0}$, let

$$
D_{k}^{(2)}=\left\{n:(n+k)^{2} \left\lvert\,\binom{ 2 n}{n}\right.\right\} .
$$

Then for each positive integer $k$ we have $D_{k}^{(2)}+k \simeq D_{0}$. The proof is similar to that of Theorem 4. So you'd like to know how often the $n$th Catalan number $C(n)$ is divisible by $n+1$ ? You are again led back to the governor set $D_{0}$. See too [1] and http://oeis.org, sequence A002503. (Notice that $n+1 \mid C(n)$ if and only if $n+1 \left\lvert\,\binom{ 2 n+1}{n}\right.$. This latter condition is considered in [13], where in Question 2.8 (a), the author asks about the distribution of such $n$. Our Theorem 3 in the case $k=0$ is thus relevant to this query.)

You may have noticed that any $D_{k}$ with $k \leq 0$ could have played the special role of the governor set, but surely $D_{0}$ is the most pleasing.
8. OTHER BINOMIAL COEFFICIENTS. One might wonder what the fuss is about the middle entry $\binom{2 n}{n}$ in the $2 n$th row of Pascal's triangle. What about oddnumbered rows, where there are the twin peaks $\binom{2 n+1}{n}=\binom{2 n+1}{n+1}$ ? It is possible to prove corresponding results on divisibility here. For example, for $k \geq 2,(n+2)(n+$ $3) \ldots(n+k)$ usually divides $\binom{2 n+1}{n}$. If you are interested, you should try to prove this.

Looking at just the case $k=2$, we have a near miss for $n+2$ always dividing $\binom{2 n+1}{n}$. In fact, unless $n+2$ is a power of 2 , it divides, and even in this case, it divides $2\binom{2 n+1}{n}$. Since there is a tie for the maximum entry in this row of Pascal's triangle, it makes sense to include both of them, and as mentioned we always have

$$
n+2 \left\lvert\, 2\binom{2 n+1}{n}\right.
$$

Thus, we might wonder, as with Catalan numbers, if the integer

$$
\begin{equation*}
\frac{2}{n+2}\binom{2 n+1}{n} \tag{6}
\end{equation*}
$$

has combinatorial significance. Indeed it does. We know that the Catalan number $C(n)$ counts the number of paths from $(0,0)$ to $(n, n)$ that do not cross below the line $y=x$, and where each step of the path is one unit to the right or one unit up. The number in (6) is similar, but now we are counting paths from $(0,0)$ to $(n, n+1)$, a so-called ballot number.

In [12], Sun has the following generalization of the fact that $n+1$ always divides $\binom{2 n}{n}$. If $k, \ell$ are positive integers and every prime dividing $k$ also divides $\ell$, then

$$
\begin{equation*}
\ell n+1 \left\lvert\,\binom{(\ell+k) n}{k n}\right. \text { for every positive integer } n \tag{7}
\end{equation*}
$$

The case $k=\ell=1$ is the situation with Catalan numbers. Sun asks if the condition that every prime factor of $k$ divides $\ell$ is necessary for (7) to hold.

For other problems and results concerning divisibility properties of binomial coefficients, the reader is referred to $[\mathbf{3 , 5}, \mathbf{1 0}]$.

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