

ON THE COMPOSITION
OF THE ARITHMETIC FUNCTIONS σ AND φ

BY

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1. Introduction. Let φ denote Euler's function and let σ denote the sum of the divisors function. In [2], Małowski and Schinzel consider the function $\sigma(\varphi(n))/n$, showing that

$$\liminf \frac{\sigma(\varphi(n))}{n} \leq \inf_{4|n} \frac{\sigma(\varphi(n))}{n} \leq \frac{1}{2} + \frac{1}{2^{34}-4}.$$

They ask if $\sigma(\varphi(n))/n \geq 1/2$ is true for all n , stating that Mrs. K. Kuhn has shown this inequality for all n with at most 6 prime factors. They also remark that even the weaker inequality

$$(1.1) \quad \inf \frac{\sigma(\varphi(n))}{n} > 0$$

remains open.

In this note* we prove (1.1). The proof is elementary, the principal tool being Brun's method.

Throughout, the letter p denotes a prime.

2. The proof of (1.1). If T is a set of primes, let

$$s(T, x) = \sum'_{x^{1/\sigma} < p \leq x} 1/p,$$

where \sum' signifies that the primes p in the sum also have the property that $p-1$ is free of prime factors from T . We have the following result:

LEMMA 2.1. *There is an absolute constant $c_1 \geq 1$ such that for any set of primes T and any x we have*

$$s(T, x) \leq c_1 \exp\left(-\sum_{\substack{p \leq x \\ p \in T}} 1/p\right).$$

Indeed, this follows from [1] (Theorem 2.2) and a partial summation.

* Supported in part by an NSF grant.

Note that if m is a natural number, then

$$\log \frac{\sigma(m)}{m} = \sum_{p^a \parallel m} \log(1 + 1/p + \dots + 1/p^a) = \sum_{p|m} 1/p + O(1),$$

$$\log \frac{\varphi(m)}{m} = \sum_{p|m} \log(1 - 1/p) = - \sum_{p|m} 1/p + O(1).$$

Given a natural number n , let $S(n)$ denote the set of prime factors of $\varphi(n)$. Thus

$$(2.1) \quad \log \frac{\sigma(\varphi(n))}{n} = \log \frac{\sigma(\varphi(n))}{\varphi(n)} + \log \frac{\varphi(n)}{n} = \sum_{p \in S(n)} 1/p - \sum_{p|n} 1/p + O(1).$$

Let $S'(n)$ denote the set of primes p such that every prime in $p-1$ lies in $S(n)$. Thus from (2.1) we have

$$\log \frac{\sigma(\varphi(n))}{n} \geq \sum_{p \in S(n)} 1/p - \sum_{p \in S'(n)} 1/p + O(1).$$

We conclude that (1.1) will follow from the following theorem:

THEOREM 2.2. *There is an absolute constant c_2 such that*

$$\sum_{p \in S'} 1/p - \sum_{p \in S} 1/p \leq c_2$$

for any finite set of primes S , where S' denotes the set of primes p such that every prime in $p-1$ lies in S .

Proof. Let S be an arbitrary finite set of primes, let k be a natural number such that $S \subset [1, \exp(e^k)]$, and let T be the (infinite) set of primes that do not belong to S . For each natural number i , let

$$\beta_i = \sum_{\substack{p \in S' \\ \exp(e^{i-1}) < p \leq \exp(e^i)}} 1/p,$$

so that

$$(2.2) \quad \sum_{p \in S'} 1/p = \frac{1}{2} + \sum_{i=1}^{\infty} \beta_i.$$

Note that from Lemma 2.1 we have

$$(2.3) \quad \beta_i = s(T, \exp(e^i)) \leq c_1 \exp\left(- \sum_{\substack{p \in T \\ p \leq \exp(e^i)}} 1/p\right)$$

for each natural number i . Let

$$b_i = \sum_{\substack{p \in T \\ \exp(e^{i-1}) < p \leq \exp(e^i)}} 1/p \quad \text{for } i = 1, 2, \dots$$

Note that we may assume $2 \notin T$, for otherwise $S' = \{2\}$ and the theorem is trivially true. Thus

$$(2.4) \quad \sum_{\substack{p \in T \\ p \leq \exp(e^i)}} 1/p = \sum_{j=1}^i b_j \quad \text{for } i = 1, 2, \dots$$

Since

$$(2.5) \quad b'_i := \sum_{\exp(e^{i-1}) < p \leq \exp(e^i)} 1/p = 1 + O(e^{-i}), \quad i = 1, 2, \dots,$$

we infer from (2.3) and (2.4) that

$$\beta_i \leq \min(1, c_1 \exp(-(b_1 + \dots + b_i))) + O(e^{-i})$$

for every natural number i . We conclude from (2.2) that

$$(2.6) \quad \sum_{p \in S'} 1/p - \sum_{p \in S} 1/p = \frac{1}{2} + \sum_{i=1}^{\infty} \beta_i - \sum_{p \leq \exp(e^k)} 1/p + \sum_{\substack{p \in T \\ p \leq \exp(e^k)}} 1/p \\ \leq \sum_{i=1}^{\infty} \min(1, c_1 \exp(-(b_1 + \dots + b_i))) - k + b_1 + \dots + b_k + O(1).$$

Now, for $i > k$, we infer from (2.5) that $b_i = b'_i = 1 + O(e^{-i})$, so that

$$(2.7) \quad \sum_{i>k} \min(1, c_1 \exp(-(b_1 + \dots + b_i))) \ll \sum_{i>k} e^{-(i-k)} \ll 1.$$

Thus from (2.5)–(2.7) we will have Theorem 2.2 once we prove the following lemma:

LEMMA 2.3. *If $c \geq 1$ is fixed and if*

$$f(b_1, \dots, b_k) := \sum_{i=1}^k \min(1, c \exp(-(b_1 + \dots + b_i))) - k + b_1 + \dots + b_k,$$

then the maximum value of f on $[0, 1]^k$ is less than $e/(e-1) + \log c$ for any natural number k .

Note that the letters b_1, \dots, b_k appearing in (2.6) satisfy $0 \leq b_i \leq b'_i$ for each i , rather than $0 \leq b_i \leq 1$. However, the maximum value of f on $[0, b'_1] \times \dots \times [0, b'_k]$ is less than or equal to the maximum value of f on

$$[0, \max(1, b'_1)] \times \dots \times [0, \max(1, b'_k)],$$

which, by (2.5), is within $O(1)$ of the maximum value of f on $[0, 1]^k$. Thus Lemma 2.3 suffices for the completion of the proof of Theorem 2.2 and (1.1).

Proof of Lemma 2.3. Let k be given and assume the maximum value of f is attained at $(B_1, \dots, B_k) \in [0, 1]^k$. First note we may assume that

$$B_1 \leq B_2 \leq \dots \leq B_k,$$

for if $B_{i+1} < B_i$, then interchanging these two numbers will not make f smaller. If

$$c \exp(-(B_1 + \dots + B_k)) \geq 1,$$

then

$$f(B_1, \dots, B_k) = B_1 + \dots + B_k \leq \log c.$$

So we may assume there is a first subscript i_0 such that

$$c \exp(-(B_1 + \dots + B_{i_0})) < 1.$$

Then $B_1 + \dots + B_{i_0-1} \leq \log c$, so that

$$(2.8) \quad f(B_1, \dots, B_k) \leq i_0 + \sum_{i=i_0+1}^k \exp(-(B_{i_0+1} + \dots + B_i)) - k + B_1 + \dots + B_k \\ \leq \sum_{i=i_0+1}^k \exp(-(B_{i_0+1} + \dots + B_i)) - (k - i_0) + B_{i_0+1} + \dots + B_k + 1 + \log c.$$

Let

$$g(b_{i_0+1}, \dots, b_k) = \sum_{i=i_0+1}^k \exp(-(b_{i_0+1} + \dots + b_i)) - (k - i_0) + b_{i_0+1} + \dots + b_k$$

and say the maximum of g on $[0, 1]^{k-i_0}$ is assumed at (A_{i_0+1}, \dots, A_k) . Thus from (2.8) we have

$$(2.9) \quad f(B_1, \dots, B_k) \leq g(A_{i_0+1}, \dots, A_k) + 1 + \log c.$$

As above, we may assume $A_{i_0+1} \leq \dots \leq A_k$. If $A_{i_0+1} = \dots = A_k = 1$, then from (2.9) we have

$$f(B_1, \dots, B_k) \leq \sum_{j=1}^{k-i_0} e^{-j} + 1 + \log c < \frac{e}{e-1} + \log c.$$

Thus we may assume there is a greatest index i_1 with $A_{i_1} < 1$.

If $A_{i_1} = 0$, then from (2.9) we have

$$f(B_1, \dots, B_k) \leq i_1 - i_0 + \sum_{j=1}^{k-i_1} e^{-j} - (k - i_0) + (k - i_1) + 1 + \log c \\ < \frac{e}{e-1} + \log c.$$

Thus we may assume $0 < A_{i_1} < 1$. Then

$$\frac{\partial g}{\partial b_{i_1}}(A_{i_0+1}, \dots, A_k) = 0,$$

so that

$$(2.10) \quad 1 = \sum_{i=i_1}^k \exp(-(A_{i_0+1} + \dots + A_i)) \\ = [\exp(-(A_{i_0+1} + \dots + A_{i_1}))](1 + \exp(-1) + \dots + \exp(-(k-i_1))).$$

This implies

$$A_{i_0+1} + \dots + A_{i_1} < \log \frac{e}{e-1}.$$

Thus from (2.9) and (2.10) we have

$$f(B_1, \dots, B_k) < (i_1 - i_0 - 1) + 1 - (k - i_0) + \log \frac{e}{e-1} + (k - i_1) + 1 + \log c \\ = \log \frac{e}{e-1} + 1 + \log c < \frac{e}{e-1} + \log c,$$

which proves the lemma.

REFERENCES

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 [2] A. Mąkowski and A. Schinzel, *On the functions $\varphi(n)$ and $\sigma(n)$* , Colloq. Math. 13 (1964), pp. 95-99.

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Reçu par la Rédaction le 26.2.1988