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Counterexamples to the modified Weyl–Berry conjecture on fractal drums

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1. *Introduction*

Let Ω be a non-empty open set in $\mathbb{R}^n$ with finite 'volume' (n-dimensional Lebesgue measure). Let $\Delta = \sum_{k=1}^n \partial^2/\partial x_k^2$ be the Laplacian operator. Consider the eigenvalue problem (with Dirichlet boundary conditions):

$$-\Delta u = \lambda u \quad \text{on} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{1·1}$$

where $\lambda \in \mathbb{R}$ and u is a non-zero member of $H_0^1(\Omega)$ (the closure in the Sobolev space $H^1(\Omega)$ of the set of smooth functions with compact support contained in Ω). It is well known that the values of $\lambda \in \mathbb{R}$ for which (1·1) has a non-zero solution $u \in H_0^1(\Omega)$ are positive and form a discrete set. Moreover, for each λ, the associated eigenspace is finite dimensional. Let the spectrum of (1·1) be denoted $(\lambda_m)_{m=1}^\infty$ where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ and where the multiplicity of each λ in the sequence is the dimension of the associated eigenspace. Let

$$N(\lambda) = N(\lambda; \Omega) = \#\{m \geqslant 1 : \lambda_m \leqslant \lambda\}.$$

This paper is concerned with estimations for the counting function $N(\lambda)$.

Weyl's classical asymptotic formula – now known for an arbitrary domain Ω (see [**1**]) – states that

$$N(\lambda) = (1+o(1))\,\phi(\lambda) \quad \text{for} \quad \lambda \to +\infty,$$

where

$$\phi(\lambda) = (2\pi)^{-n}\mathfrak{B}_n|\Omega|_n\,\lambda^{n/2}.$$

Here $\mathfrak{B}_n$ is the volume of the unit ball in $\mathbb{R}^n$ and $|\Omega|_n$ is the volume of Ω.

One may wonder if anything interesting can be said about the error term $N(\lambda)-\phi(\lambda)$. Weyl conjectured that if $\partial\Omega$ is sufficiently 'regular', then $N(\lambda)-\phi(\lambda)$ is asymptotically a constant times $\lambda^{(n-1)/2}$, where the constant is proportional to $|\partial\Omega|_{n-1}$, the $(n-1)$-dimensional 'area' of $\partial\Omega$, the proportionality constant depending only on n. This was proved by Ivrii in 1980 when $\partial\Omega$ is smooth and Ω satisfies a certain technical assumption (see [**6**]).

In an attempt to generalize Weyl's conjecture to the case when $\partial\Omega$ is 'fractal', Berry conjectured in 1979 that if $\partial\Omega$ has Hausdorff dimension H, then $N(\lambda)-\phi(\lambda)$ is asymptotically a constant times $\lambda^{H/2}$, where the constant is proportional to the

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(normalized) Hausdorff measure of $\partial\Omega$ and the proportionality constant depends only on n and H.

However, Brossard and Carmona[2] showed Berry's conjecture to be false and suggested instead that the Minkowski dimension is more appropriate than the Hausdorff dimension. In particular, in [8] one of us stated his 'modified Weyl–Berry conjecture' (MWB conjecture) which is the same as Berry's conjecture, but for an additional assumption of 'Minkowski measurability' of $\partial\Omega$ and with 'Minkowski' replacing each occurrence of 'Hausdorff' (see (1·2) below). Moreover, he obtained some partial results that supported the MWB conjecture. In [12, 13], we proved the MWB conjecture in the case $n = 1$ and established in the process a connection with the Riemann zeta function. More recently, Falconer[4] gave a simplified proof of the result of [13] characterizing the Minkowski measurable sets when $n = 1$. This result is key to the proof in [13] of the MWB conjecture in dimension 1.

In this paper we shall disprove the MWB conjecture in all dimensions exceeding 1. In particular we will give two families of examples that show the spectrum of (1·1) must depend on more geometry of Ω than just its volume and the Minkowski dimension and content of its boundary.

To state our results properly we first review the definition of the notion of Minkowski dimension and content. For any set $S \subset \mathbb{R}^n$, let S_ϵ denote the ϵ-neighbourhood of S; that is, $S_\epsilon = \{x \in \mathbb{R}^n : |x-y| < \epsilon \text{ for some } y \in S\}$. Let

$$\mathscr{M}^*(d;\partial\Omega) = \limsup_{\epsilon\to 0^+} \epsilon^{-(n-d)}|(\partial\Omega)_\epsilon \cap \Omega|_n$$

and let $\mathscr{M}_*(d;\partial\Omega)$ denote the corresponding lower limit. Then

$$D = D(\partial\Omega) = \inf\{d \geqslant 0 : \mathscr{M}^*(d;\partial\Omega) < +\infty\}$$

is the *Minkowski dimension* of the boundary of Ω with respect to Ω. We say that $\partial\Omega$ is *Minkowski measurable* (with respect to Ω) if $0 < \mathscr{M}_*(D;\partial\Omega) = \mathscr{M}^*(D;\partial\Omega) < +\infty$ and we let the common number be denoted $\mathscr{M}(D;\partial\Omega)$, calling it the *Minkowski content* of $\partial\Omega$ (with respect to Ω). Recall from [8] that for $D = D(\partial\Omega)$, we have $n-1 \leqslant D \leqslant n$.

The MWB conjecture of [8], p. 520, asserts that if $\partial\Omega$ is Minkowski measurable with Minkowski dimension D, with $n-1 < D < n$, then

$$N(\lambda) = \phi(\lambda) - c_{n,D}\mathscr{M}(D;\partial\Omega)\lambda^{D/2} + o(\lambda^{D/2}) \quad \text{for} \quad \lambda \to +\infty, \tag{1·2}$$

where $c_{n,D}$ is a positive constant depending only on n and D. Towards this conjecture it was shown in [8] that if $D = D(\partial\Omega)$, $n-1 < D < n$ and $\mathscr{M}^*(D;\partial\Omega) < +\infty$, then Weyl's formula with sharp error term holds:

$$N(\lambda) = \phi(\lambda) + O(\lambda^{D/2}) \quad \text{for} \quad \lambda \geqslant 0, \tag{1·3}$$

where the implied constant depends on n, D and Ω. (Also, see [11] and, for corresponding pre-Tauberian estimates, [2].) It was also conjectured in [8], p. 521, that if $n-1 < D < n$ and $0 < \mathscr{M}_*(D;\partial\Omega), \mathscr{M}^*(D;\partial\Omega) < +\infty$, then there is some positive constant c, depending on Ω, n and D such that

$$|N(\lambda) - \phi(\lambda)| \geqslant c\lambda^{D/2} \tag{1·4}$$

for all sufficiently large values of λ.

Both of the conjectures (1·2) and (1·4) were proved in [13] in the case $n = 1$ (i.e. for 'fractal strings' rather than 'fractal drums'). In this paper we disprove both conjectures in each dimension $n \geqslant 2$. We do this by means of two families of examples. The first family, discussed in Sections 2–4, gives pairs of sets Ω_1, Ω_2 in $\mathbb{R}^2$ with $|\Omega_1|_2 = |\Omega_2|_2$, with both $\partial\Omega_1$ and $\partial\Omega_2$ being Minkowski measurable in dimension D, with $1 < D < 2$, and having the same Minkowski content. Thus conjecture (1·2) would imply that $N(\lambda) - \phi(\lambda)$ is asymptotically the same for the two examples. However, we show that though in both cases we have $N(\lambda) - \phi(\lambda)$ asymptotically a constant times $\lambda^{D/2}$, the two constants are not the same.

Our second family of examples, discussed in Sections 5 and 6, can be used to disprove both (1·2) and (1·4). Both families of examples are given in dimension $n = 2$, and so a simple Cartesian product construction can be used to get counterexamples in all higher dimensions.

The sets in the first family are not connected, but the connected components are simply connected. In particular, the boundaries do not contain any isolated points. The sets in the second family are connected, but not simply connected. In fact, each set consists of an open disc with a certain countable closed set removed. Thus the conjectures (1·2) and (1·4) remain open for simply connected domains in $\mathbb{R}^2$.

An example is given in [**5**] consisting of an infinite sequence of square regions in $\mathbb{R}^2$, where the boundary has Minkowski dimension D and $1 < D < 2$, but the second term of the eigenvalue distribution function oscillates between two constant multiples of $\lambda^{D/2}$. However, as is pointed out, the boundary is not Minkowski measurable. In [**12**, §4·2; **13**, §4·3} we gave such an example in $\mathbb{R}$; it is the complement in $[0, 1]$ of the ternary Cantor set. Such an example can be lifted to dimension 2 by taking the Cartesian product with the open unit interval, just as in [**8**, example 5·1′]. It is claimed in [**5**] that the infinite sequence of square regions can be altered to produce a counterexample to (1·2) by removing an appropriate sequence of isolated points. If one wishes to remove countable sequences from open sets so as to create counterexamples to (1·2), the procedure we suggest below in Sections 5 and 6 is both simpler, and also serves to disprove the conjecture (1·4). However, as we suggest in Section 6, this type of counterexample should not be considered as fundamental as the kind we give in Section 4.

Counterexamples to the MWB conjecture along the lines of Sections 2–4 were discussed in the lecture associated with [**14**] and were announced in [**9**] and several other places. We take this opportunity to thank Tom Gard, Leonid Friedlander, José Santos and Paul Wenston for some helpful discussions.

2. *Sprays*

In $\mathbb{R}$ any open set is a disjoint union of open intervals. The eigenvalue problem (1·1) does not depend on the placement of these open intervals on the number line. Rather, it depends only on the sequence of lengths of these intervals. (See [**8**, example 5·1] and [**12**, **13**] where the one-dimensional situation is studied in detail.) Our first class of examples mimics this situation in higher dimensions.

Let Ω be a bounded open set in $\mathbb{R}^n$. If r is a positive real, let $r\Omega$ be the image of Ω under the homothety $x \mapsto rx$ on $\mathbb{R}^n$. If $\mathscr{L} = (l_j)_{j=1}^{\infty}$ is a non-increasing sequence of positive reals, we call a *spray of* $\mathscr{L}$ *on* Ω any open set Ω' in $\mathbb{R}^n$ which is a disjoint union of open sets Ω_j for $j = 1, 2, \ldots$, where Ω_j is congruent to $l_j \Omega$ for each j. Any open set which may be called a spray of $\mathscr{L}$ on Ω has the same spectrum for the eigenvalue problem (1·1) as any other such set.

If the sequence $\mathscr{L}$ has the property that $l = \sum_{j=1}^{\infty} l_j^n < +\infty$, then any spray of $\mathscr{L}$ on Ω has finite volume equal to $l|\Omega|_n$. It is interesting to remark that in this case it is always possible to choose a spray of $\mathscr{L}$ on Ω so that it is bounded; we leave the simple verification to the reader.

In this section we shall discuss the Minkowski dimension and content of the boundaries of sprays. We shall do this in the specific setting of $\mathbb{R}^2$ and for a particular sequence $\mathscr{L}$. It is clear that the methods might be used in a more general setting.

For any bounded open set Ω in $\mathbb{R}^2$, define for $t > 0$,

$$\Gamma(t;\Omega) = |(\partial\Omega)_t \cap \Omega|_2,$$

and let $t_\Omega = \inf\{t > 0 \colon \Gamma(t;\Omega) = |\Omega|_2\}$. If $r > 0$, then clearly $t_{r\Omega} = rt_\Omega$. We have for $\epsilon > 0, r > 0$,

$$\Gamma(\epsilon; r\Omega) = \begin{cases} r^2\Gamma(\epsilon/r;\Omega), & \text{if } \epsilon \leqslant rt_\Omega, \\ r^2|\Omega|_2, & \text{if } \epsilon > rt_\Omega. \end{cases}$$

Thus we always have $\Gamma(\epsilon; r\Omega) = r^2\Gamma(\epsilon/r;\Omega)$.

Suppose now that $1 < D < 2$ and that $\mathscr{L}$ is the sequence $(j^{-1/D})_{j=1}^{\infty}$. (This choice of $\mathscr{L}$ is motivated by [**8**, example 5·1 and appendix C] as well as by [**13**, theorem 2·2].) Let Ω be a bounded open set in $\mathbb{R}^2$ and let S be a spray of $\mathscr{L}$ on Ω. We have

$$\begin{aligned}\Gamma(\epsilon;S) &= \sum_{j=1}^{\infty} \Gamma(\epsilon; j^{-1/D}\Omega) = \sum_{j=1}^{\infty} j^{-2/D}\Gamma(\epsilon j^{1/D};\Omega) \\ &= \sum_{j \leqslant (t_\Omega/\epsilon)^D} j^{-2/D}\Gamma(\epsilon j^{1/D};\Omega) + \sum_{j > (t_\Omega/\epsilon)^D} j^{-2/D}|\Omega|_2 \\ &= \sum_{j \leqslant (t_\Omega/\epsilon)^D} j^{-2/D}\Gamma(\epsilon j^{1/D};\Omega) + (1+o(1))\frac{D}{2-D} t_\Omega^{D-2}|\Omega|_2\,\epsilon^{2-D} \qquad (2·1)\end{aligned}$$

for $\epsilon \to 0^+$.

To compute the Minkowski dimension of ∂S or to determine if ∂S is Minkowski measurable, it seems we should know more about the underlying open set Ω so that the remaining sum in (2·1) may be estimated. We now specify several simple choices for Ω and complete the computation for these choices.

Example 2·1. Let Ω be the interior of an $a \times b$ rectangle, where $0 < a \leqslant b$. Then $t_\Omega = a/2$. Let $\mathscr{L} = (j^{-1/D})_{j=1}^{\infty}$, where $1 < D < 2$ and let S be a spray of $\mathscr{L}$ on Ω. Then ∂S has Minkowski dimension D, it is Minkowski measurable and

$$\mathscr{M}(D;\partial S) = \left(\frac{a}{2}\right)^{D-2}\left(ab\frac{D}{2-D} + (a^2+ab)\frac{D}{D-1} - a^2\right). \qquad (2·2)$$

To see this, note that for $0 < t \leqslant t_\Omega = a/2$, we have

$$\Gamma(t;\Omega) = ab - (a-2t)(b-2t) = (2a+2b)t - 4t^2.$$

Thus the remaining sum in (2·1) is

$$\begin{aligned}&\sum_{j \leqslant (a/2\epsilon)^D} ((2a+2b)j^{-1/D}\epsilon - 4\epsilon^2) \\ &\qquad = (1+o(1))(2a+2b)\frac{D}{D-1}\left(\frac{a}{2}\right)^{D-1}\epsilon^{2-D} - (1+o(1))4\left(\frac{a}{2}\right)^{D}\epsilon^{2-D}\end{aligned}$$

for $\epsilon \to 0^+$. Combining this with (2·1) we have

$$\lim_{\epsilon\to 0^+} \epsilon^{-(2-D)}\Gamma(\epsilon; S) = ab\frac{D}{2-D}\left(\frac{a}{2}\right)^{D-2} + (2a+2b)\frac{D}{D-1}\left(\frac{a}{2}\right)^{D-1} - 4\left(\frac{a}{2}\right)^{D},$$

which proves (2·2).

Example 2·2. Let Ω be the interior of a disc of radius r. Then $t_\Omega = r$. Let $D, \mathscr{L}$ be as in Example 2·1 and let S be a spray of $\mathscr{L}$ on Ω. Then ∂S has Minkowski dimension D, it is Minkowski measurable and

$$\mathscr{M}(D; \partial S) = \pi r^{D-2}\left(\frac{D}{2-D} + \frac{2D}{D-1} - 1\right). \tag{2·3}$$

We leave the computation to the reader.

3. *Eigenvalues for sprays*

In this section we consider the distribution of eigenvalues for the problem (1·1) on a spray. We do not consider the most general case, but rather consider only special sequences $\mathscr{L}$ and special bounded open sets Ω. In addition we restrict attention to dimension $n = 2$.

Say a bounded open set Ω in $\mathbb{R}^2$ is *regular* if the function $N(\lambda; \Omega)$ of Section 1 satisfies

$$N(\lambda; \Omega) = \frac{1}{4\pi}|\Omega|_2\,\lambda + O(\lambda^{\frac{1}{2}}) \tag{3·1}$$

for $\lambda \geqslant 0$. The O-constant may depend on the set Ω. From results of Seeley and Pham The Lai any bounded domain with a smooth boundary is regular (see [**6**]). Moreover, as can be checked directly, rectangular regions are regular (cf. [**7**]).

Say $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$ are the eigenvalues for the problem (1·1) for a *regular* open set Ω. Let $\zeta_\Omega(s)$ denote the spectral zeta function for Ω, so that

$$\zeta_\Omega(s) = \sum_{m=1}^{\infty} \lambda_m^{-s}.$$

It follows immediately from (3·1) that this series converges uniformly on compact subsets of $\operatorname{Re} s > 1$. Indeed, by partial summation,

$$\zeta_\Omega(s) = \int_{\lambda_1}^{\infty} s\lambda^{-s-1}N(\lambda; \Omega)\,d\lambda, \tag{3·2}$$

from which our assertion is transparent. In fact (3·1) can be used to continue $\zeta_\Omega(s)$ meromorphically to $\operatorname{Re} s > \frac{1}{2}$, as we now show.

LEMMA 3·1. *For Ω a regular open set in $\mathbb{R}^2$, the spectral zeta function $\zeta_\Omega(s) = \sum_{m=1}^{\infty} \lambda_m^{-s}$ of Ω is a meromorphic function in the region $\operatorname{Re} s > \frac{1}{2}$, its only singularity there being a simple pole at $s = 1$ with residue $|\Omega|_2/(4\pi)$.*

Proof. Let $c = |\Omega|_2/(4\pi)$. From (3·2)

$$\int_{\lambda_1}^{\infty} (cs\lambda^{-s} - s\lambda^{-s-1}N(\lambda; \Omega))\,d\lambda = \frac{cs\lambda_1^{1-s}}{s-1} - \zeta_\Omega(s)$$

for $\operatorname{Re} s > 1$. It remains to note that the integral converges uniformly on compact subsets of $\operatorname{Re} s > \frac{1}{2}$. Indeed, from (3·1) the integrand is $O(|s\lambda^{-s-\frac{1}{2}}|)$ and so our assertion follows. This completes the proof of the lemma.

We are now ready for the principal result of this section.

THEOREM 3·2. *Let Ω be a regular open set in $\mathbb{R}^2$, let D be a real number with $1 < D < 2$ and let S be a spray of $(j^{-1/D})_{j=1}^{\infty}$ on Ω. Then*

$$N(\lambda;S) = \frac{1}{4\pi}\zeta(2/D)|\Omega|_2\,\lambda + (\zeta_\Omega(D/2)+o(1))\,\lambda^{D/2}$$

for $\lambda \to +\infty$. Here ζ denotes the Riemann zeta function and ζ_Ω denotes the spectral zeta function of Ω.

Proof. For any positive real number r, we have $N(\lambda;r\Omega) = N(r^2\lambda;\Omega)$. Thus

$$N(\lambda;S) = \sum_{j=1}^{\infty} N(\lambda;j^{-1/D}\Omega) = \sum_{j=1}^{\infty} N(j^{-2/D}\lambda;\Omega).$$

For each positive integer j, let $\delta_j(\lambda) = cj^{-2/D}\lambda - N(j^{-2/D}\lambda;\Omega)$, where $c = |\Omega|_2/(4\pi)$. Let $\delta(\lambda) = \sum_{j=1}^{\infty}\delta_j(\lambda)$ so that

$$\delta(\lambda) = c\zeta(2/D)\,\lambda - N(\lambda;S).$$

Thus it is sufficient to show that

$$\delta(\lambda) = (-1+o(1))\,\zeta_\Omega(D/2)\,\lambda^{D/2} \quad \text{for} \quad \lambda \to +\infty. \tag{3·3}$$

Let the eigenvalues for the problem (1·1) on Ω be $(\lambda_m)_{m=1}^{\infty}$, where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$. Note that if m is a positive integer with $\lambda_m \leqslant j^{-2/D}\lambda < \lambda_{m+1}$, then $N(j^{-2/D}\lambda;\Omega) = m$, while if $j^{-2/D}\lambda < \lambda_1$, then $N(j^{-2/D}\lambda;\Omega) = 0$. Let $x_m = (\lambda/\lambda_m)^{D/2}$ for $m = 1, 2, \dots$. Thus for any real number λ and any integer $k > 0$, we have

$$\begin{aligned}
\delta(\lambda) &= \sum_{j=1}^{\infty}\delta_j(\lambda) = \sum_{j=1}^{\infty}(cj^{-2/D}\lambda - N(j^{-2/D}\lambda;\Omega)) \\
&= \sum_{j>x_1} cj^{-2/D}\lambda + \sum_{m=2}^{k}\sum_{x_m<j\leqslant x_{m-1}}(cj^{-2/D}\lambda-(m-1)) + \sum_{j\leqslant x_k}\delta_j(\lambda) \\
&= \sum_{j>x_k} cj^{-2/D}\lambda + \sum_{m=2}^{k}([x_m]-[x_{m-1}])(m-1) + \sum_{j\leqslant x_k}\delta_j(\lambda) \\
&= A+B+C,\ \text{say},
\end{aligned} \tag{3·4}$$

where $[x]$ denotes the integer part of x.

We have

$$A = c\lambda\left(\frac{1}{2/D-1}x_k^{1-2/D} + O(x_k^{-2/D})\right) = c\lambda^{D/2}\frac{D}{2-D}\lambda_k^{1-D/2} + O(\lambda_k),$$

where the O-constant depends only on Ω. Rearranging the sum in B, we have

$$B = k[x_k] - \sum_{m=1}^{k}[x_m] = (k\lambda_k^{-D/2} - \sum_{m=1}^{k}\lambda_m^{-D/2})\,\lambda^{D/2} + O(k),$$

where the O-constant is absolute. Using (3·1), we have $\delta_j(\lambda) = O(\lambda^{\frac{1}{2}} j^{-1/D})$, where the O-constant depends only on Ω. Hence

$$C = O\left(\lambda^{\frac{1}{2}} \sum_{j \leqslant x_k} j^{-1/D}\right) = O\left(\lambda^{D/2} \frac{D}{D-1} \lambda_k^{-(D-1)/2}\right).$$

We thus deduce from (3·4) that

$$\delta(\lambda)\,\lambda^{-D/2} = \frac{cD}{2-D}\lambda_k^{1-D/2} + k\lambda_k^{-D/2} - \sum_{m=1}^{k} \lambda_m^{-D/2} + O\left(k\lambda^{-D/2} + \frac{D}{D-1} k^{-(D-1)/2}\right), \quad (3·5)$$

using, from (3·1), that $k = O(\lambda_k)$ and $\lambda_k = O(k)$. The O-constant in (3·5) depends only on Ω.

The first three terms on the right of (3·5) add to

$$\frac{D}{2}\int_0^{\lambda_k} (ct^{-D/2} - t^{-1-D/2} N(t;\Omega))\,dt.$$

From the proof of Lemma 3·1, this integral converges to $-\zeta_\Omega(D/2)$ as $k \to +\infty$. Thus (3·3) and the theorem follow by first letting $\lambda \to +\infty$ in (3·5) and then letting $k \to +\infty$.

Remark 3·3. In view of the results in [**13**, §4·2] in the case $n = 1$, a refinement of this proof shows that a generalization of Theorem 3·2 (with the main term suitably adjusted) still holds if S is a spray of $(l_j)_{j=1}^{\infty}$ on Ω, where $l_1 \geqslant l_2 \geqslant \ldots > 0$ and $l_j = j^{-1/D}(1 + o(1))$ for $j \to +\infty$.

4. *Counterexamples, I*

In this section we give an example, using sprays, which disproves the MWB conjecture (see (1·2)).

Example 4·1. We describe two bounded open sets Ω_1, Ω_2 in $\mathbb{R}^2$ for which $|\Omega_1|_2 = |\Omega_2|_2$, $\partial\Omega_1, \partial\Omega_2$ are both Minkowski measurable in dimension D, $1 < D < 2$, $\mathcal{M}(D;\partial\Omega_1) = \mathcal{M}(D;\partial\Omega_2)$, yet

$$N(\lambda;\Omega_1) - N(\lambda;\Omega_2) \sim c\lambda^{D/2} \quad \text{for} \quad \lambda \to +\infty,$$

for some non-zero constant c. We thus have a contradiction to (1·2) which would imply that $N(\lambda;\Omega_1) - N(\lambda;\Omega_2) = o(\lambda^{D/2})$ for $\lambda \to +\infty$. We actually describe a family of pairs Ω_1, Ω_2 parametrized by the letter D, so that we have counterexamples to the MWB conjecture for most Minkowski dimensions D, $1 < D < 2$.

For each D, $1 < D < 2$, let $\Omega_1 = \Omega_1(D)$ denote a spray of $(j^{-1/D})_{j=1}^{\infty}$ on the unit square and let $\Omega_2' = \Omega_2'(D)$ denote a spray of $(j^{-1/D})_{j=1}^{\infty}$ on the $a \times 2a$ rectangle, where $a = (2/(D+2))^{1/D}$. Then let Ω_2 be the disjoint union of Ω_2' and the interior of a square of area $(1-2a^2)\,\zeta(2/D)$.

The reason we perturb Ω_2' with an additional square is to arrange for Ω_1 and Ω_2 to have the same area. (Note that for all D, $1 < D < 2$, we have $2a^2 < 1$.) This perturbation does not affect the Minkowski dimension of $\partial\Omega_2$, its Minkowski content or the asymptotics of the second term of $N(\lambda;\Omega_2)$.

From (2·2) we have that Ω_1 and Ω_2 are both Minkowski measurable in dimension D and with the same Minkowski content, namely $2^{3-D}(2-D)^{-1}(D-1)^{-1}$. Let ζ_1 denote the spectral zeta function for the unit square and let ζ_2 denote the spectral zeta function for the $a\times 2a$ rectangle. Then from Theorem 3·2 we have

$$N(\lambda;\Omega_1)-N(\lambda;\Omega_2) = (1+o(1))(\zeta_1(D/2)-\zeta_2(D/2))\lambda^{D/2}$$

for $\lambda\to+\infty$. Consider this coefficient of $\lambda^{D/2}$. It is

$$C(D) := \zeta_1(D/2)-\zeta_2(D/2). \tag{4·1}$$

Let us look a little more closely at the spectral zeta functions ζ_1 and ζ_2. The eigenvalues for the problem (1·1) on the unit square are the numbers $\pi^2(m_1^2+m_2^2)$, where m_1, m_2 run over the positive integers. Thus the least eigenvalue is $2\pi^2$ and it has multiplicity 1, so that

$$\zeta_1(s) = (1+o(1))(2\pi^2)^{-s} \quad \text{for} \quad \operatorname{Re} s\to+\infty.$$

The situation for ζ_2 is more delicate since it depends on a, which in turn depends on D. Let $\zeta_{1\times 2}$ be the spectral zeta function for the 1×2 rectangle. Thus

$$\zeta_2(D/2) = a^D\zeta_{1\times 2}(D/2) = \frac{2}{D+2}\zeta_{1\times 2}(D/2). \tag{4·2}$$

The eigenvalues for the problem (1·1) on the 1×2 rectangle are the numbers $\pi^2(m_1^2+m_2^2/4)$, where m_1, m_2 run over the positive integers. Thus the least eigenvalue is $5\pi^2/4$ and it has multiplicity 1, so that

$$\zeta_{1\times 2}(s) = (1+o(1))(5\pi^2/4)^{-s} \quad \text{for} \quad \operatorname{Re} s\to+\infty.$$

We now assume that the variable D in (4·1) is not restricted to be a real number in the interval $1 < D < 2$, but is allowed to be a complex number in the half plane $\operatorname{Re} D > 1$. The function $C(D)$ is, by (4·1) and (4·2), given by

$$C(D) = \zeta_1(D/2)-\frac{2}{D+2}\zeta_{1\times 2}(D/2),$$

and so from Lemma 3·1 we see that $C(D)$ is meromorphic (in fact, it is holomorphic) in the half plane $\operatorname{Re} D > 1$. From the considerations above we deduce that

$$C(D) = -(1+o(1))\frac{2}{D+2}\left(\frac{5\pi^2}{4}\right)^{-D/2} \quad \text{for} \quad \operatorname{Re} D\to+\infty.$$

In particular, we see that the function $C(D)$ is not identically zero. Thus it is 0 at most finitely often in any compact subset of the real interval $(1, 2]$. For each D in $(1, 2)$ with $C(D) \neq 0$, we thus have a counterexample to the MWB conjecture.

Since the 1×2 rectangle has area double that of the unit square, and since $2/(D+2) > \frac{1}{2}$ for all D, $1 < D < 2$, it may be possible to show that $C(D)$ is never 0 in this interval, which we conjecture to be the case.

By choosing a as $(\zeta_1(D/2)/\zeta_{1\times 2}(D/2))^{1/D}$, we see that $N(\lambda;\Omega_1)-N(\lambda;\Omega_2) = o(\lambda^{D/2})$ for $\lambda\to+\infty$, but now the Minkowski contents of $\partial\Omega_1$ and $\partial\Omega_2$ are unequal for all but at most finitely many choices for D in any compact subset of $(1, 2]$ (the same possible exceptional dimensions D as before). For the unexceptional fractal dimensions D in

(1, 2) (which we conjecture to be every D in this interval) one cannot 'hear' the shape of a fractal drum, at least if one's hearing is limited to the first two terms in the asymptotic expansion of $N(\lambda;\Omega)$.

Remark 4·2. One could similarly obtain counterexamples to (1·2) based on a spray on the unit square and a spray on the unit disc (with the first square adjusted to equalize the areas). Such an example would use Examples 2·1, 2·2, Theorem 3·2, and the known distribution of the eigenvalues on the unit disc.

5. *Multiply punctured discs*

In this section we give the background for another family of counterexamples to the MWB conjecture.

For each real number a with $0 < a < 1$, let

$$S(a) = \{(m^{-a}, jm^{-a-2}) : j, m \in \mathbb{Z}, m > 1, 0 < j \leqslant m^2\} \cup \{(0,0)\}.$$

Thus $S(a)$ is a countable closed subset of $\mathbb{R}^2$. Let Ω be the open unit disc in $\mathbb{R}^2$ centred at the origin and let $\Omega(a) = \Omega \backslash S(a)$.

THEOREM 5·1. *For $\Omega(a)$ as above, we have $\partial\Omega(a)$ Minkowski measurable in dimension* $2/(a+1)$.

Proof. It will suffice to show there is some positive real number c with $|S(a)_\epsilon|_2 = (c+o(1))\,\epsilon^{2-2/(a+1)}$ for $\epsilon \to 0^+$. Let $\epsilon > 0$. Let $M_1 = (2\epsilon)^{-1/(a+2)}$ and let $M_2 = (2\epsilon/a)^{-1/(a+1)}$. For ϵ sufficiently small we have $M_1 < M_2$. The ϵ-neighbourhood of $S(a)$ is divided into three parts by M_1, M_2. The first part corresponds to points in $S(a)$ with $m \leqslant M_1$, and the ϵ-neighbourhood consists of disjoint discs of radius ϵ. The second part corresponds to points in $S(a)$ with $M_1 < m \leqslant M_2$, and the ϵ-neighbourhood changes from columns of 'kissing' discs for m near M_1 to rectangles with very small notches removed for m near M_2. The third part corresponds to the points of $S(a)$ with $m > M_2$ and the ϵ-neighbourhood is now filled in – it is essentially the region under a part of the graph of $y = x$ in the first quadrant.

We have $|S(a)_\epsilon|_2 = A_1 + A_2 + A_3$, where A_1, A_2, A_3 are the areas of the ϵ-neighbourhoods of the three parts of $S(a)$ described above. Evidently, for $\epsilon \to 0^+$,

$$A_1 = O\left(\epsilon^2 \sum_{m \leqslant M_1} m^2\right) = O(\epsilon^2 M_1^3) = O(\epsilon^{2-3/(a+2)}). \tag{5·1}$$

The values of A_2 and A_3 are a little harder to estimate.

Let A_2' be the sum of the areas of rectangles of width 2ϵ and height m^{-a} for $M_1 < m \leqslant M_2$. Then for $\epsilon \to 0^+$,

$$\begin{aligned} A_2' &= 2\epsilon \sum_{M_1 < m \leqslant M_2} m^{-a} = \frac{2\epsilon}{1-a} M_2^{1-a} + O(\epsilon M_1^{1-a}) \\ &= \frac{2}{1-a}\left(\frac{2}{a}\right)^{1-2/(a+1)} \epsilon^{2-2/(a+1)} + O(\epsilon^{2-3/(a+2)}). \end{aligned}$$

If these rectangles extend by ϵ from both sides of the points $(m^{-a}, 0)$ for $M_1 < m \leqslant M_2$, then by our choice of M_2, they do not overlap. The principal error in going from A_2

to A_2' is caused by the fact that a column of overlapping discs does not quite have straight sides. The depth of an indentation between two discs is

$$\epsilon-\sqrt{\epsilon^2-(4m^{2a+4})^{-1}} = \epsilon(1-\sqrt{1-(4\epsilon^2m^{2a+4})^{-1}}) = O(\epsilon(\epsilon^2m^{2a+4})^{-1}) = O(\epsilon^{-1}m^{-2a-4}).$$

Thus

$$\begin{aligned} A_2 &= A_2'+O\Bigg(\sum_{M_1<m\leqslant M_2}\epsilon^{-1}m^{-2a-4}\cdot m^{-a}\Bigg) \\ &= A_2'+O(\epsilon^{-1}M_1^{-3a-3}) = A_2'+O(\epsilon^{2-3/(a+2)}) \\ &= \frac{2}{1-a}\left(\frac{2}{a}\right)^{1-2/(a+1)}\epsilon^{2-2/(a+1)}+O(\epsilon^{2-3/(a+2)}), \end{aligned}$$

by our estimate for A_2'.

The value of A_3 is approximated by the area under the curve $y = x$ from $x = 0$ to $x = M_2^{-a}$. Specifically,

$$\begin{aligned} A_3 &= \tfrac{1}{2}M_2^{-2a}+O(\epsilon M_2^{-a}) = \frac{1}{2}\left(\frac{2}{a}\right)^{2a/(a+1)}\epsilon^{2a/(a+1)}+O(\epsilon^{1+a/(a+1)}) \\ &= \frac{1}{2}\left(\frac{2}{a}\right)^{2-2/(a+1)}\epsilon^{2-2/(a+1)}+O(\epsilon^{2-1/(a+1)}). \end{aligned}$$

Combining our estimates for A_1, A_2, A_3, we have

$$\begin{aligned} |S(a)_\epsilon|_2 &= \left(\frac{1}{2}+\frac{a}{1-a}\right)\left(\frac{2}{a}\right)^{2-2/(a+1)}\epsilon^{2-2/(a+1)}+O(\epsilon^{2-3/(a+2)}) \\ &= (c+o(1))\,\epsilon^{2-2/(a+1)} \quad \text{as} \quad \epsilon\to 0^+, \end{aligned}$$

for an appropriate positive constant c. This completes the proof of the theorem.

We remark that the Minkowski dimension of $\partial\Omega(a)$, that is, $2/(a+1)$, assumes all of the values in the interval $(1,2)$ as a runs over $(0,1)$.

6. *Counterexamples, II*

Note that since the set $S(a)$ of the previous section is countable and closed, it is a closed set of (Newtonian) capacity 0 in $\mathbb{R}^2$. (See, e.g. [**15**, example 1, p. 35] and [**3**, §V·3, p. 59].) Thus the eigenvalues for the Dirichlet problem (1·1) on $\Omega(a)$ are exactly the same as the eigenvalues for the problem (1·1) on Ω, the open unit disc; and the eigenfunctions on $\Omega(a)$ are the restrictions of the eigenfunctions on Ω. (See, e.g., [**15**, proposition 2·2, p. 36].) Consequently

$$N(\lambda;\Omega(a)) = \tfrac{1}{4}\lambda+O(\lambda^{\frac{1}{2}}) \tag{6·1}$$

for any choice of a with $0 < a < 1$. However, by Theorem 5·1, $\partial\Omega(a)$ is Minkowski measurable in dimension $D = 2/(a+1)$. Since $1 < D < 2$, the assumptions of conjecture (1·2) are satisfied and thus, if (1·2) is true, $N(\lambda;\Omega(a))-\lambda/4$ is asymptotic to a non-zero constant times $\lambda^{D/2}$, for $\lambda\to+\infty$. This clearly contradicts (6·1) (since $D > 1$) so that (1·2) (the MWB conjecture) is false. Note that (6·1) also disproves the conjecture (1·4). Further, in [**13**, p. 44] (see also [**9**, conjecture $\tilde{1}$, §5·3]), it is conjectured that if the boundary of a bounded open set in $\mathbb{R}^n$ is Minkowski measurable in dimension D in $(n-1, n)$, then $N(\lambda)-\phi(\lambda)$ is asymptotically proportional to $\lambda^{D/2}$. Thus (6·1) also disproves this conjecture.

To our minds, this family of counterexamples does not seem as fundamental as the one of Section 4. For example, one may say two open sets are *equivalent* if they differ by a (closed) set of (Newtonian) capacity 0 (a 'negligible' or 'polar' set in potential theory [3, chapter V]). Then one might define the *intrinsic Minkowski dimension* of $\partial\Omega$ as the infimum of the Minkowski dimensions of boundaries of all open sets equivalent to Ω. If so, then the intrinsic Minkowski dimension of $\partial\Omega(a)$ is 1, so that $\Omega(a)$ does not have an intrinsically fractal boundary and the counterexamples of this section disappear. There does not seem to be an easy way to bar the 'monsters' from Section 4 and that is one reason we think they have more fundamental interest.

Another reason is that the counterexamples of Section 4 (and the positive results of Section 3) extend without difficulty to Neumann (rather than Dirichlet) boundary conditions. This does not seem to be the case, however, of the counterexamples constructed in this section.

7. *Remarks*

We close this paper by several comments that are directly related to our present work and may help to put it in a broader context.

7·1. We can show (see [**9**, remark 5·6 (*c*)]) that if $N(\lambda)$ admits an asymptotic second term proportional to $\lambda^{D/2}$, with $n-1 \leqslant D < n$, then the proportionality constant C is given by

$$C = \lim_{s\to(D/2)^+} \frac{2}{D}\left(s-\frac{D}{2}\right)\zeta_\Omega(s), \tag{7·1}$$

where $\zeta_\Omega(s)$ is the spectral zeta function of Ω. A similar statement holds for Neumann as well as mixed boundary conditions. Hence C in (7·1) depends not only on Ω and $\partial\Omega$, but also on the boundary conditions; i.e. both on geometric and analytic data.

7·2. It seems unlikely that there are results along the lines of conjectures (1·2) and (1·4) above that are true for all bounded open sets in $\mathbb{R}^n$ with Minkowski measurable boundary in dimension D in $(n-1, n)$, when $n \geqslant 2$. However, the first author [**10**, part II] has recently proposed several conjectures that would apply to certain special classes of domains, such as domains with (approximately) self-similar boundary. For example, for snowflake-type (simply connected planar) domains, Conjectures 2 and 3 in [**10**], pp. 159 and 163, imply that 'generically', $\partial\Omega$ is Minkowski measurable and $N(\lambda)$ admits an asymptotic second term proportional to $\lambda^{D/2}$. Moreover, many related open problems remain to be investigated.

REFERENCES

[1] M. S. Birman and M. Z. Solomjak. The principal term of the spectral asymptotics for nonsmooth elliptic problems. *Funktsional. Anal. i Prilhozen* **4**, no. 4 (1970), 1–13.
[2] J. Brossard and R. Carmona. Can one hear the dimension of a fractal? *Comm. Math. Phys.* **104** (1986), 103–122.
[3] J. L. Doob. *Classical potential theory and its probabilistic counterpart* (Springer-Verlag, 1984).
[4] K. Falconer. On the Minkowski measurability of fractals. *Proc. Amer. Math. Soc.*, to appear.
[5] J. Fleckinger-Pellé and D. G. Vassiliev. An example of a two-term asymptotics for the 'counting function' of a fractal drum. *Trans. Amer. Math. Soc.* **337** (1993), 99–116.

[6] L. HÖRMANDER. *The analysis of linear partial differential operators*, vols. III, IV (Springer-Verlag, 1985).

[7] N. V. KUZNETSOV. Asymptotic distribution of the eigenfrequencies of a plane membrane in the case when the variables can be separated. *Differential Equations* **2** (1966), 715–723.

[8] M. L. LAPIDUS. Fractal drum, inverse spectral problems for elliptic operators and a partial resolution of the Weyl–Berry conjecture. *Trans. Amer. Math. Soc.* **325** (1991), 465–529.

[9] M. L. LAPIDUS. Spectral and fractal geometry: from the Weyl–Berry conjecture for the vibrations of fractal drums to the Riemann zeta-function; in *Differential Equations and Mathematical Physics*. Proc. Fourth UAB Intern. Conf., Birmingham, Alabama, March 1990 (ed. C. Bennewitz, Academic Press, 1992), pp. 151–182.

[10] M. L. LAPIDUS. Vibrations of fractal drums, the Riemann hypothesis, waves in fractal media, and the Weyl–Berry conjecture; in *Ordinary and partial differential equations*, vol. IV, Proc. Twelfth Dundee Intern. Conf., Dundee, Scotland, UK, June 1992 (eds. B. D. Sleeman and R. J. Jarvis), Pitman Research Notes in Math. Series **289** (Longman Scientific and Technical, 1993), pp. 126–209.

[11] M. L. LAPIDUS and J. FLECKINGER-PELLÉ. Tambour fractal: vers une résolution de la conjecture de Weyl–Berry pour les valeurs propres du laplacien. *C.R. Acad. Sci. Paris Sér. I Math.* **306** (1988), 171–175.

[12] M. L. LAPIDUS and C. POMERANCE. Fonction zêta de Riemann et conjecture de Weyl–Berry pour les tambours fractals. *C.R. Acad. Sci. Paris Sér. I Math.* **310**, No. 6 (1990), 343–348.

[13] M. L. LAPIDUS and C. POMERANCE. The Riemann zeta-function and the one-dimensional Weyl–Berry conjecture for fractal drums. *Proc. London Math. Soc.* (3) **66**, No. 1 (1993), 41–69.

[14] M. L. LAPIDUS and C. POMERANCE. Abstract no. 865-11-73 (865th Meeting of the Amer. Math. Soc., Tampa, March 1991), *Abstracts Amer. Math. Soc.* **12**, No. 2 (1991), p. 238.

[15] J. RAUCH and M. TAYLOR. Potential and scattering theory on wildly perturbed domains. *J. Funct. Anal.* **18** (1975), 25–59.