Cyclotomic primes

Carl Pomerance, Dartmouth College

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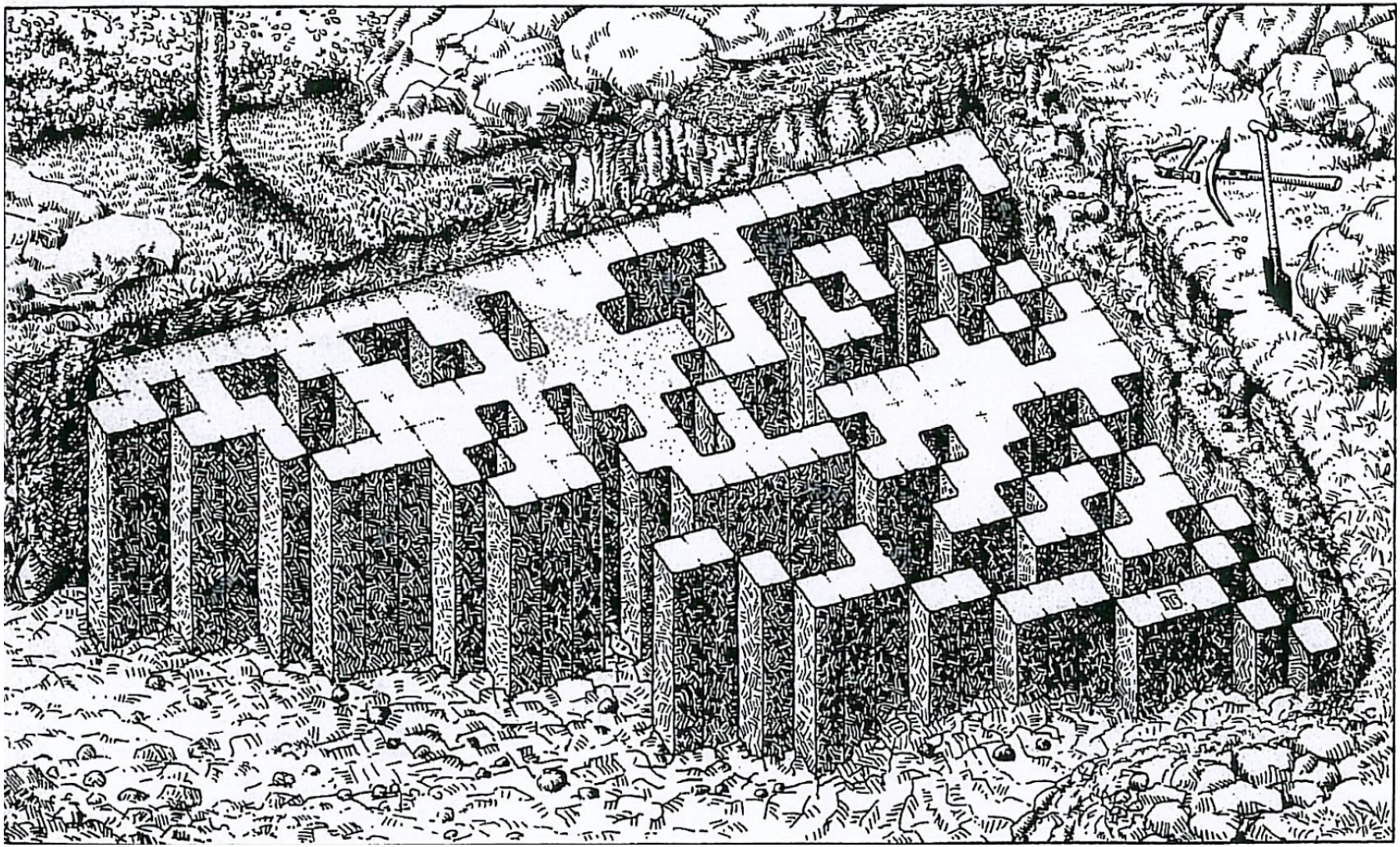
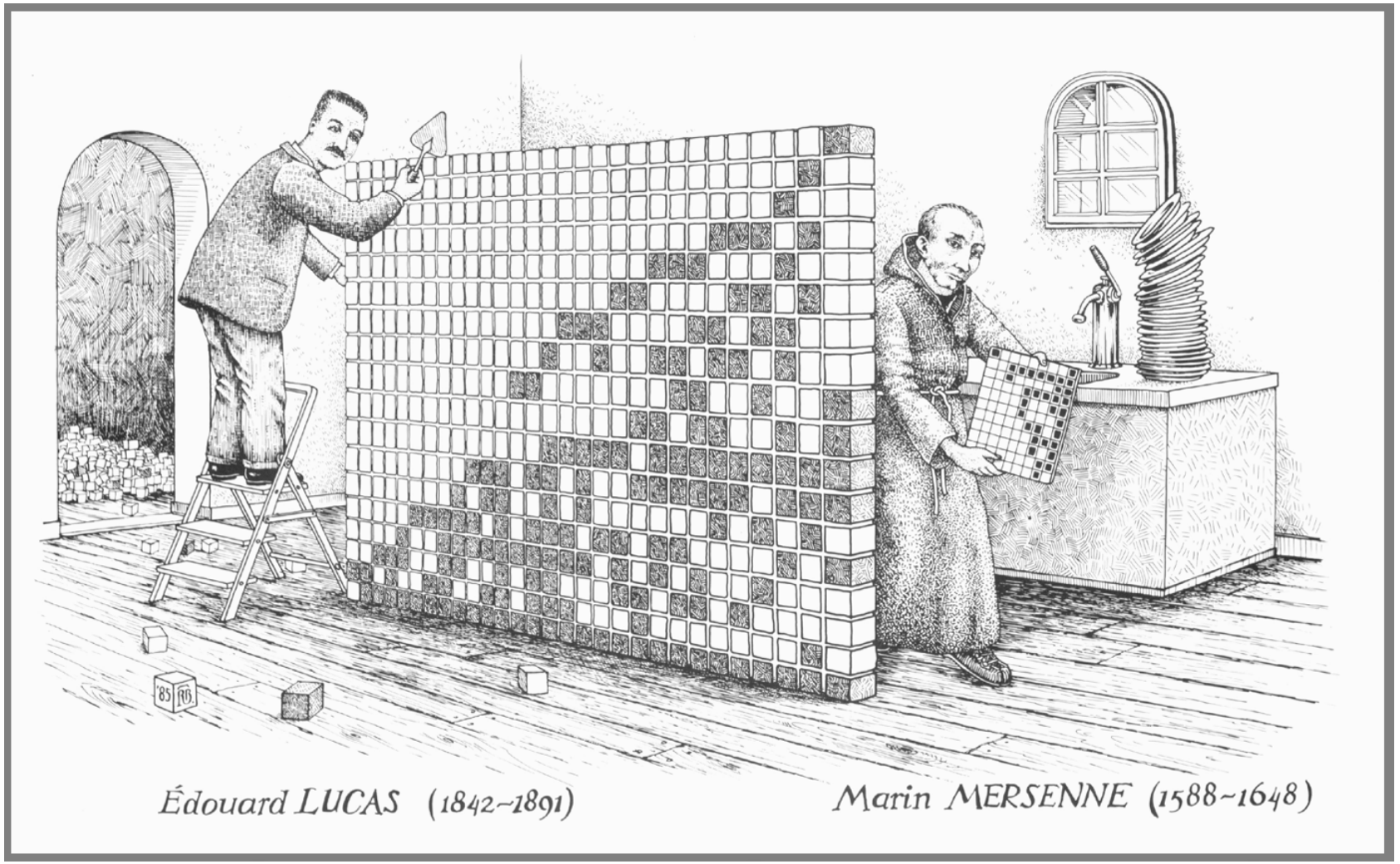



Édouard LUCAS (1842-1891)
Marin MERSENNE (1588-1648)

(Graphic art by Tobias Baanders, based on some needlepoint of Willemien Lenstra and a concept of Hendrik Lenstra)

These are in fact the sequence of exponents n, written in binary, for which $2^n - 1$ is prime.

A prime of the form $2^n - 1$ must have n itself prime. They go back to Pythagoras and Euclid, and are currently known as Mersenne primes.

They have certainly grabbed the public's imagination!

TIME Magazine's 29th greatest invention of 2008.

The exponent in binary: 10100100011101100010100001.

(While perhaps the 29th greatest invention, it is the 47th prime of the form $2^p - 1$.)

The current largest known (Mersenne) prime is

$$2^{136279841} - 1.$$

The exponent in binary is 1000000111110111011100100001.

It was unearthed last October (after a hiatus of 6 years) and is the 52nd known Mersenne prime.

The current search: One runs through candidate exponents p. If $2^p - 1$ survives a search for possible small prime factors $q \equiv 1 \pmod{p}$ with $(2/q) = 1$, it is checked if

$$3^{2^{p-1}} \equiv -3 \pmod{2^p - 1}.$$

If it is not, the calculation is checked (in much less time than it took to compute the first time). If the congruence holds, the exponent p is then subjected to the Lucas–Lehmer test: Starting with $x = 4$, iterate $x \mapsto x^2 - 2 \pmod{2^p - 1}$ $p - 2$ times. This is 0 if and only if $2^p - 1$ is prime.

The Lucas–Lehmer test "lives" in the quadratic field $\mathbb{Q}[\sqrt{-3}]$. Similar primality tests work for n when $n + 1$ has a fully known (or mostly known) prime factorization. Though in general we have the polynomial time primality test of Agrawal, Kayal, and Saxena (with improvements by Lenstra and P), it is not competitive with Lucas–Lehmer when the latter is appropriate.

Here is a heuristic that there are infinitely many Mersenne primes: The number $2^p - 1$ has least prime factor $> p$. A random number n with all prime factors $> \log n$ is prime with probability $\sim e^\gamma \log\log n / \log n$, where γ is Euler's constant. Applying this with $n = 2^p - 1$, the likelihood it is prime is $(e^\gamma / \log 2)(\log p)/p$. It remains to note that the series $\sum(\log p)/p$ diverges. The numerical evidence seems to support this reasoning, even with the special constant $e^\gamma / \log 2$.

We also have the Fermat numbers $F_n = 2^{2^n} + 1$. (An odd prime of the form $2^k + 1$ must have k a power of 2.) Fermat thought all of these numbers are prime, and he was right for $n = 0, 1, 2, 3, 4$. Consider the case $n = 5$. If $p \mid F_5$, then $2^{2^5} \equiv -1 \pmod{p}$, so that $p \equiv 1 \mod 2^6$. Then $(2/p) = 1$, so that $2^{(p-1)/2} \equiv 1 \pmod{p}$, and hence $p \equiv 1 \pmod{2^7}$. The candidates for p are

$$129,\ 257,\ 385,\ 513,\ 641, \ldots.$$

But 129, 385, and 513 are obviously not prime, and 257, which is F_3 cannot divide F_5 since the Fermat numbers are easily seen to be pairwise coprime. So, the very first candidate is 641, and in fact, as Euler showed, it is indeed a proper factor of F_5.

The Fermat numbers have been tested up to $n = 32$, and all of them after $n = 4$ are composite. Many of these were found composite after a nontrivial prime factor was found. We also have Pepin's test: For $n \geq 1$, $F_n = 2^{2^n} + 1$ is prime if and only if $3^{(F_n-1)/2} \equiv -1 \pmod{F_n}$. The largest n tested this way is $n = 24$. Pepin's test generalizes to the case when the number m to be tested has $m - 1$ completely (or mostly) factored, and is simpler than the Lucas–Lehmer test.

A heuristic suggests there are only finitely many Fermat primes, since $\sum(\log\log F_n)/\log F_n$ converges. In fact it converges so rapidly, it is thought there are no more Fermat primes after $n = 4$.

Here are some possibly easier questions:

Are there infinitely many primes p with $2^p - 1$ composite?

Are there infinitely many n with $2^{2^n} + 1$ composite?

Actually these are also not known!

Perhaps we can follow the maxim: If you can't solve the problem, generalize it.

Let $\Phi_m(x)$ denote the mth cyclotomic polynomial. This is the minimum polynomial for $e^{2\pi i/m}$. Some facts:

$$x^n - 1 = \prod_{m \,|\, n} \Phi_m(x), \quad \Phi_n(x) = \prod_{m \,|\, n} (x^m - 1)^{\mu(n/m)}.$$

Also, $\deg(\Phi_n) = \varphi(n)$, Euler's function. Some examples:

$\Phi_p(x) = (x^p - 1)/(x - 1)$, $\quad \Phi_{2^{n+1}} = (x^{2^{n+1}} - 1)/(x^{2^n} - 1) = x^{2^n} + 1$,

so that

$$\Phi_p(2) = 2^p - 1, \quad \Phi_{2^{n+1}}(2) = 2^{2^n} + 1.$$

So, here are the generalized and supposedly easier questions:

Are there infinitely many m with $\Phi_m(2)$ prime?

Are there infinitely many m with $\Phi_m(2)$ composite?

The second question has a disappointingly easy answer! In fact two disappointingly easy answers!

Say a prime factor p of $\Phi_m(2)$ is *primitive* if $\ell(p) = m$. Here, $\ell(p)$ denotes the multiplicative order of 2 in $(\mathbb{Z}/p\mathbb{Z})^\times$.
Examples: $\ell(5) = 4,\ \ell(7) = 3,\ \ell(17) = 8,\ \ell(31) = 5$.

In fact: If $\ell(p) = m$, then $p \mid \Phi_m(2)$.

Must every prime factor of $\Phi_m(2)$ be primitive?

The answer is "almost." If m is of the form $p^j \ell(p)$ for some prime p, with $j \geq 1$, then $p \mid \Phi_m(2)$. In this case p is an *intrinsic* prime factor. It is unique and $p^2 \nmid \Phi_m(2)$.

Bang (1886): Each $\Phi_m(2)$ has a primitive prime factor except for $m = 1$ and $m = 6$. (Note that $\Phi_1(2) = 1$ and $\Phi_6(2) = 3$.)

Thus, every $\Phi_m(2)$, where $m = p^j \ell(p)$ for a prime $p > 3$, is composite. For example, $m = 20$, and $\Phi_{20}(2) = 205$, which has the primitive prime factor 41 and the intrinsic prime factor 5.

So, let's reword the problems. Let $\psi_m = \Phi_m(2)/p$ if m is of the form $p^j \ell(p)$, and otherwise let $\psi_m = \Phi_m(2)$.

Are there infinitely many m with ψ_m prime?, composite?

Consider the polynomial $x^4 + 4$. Is it irreducible in $\mathbb{Z}[x]$?

Well, the roots are $\pm 1 \pm i$, which each have degree 2, so it's not irreducible. In fact

$$\begin{aligned} x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 &= (x^2 + 2)^2 - (2x)^2 \\ &= (x^2 + 2x + 2)(x^2 - 2x + 2). \end{aligned}$$

Similarly,

$$4x^4 + 1 = (2x^2 + 2x + 1)(2x^2 - 2x + 1).$$

Now say $m \equiv 4 \pmod 8$, so $m = 8k+4$. Then $\Phi_m(x) \mid x^{4k+2}+1$, so

$$\Phi_m(2) \mid 4 \cdot 2^{4k}+1 = (2 \cdot 2^{2k}+2 \cdot 2^k+1)(2 \cdot 2^{2k}-2 \cdot 2^k+1).$$

When $m = 8k+4$, we have $\psi_m = \psi_m^+ \psi_m^-$, where

$$\psi_m^+ = \gcd(\psi_m, 2 \cdot 2^{2k}+2 \cdot 2^k+1), \quad \psi_m^- = \gcd(\psi_m, 2 \cdot 2^{2k}-2 \cdot 2^k+1).$$

Schinzel (1962): For $m \equiv 4 \pmod 8$ and $m > 20$, we have $\psi_m^+ > 1$ and $\psi_m^- > 1$. In particular, ψ_m is composite.

Theorem (P, 2024). There are infinitely many $m \not\equiv 4 \pmod 8$ with ψ_m composite. There are infinitely many $m \equiv 4 \pmod 8$ with not both ψ_m^+ and ψ_m^- prime.

Sketch of the proof.

The idea is that ψ_m is exponentially large, about $2^{\varphi(m)}$. So if $p \mid \psi_m$ and p is not much bigger than m, then as a large number with a modest prime factor, ψ_m must be composite.

How many primes $p \le x$ have $\ell(p) < x^{1/2}/\log x$? Well, $2^m - 1$ has fewer than m prime factors and

$$\sum_{m < x^{1/2}/\log x} m < x/(\log x)^2.$$

Since there are $\sim \frac{1}{2}x/\log x$ primes $p \in (x/2, x]$, most of them have $\ell(p) \ge x^{1/2}/\log x$. So, most of them have $\psi_{\ell(p)}$ composite. That is, there are infinitely many m with ψ_m composite.

We can insist in this argument that $p \equiv 3 \pmod 8$, which steers us away from the ψ_m^+, ψ_m^- argument. Or $p \equiv 5 \pmod 8$, which shows that infinitely often ψ_m^+, ψ_m^- are not both prime.

All well and good, but how many $m \le x$ have ψ_m composite (in the case $m \not\equiv 4 \pmod 8$). In the argument just presented, we could have the primes p crowding into just a few ψ_m's. Though unlikely, it is possible. Or is it?

We have $m \mid p-1$. For a given m, the number of choices for $p \le x$ with $m \mid p-1$ is $\le x/m \le x^{1/2} \log x$. Let S be the set of m's which occur in this argument, so that

$$\frac{x}{\log x} \ll \sum_{m \in S} \frac{x}{m} \le \#S \cdot x^{1/2} \log x,$$

and we conclude that $\#S \gg x^{1/2}/(\log x)^2$.

By being a little more careful with the estimates, we can prove the following.

Theorem (P, 2024). The number of integers $m \le x$ with $m \not\equiv 4 \pmod 8$ and ψ_m composite is $\ge x^{1/2}$ for x sufficiently large. Further, the number of $m \le x$ with $m \equiv 4 \pmod 8$ and not both ψ_m^+, ψ_m^- prime is $\ge x^{1/2}$.

Additional problems:

Can we do better than $x^{1/2}$ values of $m \leq x$ with ψ_m composite?

Are the standard conjectures, like the abc-conjecture, the Riemann Hypothesis, the Generalized Riemann Hypothesis, and the prime k-tuples conjecture helpful for the problems we're considering?

The answer is "yes" to all of these.

First, we can use a variant of the proof presented to get $> x^\theta$ values of $m \le x$ with $m \not\equiv 4 \pmod 8$ and ψ_m composite, where $\theta = 3/5$ or a little larger, and similarly for ψ_m^+ and ψ_m^- when $m \equiv 4 \pmod 8$. The key here is a result of Baker and Harman that a positive proportion of primes p have a prime factor of $p-1$ that is $> p^\theta$. We need a version where $p \equiv 3 \pmod 4$ and a version where $p \equiv 5 \pmod 8$, which takes some effort.

Assuming the Elliott–Halberstam conjecture in analytic number theory would allow for θ to be arbitrarily close to 1.

We can prove a stronger result assuming the prime k-tuples conjecture of Hardy and Littlewood. This is a grand quantitative generalization of the twin primes conjecture. (For example, it asserts that the number of twin primes up to x is asymptotically $cx/(\log x)^2$ for an explicit positive constant c.)

From this conjecture we have that the number of primes $p \leq x$ with $p \equiv 3 \pmod 4$ and $2p+1$ prime is asymptotically $cx/(\log x)^2$ (for a different explicit positive constant c).

Why is this interesting? Well, if p is such a prime, then, $2p+1 \mid 2^p - 1$, so for $p > 3$, we have $2^p - 1$ composite. (For example, $2^{11} - 1$, $2^{23} - 1$, and $2^{83} - 1$ are composite.) So, the number of $m \leq x$ with m odd and ψ_m composite is $\gg x/(\log x)^2$.

A quick proof: Let $q = 2p + 1$, so $q \equiv 7 \pmod 8$ and $(2/q) = 1$. Then $2^{(q-1)/2} \equiv 1 \pmod q$, so $q \mid 2^p - 1$.

An issue that I've not mentioned: A composite number need not be divisible by 2 different primes! I don't know how to unconditionally prove that there are infinitely many $m \not\equiv 4 \pmod 8$ with ψ_m divisible by 2 different primes, and similarly for the 4 (mod 8) case. However, using the abc-conjecture, this issue disappears. Even so, it takes some work.

One deals with the "$a+b=c$" equation: $1+(2^m-1)=2^m$. Let $\mathrm{rad}(n)$ denote the product of the distinct primes dividing n. Then $\mathrm{rad}(abc) = 2\,\mathrm{rad}(2^m-1)$. Now $\psi_m \mid 2^m-1$, and an averaging argument shows that most of the time $\psi_m > 2^{\epsilon m}$. If ψ_m is a prime power, then $\mathrm{rad}(\psi_m) \le \psi_m^{1/2}$ so that $\mathrm{rad}(2^m-1) \le 2^{(1-\epsilon/2)m}$. This implies that $\mathrm{rad}(abc) < 2c^{1-\epsilon/2}$, contradicting the abc-conjecture. Thus, we may assume ψ_m is not a prime power, and since it is composite, it must be divisible by at least 2 distinct primes.

Hooley has shown, using the Generalized Riemann Hypothesis (actually, the RH for the zeta functions of Kummerian fields) that a positive proportion of primes p have 2 as a primitive root. A routine variant shows this is true as well for primes $p \equiv 3 \pmod 8$. This then gives $\gg x/\log x$ values of $m \not\equiv 4 \pmod 8$ with ψ_m composite, and we can do even a little better than this. Similarly for the other case.

We have seen that assuming the prime k-tuples conjecture, there are infinitely many primes p with $2^p - 1$ composite. In fact, it is divisible by at least 2 different primes, since otherwise $2^p - 1$ and 2^p would be two consecutive powers, contradicting Catalan's conjecture (= Mihăilescu's theorem).

We showed earlier by a simple argument that $\ell(p) > x^{1/2}/\log x$ for almost all primes $p \le x$. For each such $p \equiv 3 \pmod 4$ we have $\psi_{\ell(p)}$ composite, and for each such $p \equiv 5 \pmod 8$ we have that not both $\psi^+_{\ell(p)}$ and $\psi^-_{\ell(p)}$ are prime. We worked harder to show that there are in fact many values of $m \not\equiv 4 \pmod 8$ with ψ_m composite, and similarly for the other case.

However, there are $\gg x/\log x$ values of p in play. Surely there should be many distinct values of $\ell(p)$ among all these p's! This seems hard to prove other than by the means mentioned above. It is amusing that if there are not so many distinct values of $\ell(p)$, then there are values of m with many distinct primitive prime factors of ψ_m. So, we have the following result.

Theorem: Either there are $\gg x/\log x$ values of $m \leq x$ with $m \not\equiv 4 \pmod 8$ and ψ_m composite or the number of primitive prime factors of ψ_m is unbounded as m varies.

Of course, both must be true!

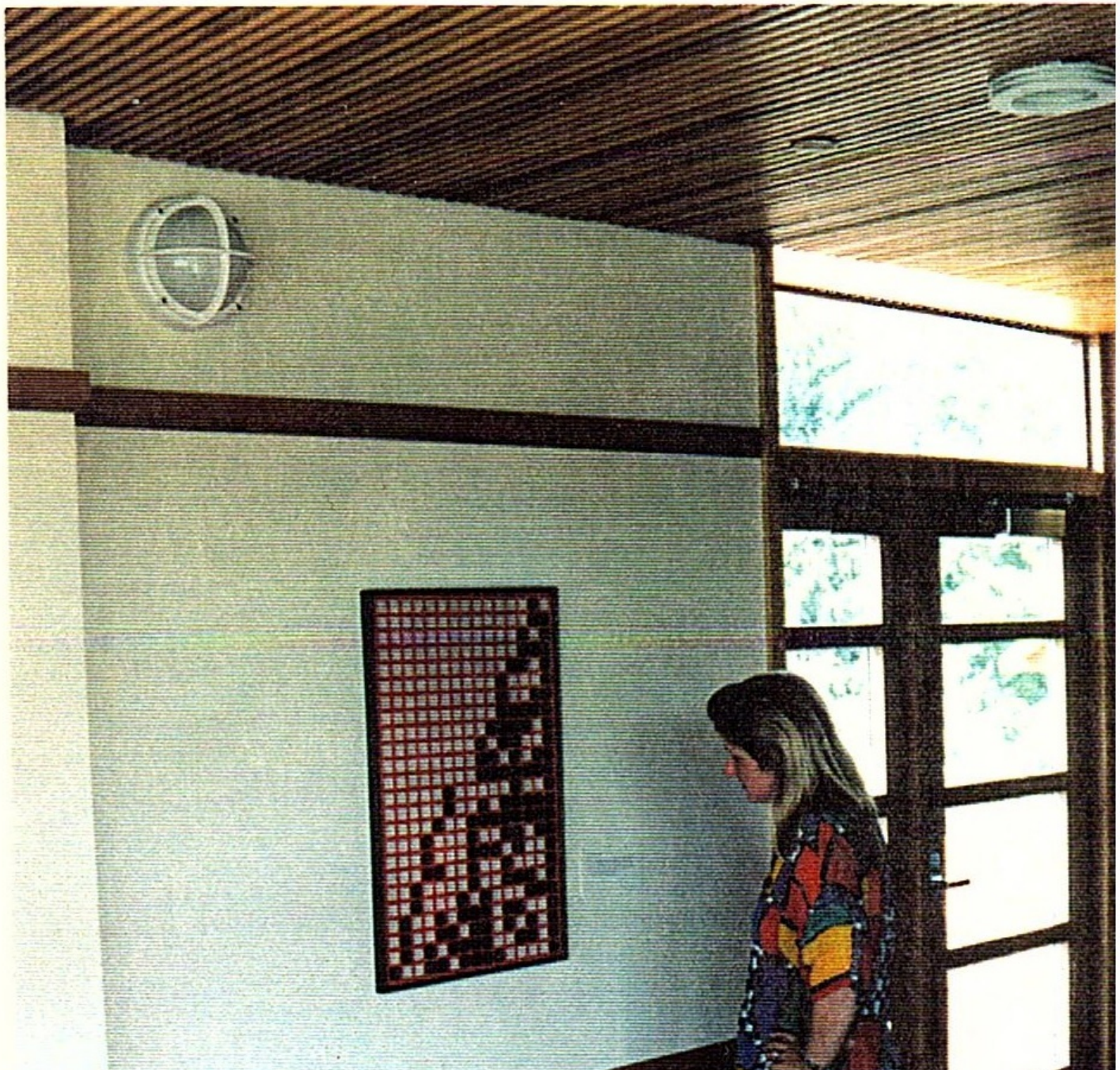
Additional problems, some tractable:

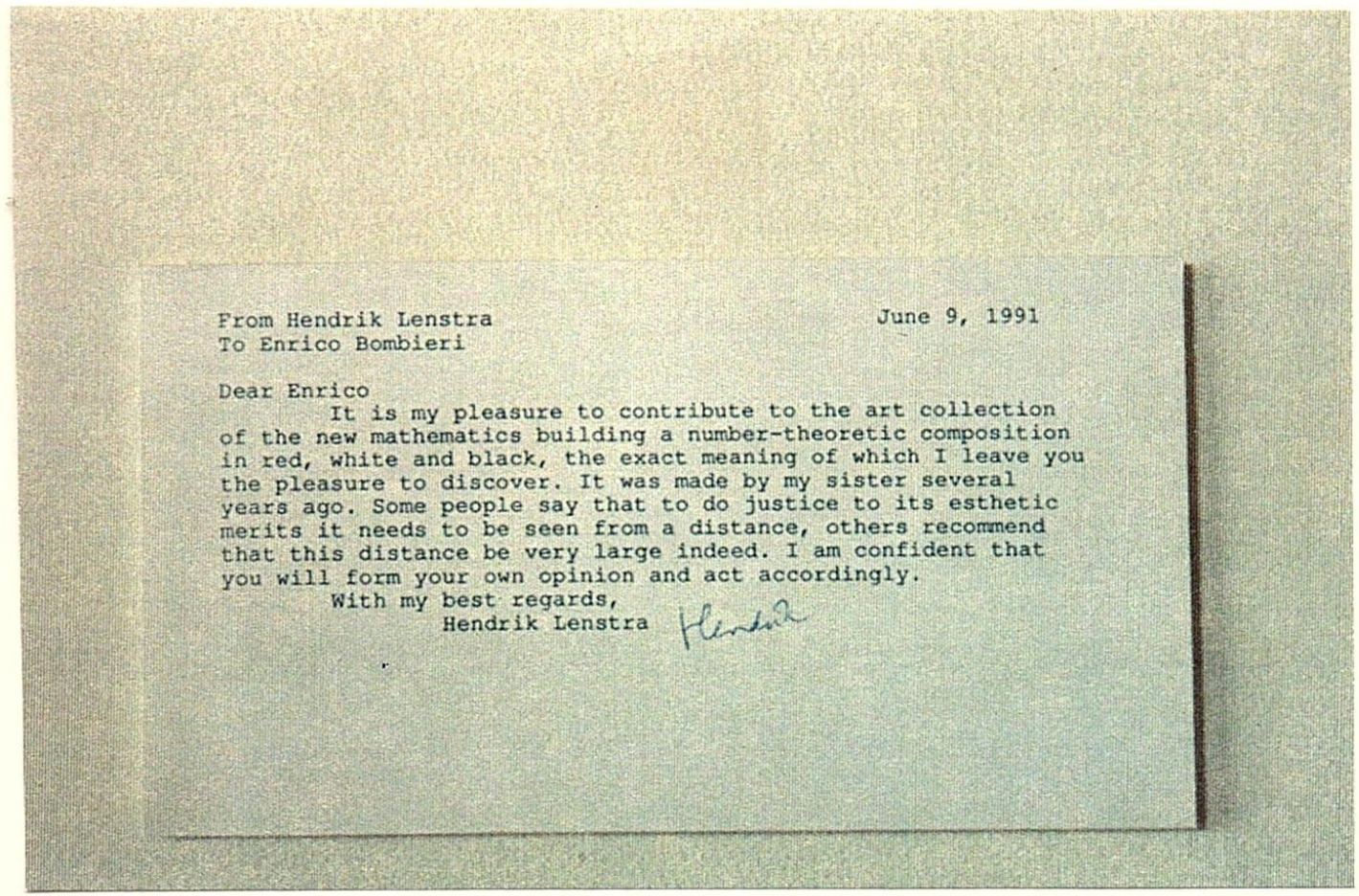
Show there are infinitely many $m \equiv 4 \pmod 8$ with both ψ_m^+ and ψ_m^- composite.

Generalize these results to $\Phi_m(a)$ where $a > 2$.

Generalize to the Fibonacci numbers, and similar Lucas sequences.

There must be an elliptic curve analogue ...



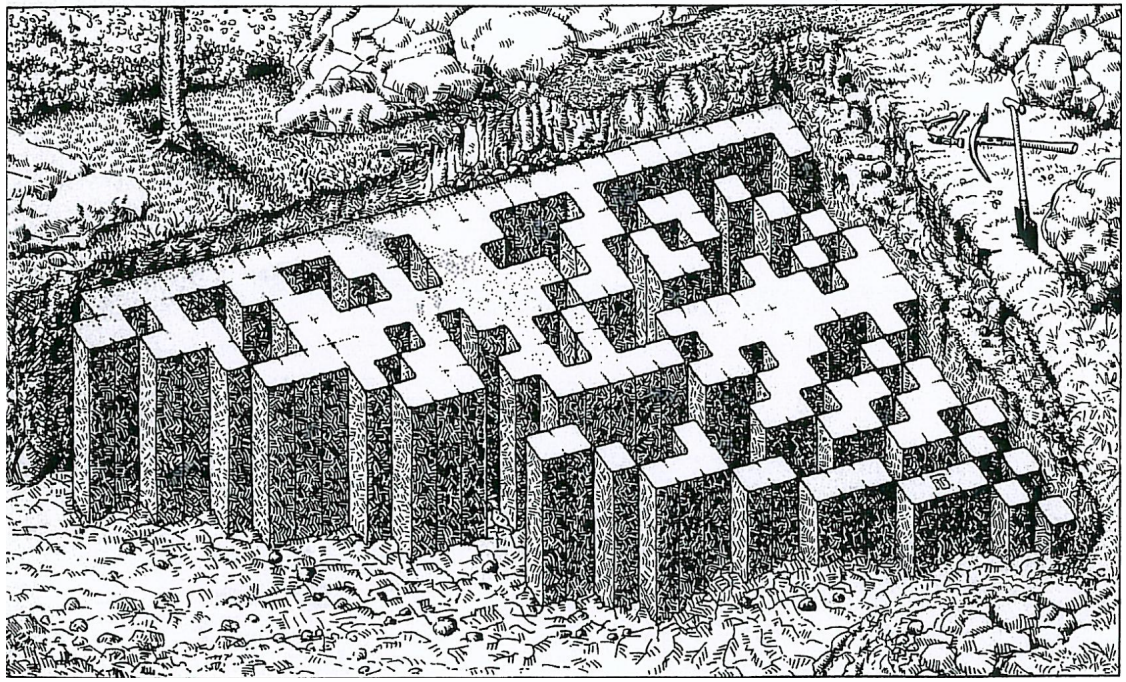


From Hendrik Lenstra June 9, 1991
To Enrico Bombieri

Dear Enrico
It is my pleasure to contribute to the art collection of the new mathematics building a number-theoretic composition in red, white and black, the exact meaning of which I leave you the pleasure to discover. It was made by my sister several years ago. Some people say that to do justice to its esthetic merits it needs to be seen from a distance, others recommend that this distance be very large indeed. I am confident that you will form your own opinion and act accordingly.
With my best regards,
Hendrik Lenstra Hendrik

Princeton, koffiekamer
van't Inst. of Adv. Technology
priemgetallen v. Mersenne

The digs at Mersenneacus



Thank you!