ON THE LONGEST SIMPLE PATH IN THE DIVISOR GRAPH[1]

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Dedicated to Paul Erdös on his seventieth birthday.

§1. Introduction.

If S is a set of natural numbers, then the divisor graph on S is the graph whose vertices are the numbers in S and whose edges are the pairs $\{a,b\}$ where $a,b \in S$ and either $a|b$ or $b|a$. Let $f(n)$ denote the length of the longest simple path in the divisor graph on $\{1,2,\ldots,n\}$. For example, the sequence

$$26, 13, 1, 25, 5, 15, 30, 10, 20, 4, 16, 8, 24, 12, 6, 18, 9, 27, 3, 21, 7, 14, 28, 2, 22, 11$$

shows that $f(30) \geq 26$. (With a little reflection about the "hard to use" numbers 11, 22, 13, 26, 17, 19, 23, and 29 one can see that $f(30) = 26$.)

It is the object of this paper to show that $f(n) = o(n)$, thus resolving a question recently put by N. Hegyvári. The proof makes intrinsic use of an asymptotic formula for the function $\psi(x,y,z)$, the number of natural numbers up to x composed solely of primes in the interval $(z,y]$. This formula was established by Friedlander [4] and is valid when $\log x/\log z$ is bounded. Our proof that $f(n) = o(n)$ does not demonstrate the existence of an explicit function $\theta(n)$ with $\theta(n)/n \to 0$ and $f(n) \leq \theta(n)$ for all n. To do so by the method of this paper would require an extension of the range of Friedlander's theorem so that $\log x/\log z$ is allowed to tend to ∞ at some explicit pace. While this is perhaps not so difficult to do, we do not undertake this exercise here.

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Concerning lower bounds for $f(n)$, A.D. Pollington [6] has recently shown that

$$(1.1) \qquad f(n) \geq n \cdot \exp\{-(\log n \operatorname{loglog} n)^{1/2}\}$$

for all large n.

Now consider the graph on the vertices $\{1,2,\ldots,n\}$ such that $\{a,b\}$ is an edge if and only if the least common multiple $[a,b] \leq n$. If $g(n)$ is the length of the longest simple path in this graph, then clearly $g(n) \geq f(n)$. We actually prove the "stronger" result that $g(n) = o(n)$. The word stronger appears in quotation marks, because it is not immediately clear that $g(n) - f(n)$ is unbounded or even if there is any n for which $g(n) > f(n)$. If there are such values of n, it may be an interesting computational problem to find the first one.

In Erdös, Freud, Hegyvári [3], the following two results are proved. There is some permutation $a_1, a_2, \ldots$ of the natural numbers such that for each $i \geq 3$,

$$(1.2) \qquad \frac{1}{i}[a_i, a_{i+1}] < \exp\{c(\log i)^{1/2} \operatorname{loglog} i\}$$

where c is a positive constant. Also, for any permutation $a_1, a_2, \ldots$ of the natural numbers,

$$(1.3) \qquad \limsup \frac{1}{i}[a_i, a_{i+1}] \geq \frac{1}{1 - \log 2}.$$

The authors conjecture that $\{\frac{1}{i}[a_i, a_{i+1}]\}$ is unbounded for any permutation $a_1, a_2, \ldots$ of the natural numbers, a strengthening of (1.3).

Our principal result immediately gives the Erdös, Freud, Hegyvári conjecture. Indeed, if $a_1, a_2, \ldots$ is a permutation of the natural numbers and if

$$\frac{1}{i}[a_i, a_{i+1}] < B \quad \text{for} \quad i = 1,2,\ldots$$

for some integer B, then for all n, each of $a_1, \ldots, a_n$ is less than Bn. Thus $g(Bn) \geq n$ for all n, contradicting $g(n) = o(n)$.

Similarly, the result (1.2) gives the lower bound

$$g(n) > n \cdot \exp\{-c(\log n)^{1/2} \operatorname{loglog} n\}$$

which is almost as strong as what Pollington's result (1.1) implies for $g(n)$.

§2. Preliminaries.

In the introduction the function $\psi(x,y,z)$, the number of natural numbers up to x composed solely of the primes in the interval $(z,y]$, was introduced. This function generalizes the function $\psi(x,y) = \psi(x,y,1)$ and the function $\varphi(x,z) = \psi(x,x,z)$.

It has been known since 1930 that there is a continuous function $\rho(u)$ on $[0,\infty]$ that is identically 1 on $[0,1]$ and tends strictly and monotonically to 0 on $[1,\infty]$ such that for fixed u,

$$\psi(x,x^{1/u}) \sim \rho(u)x .$$

Later, de Bruijn [2] showed that

$$\psi(x,y) \sim \rho(\log x/\log y)x \tag{2.1}$$

uniformly for a certain large region in the x, y plane and this region was recently extended by H. Maier.

The function $\varphi(x,z)$ represents the number of uncancelled elements in the sieve of Eratostenes on the interval $[1,x]$ after sifting with the primes up to z. The fundamental lemma of Brun's sieve (cf. Halberstam and Richert [5], Theorem 2.5) implies that if $\log z = o(\log x)$, then

$$\varphi(x,z) \sim x \prod_{p\le z} (1 - \frac{1}{p}) \sim \frac{x}{e^{\gamma}\log z} , \tag{2.2}$$

where γ is Euler's constant. Removing the restriction $\log z = o(\log x)$, we have the following result of de Bruijn [1]. There is a continuous function $\omega(v)$ on $(1,\infty)$ such that

$$\lim_{v\to\infty} \omega(v) = e^{-\gamma} \tag{2.3}$$

and

$$\varphi(x,z) \sim \omega(\log x/\log z) \frac{x}{\log z} \qquad \text{as} \qquad z \to \infty \tag{2.4}$$

uniformly for $x > z^{1+\varepsilon}$ for any $\varepsilon > 0$ fixed.

In [4], Friedlander worked out the necessary details to combine both (2.1) and (2.4) into one theorem. He defined a function $\sigma(u,v)$, continuous where $u \neq 1$, $v \neq 1$, such that

$$\psi(x,y,z) = \sigma(u,v)\frac{x}{\log z} + O\left(\frac{x}{\log^2 z}\right), \tag{2.5}$$

$$u = \log x/\log y, \quad v = \log x/\log z,$$

uniformly for $1+\varepsilon \leq u < v \leq M$ for any fixed $\varepsilon > 0, M$. Moreover, Friedlander showed that for fixed $u > 0$,

$$\lim_{v\to\infty} \sigma(u,v) = e^{-\gamma}\rho(u). \tag{2.6}$$

Since $\sigma(u,v)$ is continuous where neither u nor v is 1, (2.6) can be made uniformly true on compact sets of the variable u that do not contain 1.

Although stated in the technical forms needed in section 3, Propositions 1, 2, and 3 are really not very deep. Proposition 1 essentially states that for each $\delta > 0$, there is an $\varepsilon > 0$ such that of the integers up to n none of whose primes exceed n^δ, at least half have a divisor $> n^{1-\varepsilon}$ all of whose primes exceed n^ε. Proposition 2 essentially states that for each $\delta > 0$ there is an $\varepsilon > 0$ such that almost all of the integers up to n have a divisor $> n^{1-\delta}$ all of whose primes exceed n^ε. Finally, Proposition 3 essentially states that for each $\delta > 0$ there is an $\varepsilon > 0$ such that almost all integers up to n are not divisible by a number $> n^{1-\varepsilon}$ all of whose primes exceed n^δ.

<u>Proposition 1</u>. Suppose $1 \geq \delta > \alpha > 0$ are fixed. Then there is an $\varepsilon' > 0$ such that for each ε, $0 < \varepsilon \leq \varepsilon'$, there is an $n_o = n_o(\delta,\alpha,\varepsilon)$ with the following property. For each $n \geq n_o$, if $1 = a_1 < a_2 < \ldots$ are the integers all of whose primes come from $(n^\varepsilon, n^\delta]$, then

$$\sum_{i>2yn^{-\varepsilon}} \left[\frac{y}{a_i}\right] > \frac{1}{2}\psi(y,n^\delta)$$

for any y, $n^\alpha \leq y \leq n$.

Proof. Since $e^{-\gamma} \doteq .56146$, from (2.6) and (2.3) there is an ε' with $0 < \varepsilon' < \min\{.01, \alpha\}$ such that if

$$v \geq \frac{\alpha}{\varepsilon'} - 1 , \quad 1 + \varepsilon' \leq u \leq \frac{1}{\delta} ,$$

then

$$\sigma(u,v) \geq (.53)\rho(u) \quad \text{and} \quad \omega(v) \geq .53 . \tag{2.7}$$

Suppose $0 < \varepsilon \leq \varepsilon'$. By (2.5) we have

$$\psi(t,n^{\delta},n^{\varepsilon}) = \sigma(u,v) \frac{t}{\log n^{\varepsilon}} + O\left(\frac{t}{\log^2 n^{\varepsilon}}\right)$$

uniformly for $n^{\delta+\delta\varepsilon'} \leq t \leq n$. Thus there is an $n_1 = n_1(\delta,\alpha,\varepsilon)$ such that if $n \geq n_1$, then

$$\psi(t,n^{\delta},n^{\varepsilon}) > (.52)\ \rho\left(\frac{\log t}{\log n^{\delta}}\right) \frac{t}{\log n^{\varepsilon}} . \tag{2.8}$$

We wish to extend the domain of validity of (2.8) for also those values of t with $n^{\alpha-\varepsilon} \leq t < n^{\delta+\delta\varepsilon'}$. First note that if $n^{\alpha-\varepsilon} \leq t \leq n^{\delta}$, then

$$\psi(t,n^{\delta},n^{\varepsilon}) = \varphi(t,n^{\varepsilon}) \sim \omega\left(\frac{\log t}{\log n^{\varepsilon}}\right) \frac{t}{\log n^{\varepsilon}}$$

uniformly. Thus by (2.7), there is an $n_2 = n_2(\delta,\alpha,\varepsilon)$ such that if $n \geq n_2$, we have the inequality (2.8) for $n^{\alpha-\varepsilon} \leq t \leq n^{\delta}$. (Note that $\rho(\log t/\log n^{\delta}) = 1$ in this range.)

Now assume $n^{\delta} < t < n^{\delta+\delta\varepsilon'}$. Then

$$\begin{aligned}
\psi(t,n^{\delta},n^{\varepsilon}) &\geq \varphi(t,n^{\varepsilon}) - (t - \psi(t,n^{\delta})) \\
&= \varphi(t,n^{\varepsilon}) - \sum_{n^{\delta}<p\leq t} \left[\frac{t}{p}\right] \\
&\geq \varphi(t,n^{\varepsilon}) - t \sum_{n^{\delta}<p\leq t} \frac{1}{p}
\end{aligned}$$

$$= \varphi(t,n^{\varepsilon}) - t(\log\log t - \log\log n^{\delta} + O(e^{-\sqrt{\log t}}))$$

$$\geq \varphi(t,n^{\varepsilon}) - t \log (1 + \varepsilon') + O(te^{-\sqrt{\log t}})$$

$$\geq \varphi(t,n^{\varepsilon}) - t \log (1.01) + O(te^{-\sqrt{\log t}}) ,$$

by the prime number theorem. Since $\varphi(t,n^{\varepsilon}) \sim \omega(\log t/\log n^{\varepsilon})t/\log n^{\varepsilon}$, by (2.7) there is an $n_3 = n_3(\delta,\varepsilon)$ such that if $n \geq n_3$, we have (2.8) for $n^{\delta} < t < n^{\delta+\delta\varepsilon'}$. (Note that $\rho(\log t/\log n^{\delta}) < 1$ in this range.)

We conclude that if $0 < \varepsilon \leq \varepsilon'$ and $n \geq n_4 = \max \{n_1,n_2,n_3\}$, then (2.8) holds for all t with $n^{\alpha-\varepsilon} \leq t \leq n$.

We are nearly ready to consider the sum in the proposition, but first we must estimate the starting point $a_{[2yn^{-\varepsilon}+1]}$. Let $\beta = \beta(\delta,\varepsilon,y,n)$ be defined by

$$a_{[2yn^{-\varepsilon}]} = yn^{-\beta} .$$

Then $\beta < \varepsilon$ and by (2.5), $\beta \to \varepsilon$ as $n \to \infty$ uniformly in y satisfying the hypothesis of the proposition and for each fixed δ, ε.

By partial summation we have

$$\sum_{i>2yn^{-\varepsilon}} [\frac{y}{a_i}] = \sum_{yn^{-\beta}<a_i\leq y} [\frac{y}{a_i}] = \sum_{yn^{-\beta}<a_i\leq y} \frac{y}{a_i} + O(\psi(y,n^{\delta},n^{\varepsilon}))$$

(2.9)

$$= \psi(y,n^{\delta},n^{\varepsilon}) - \frac{y}{yn^{-\beta}} \psi(yn^{-\beta},n^{\delta},n^{\varepsilon}) + \int_{yn^{-\beta}}^{y} \frac{y}{t^2} \psi(t,n^{\delta},n^{\varepsilon})dt + O(\psi(y,n^{\delta},n^{\varepsilon}))$$

$$= \int_{yn^{-\beta}}^{y} \frac{y}{t^2} \psi(t,n^{\delta},n^{\varepsilon})dt + O(\frac{y}{\log n}) ,$$

by (2.5). Assuming $0 < \varepsilon \leq \varepsilon'$ and $n \geq n_4$, we have by (2.8)

$$\int_{yn^{-\beta}}^{y} \frac{y}{t^2}\,\psi(t,n^{\delta},n^{\varepsilon})\,dt > \frac{(.52)y}{\log n^{\varepsilon}} \int_{yn^{-\beta}}^{y} \frac{1}{t}\,\rho\left(\frac{\log t}{\log n^{\delta}}\right)dt$$

$$> \frac{(.52)y\,\rho(\log y/\log n^{\delta})}{\log n^{\varepsilon}} \int_{yn^{-\beta}}^{y} \frac{1}{t}\,dt$$

$$= (.52\,\beta/\varepsilon)y\,\rho(\log y/\log n^{\delta}) \ .$$

Thus by (2.9) there is an $n_5 = n_5(\delta,\alpha,\varepsilon) \geq n_4$ such that if $n \geq n_5$ then

$$\sum_{i>2yn^{-\varepsilon}} \left[\frac{y}{a_i}\right] > (.51)y\,\rho(\log y/\log n^{\delta})$$

Finally, by (2.1) there is an $n_o = n_o(\delta,\alpha,\varepsilon) \geq n_5$ such that if $n \geq n_o$, then

$$\sum_{i>2yn^{-\varepsilon}} \left[\frac{y}{a_i}\right] > \frac{1}{2}\,\psi(y,n^{\delta}) \ .$$

<u>Proposition 2</u>. Given $1 \geq \delta$, $\alpha > 0$, there is an $\varepsilon' > 0$ with the following property. If $0 < \varepsilon \leq \varepsilon'$, there is an $n_o = n_o(\delta,\alpha,\varepsilon)$ such that if $n \geq n_o$ and $1 = b_1 < b_2 < \ldots$ are the integers all of whose primes exceed n^{ε}, then

$$\sum_{i<2n^{1-\delta}} \psi\left(\frac{n}{b_i},n^{\varepsilon}\right) < \alpha n \ .$$

<u>Proof</u>. We shall take $\varepsilon' < 1/2$. Then b_i does not exceed the i-th prime above $n^{1/2}$, so that

$$b_{[2n^{1-\delta}]} < n^{1-\frac{1}{2}\delta} \tag{2.10}$$

for all $n \geq n_1$. Let

$$h_{\varepsilon}(m) = \sum_{\substack{p^a \| m \\ p \leq n^{\varepsilon}}} a \log p = \sum_{\substack{p^a | m \\ p \leq n^{\varepsilon}}} \log p \quad ,$$

where p denotes primes. Then

$$\sum_{m=1}^{n} h_\varepsilon(m) = \sum_{p \le n^\varepsilon} \sum_{a \ge 1} [\frac{n}{p^a}] \log p$$

$$= \sum_{p \le n^\varepsilon} \frac{n \log p}{p} + O(n)$$

$$= \varepsilon n \log n + O(n)$$

for any ε. Thus there is an $n_2(\varepsilon)$ such that if $n \ge n_2(\varepsilon)$, then

$$\sum_{m=1}^{n} h_\varepsilon(m) < 2\varepsilon n \log n .$$

Thus for $n \ge n_2(\varepsilon)$,

$$\sum_{\substack{m<n \\ h_\varepsilon(m) \ge \frac{1}{4}\delta \log n}} 1 < \frac{8\varepsilon}{\delta} n \quad . \tag{2.11}$$

Let $\varepsilon' = \delta\alpha/9$. For $n \ge n_3(\delta,\alpha)$, we have

$$n^{1-\delta/4} < \frac{1}{9} \alpha n \quad . \tag{2.12}$$

Let $0 < \varepsilon \le \varepsilon'$ be arbitrary, let $n_o(\delta,\alpha,\varepsilon) = \max \{n_1, n_2, n_3\}$, and let $n \ge n_o$. From (2.10) we have

$$\sum_{i<2n^{1-\delta}} \psi(\frac{n}{b_i}, n^\varepsilon) \le \sum_{b_i<n^{1-\delta/2}} \psi(\frac{n}{b_i}, n^\varepsilon)$$

$$= \sum_{b_i<n^{1-\delta/2}} \sum_{\substack{b_i \ell \le n \\ p|\ell \to p \le n^\varepsilon}} 1 = \sum_{\substack{m<n \\ m/e^{h_\varepsilon(m)} < n^{1-\delta/2}}} 1$$

$$\le n^{1-\delta/4} + \sum_{\substack{n^{1-\delta/4} < m \le n \\ m/e^{h_\varepsilon(m)} < n^{1-\delta/2}}} 1$$

$$\leq n^{1-\delta/4} + \sum_{\substack{m\leq n \\ h_\varepsilon(m) > \frac14 \delta \log n}} 1$$

$$< \frac{1}{9}\alpha n + \frac{8\alpha}{9} n = \alpha n ,$$

where for the last estimate we use (2.11) and (2.12).

Proposition 3. Given $1 > \delta, \alpha > 0$, there is an ε' with $\delta > \varepsilon' > 0$ such that for each ε, $0 < \varepsilon \leq \varepsilon'$, there is an $n_o = n_o(\delta,\alpha,\varepsilon)$ with the following property. For every $n \geq n_o$, we have

$$\sum_{\substack{n^{1-\varepsilon}<D\leq n \\ p|D \to p>n^\delta}} \frac{1}{D} < \alpha$$

where p denotes primes.

Proof. For $\varepsilon < 1 - \delta$, we have the sum in the proposition equal to

$$\frac{1}{n}\varphi(n,n^\delta) - \frac{1}{n^{1-\varepsilon}}\varphi(n^{1-\varepsilon},n^\delta) + \int_{n^{1-\varepsilon}}^{n} \frac{1}{t^2}\varphi(t,n^\delta)dt$$

$$= \frac{\omega(v)}{\delta \log n} \int_{n^{1-\varepsilon}}^{n} \frac{1}{t}\, dt + O_{\delta,\varepsilon}\left(\frac{1}{\log n}\right)$$

$$= \frac{\varepsilon}{\delta}\omega(v) + O_{\delta,\varepsilon}\left(\frac{1}{\log n}\right)$$

for some v between $(1 - \varepsilon)/\delta$ and $1/\delta$ inclusive. Thus if ε is sufficiently small and $n \geq n_o(\delta,\alpha,\varepsilon)$, then the inequality in the proposition holds.

§3. The principal result.

Theorem. If $m_1,m_2,\ldots,m_{g(n)}$ is the longest sequence of distinct integers from $\{1,2,\ldots,n\}$ such that for each $\nu = 1,2,\ldots,g(n) - 1$ we have $[m_\nu,m_{\nu+1}] \leq n$, then $g(n) = o(n)$.

Proof. Let k be an arbitrary, but fixed positive integer. Suppose $m_1,m_2,\ldots,m_{g(n)}$ is a path that realizes $g(n)$. Let

$$1 = \delta_0 > \delta_1 > \ldots > \delta_k > 0$$

be constants where $\delta_1,\ldots,\delta_k$ will be specified shortly. Let $d_\ell(m)$ denote the largest divisor of m all of whose primes come from $(n^{\delta_\ell},n^{\delta_{\ell-1}}]$ for $\ell = 1,\ldots,k$ and let $D_\ell(m) = d_1(m)d_2(m)\ldots d_\ell(m)$. Let $D_0(m) = 1$.

To specify $\delta_1,\ldots,\delta_k$, suppose $1 \leq \ell \leq k$ and $\delta_1,\ldots,\delta_{\ell-1}$ have already been chosen. Let δ'_ℓ be the bound guaranteed by Proposition 3 with $\delta = \delta_{\ell-1}$ and $\alpha = 3^{-k}$. Let $\varepsilon_{1,\ell}$ be the bound guaranteed by Proposition 1 with $\delta = \delta_{\ell-1}$, $\alpha = \delta'_\ell$. Finally, let $\varepsilon_{2,\ell}$ be the bound guaranteed by Proposition 2 with $\delta = \delta_{\ell-1}$, $\alpha = 3^{-k}$. Then we define

$$\delta_\ell = \min \{\varepsilon_{1,\ell},\varepsilon_{2,\ell},\tfrac{1}{2}\,\delta_{\ell-1}\} .$$

There is a bound n_o such that if $n \geq n_o$ then Propositions 1, 2, and 3 hold with the above choices of parameters for each $\ell = 1,\ldots,k$. We also choose n_o so that $n_o^{\delta_1} > 3^k$. For the remainder of the proof, we assume that $n \geq n_o$.

We now examine the consecutive values $d_1(m_1),\ldots,d_1(m_{g(n)})$. Say this sequence changes value $u - 1$ times, so there are u values, perhaps with repeats. This defines a partition of the sequence $m_1,\ldots,m_{g(n)}$ into u blocks. If m_ν is in the i-th block, we write $i_1(m_\nu) = i$, $d(i) = d_1(m_\nu)$.

In the i-th block of consecutive m_ν's, say the value $d_2(m_\nu)$ changes $u(i) - 1$ times, so there are $u(i)$ not necessarily distinct values of $d_2(m_\nu)$ in this block. This then defines a partition of the i-th block into $u(i)$ sub-blocks. If m_ν is in the i-th block, j-th sub-block, we write $i_2(m_\nu) = j$, $d(i,j) = d_2(m_\nu)$.

We then use changes in the value of $d_3(m_\nu)$ to further refine the partition, etc. By the ℓ-th level, we have defined numbers in the u-family, in the i-family, and in the d-family, where u-numbers give the number of blocks in a particular subdivision, i-numbers refer to particular blocks, and d-numbers give the divisors of the m_ν's which define the next subdivision.

In computing an upper bound for $g(n)$, there are certain types of values of $m_1,\ldots,m_{g(n)}$ that cannot be conveniently handled by the principal argument and so are separated off and estimated by other arguments. These are the m_ν for which $D_{\ell-1}(m_\nu) > n^{1-\delta'_\ell}$ or $i_\ell(m_\nu) = 1$ for some $\ell = 1,\ldots,k$.

By Proposition 3, our choice of $\delta_1,\ldots,\delta_k$ and our choice of n, the number of $m \le n$ with $D_{\ell-1}(m) > n^{1-\delta'_\ell}$ is less than $3^{-k}n$. Therefore, the number of m_ν with $D_{\ell-1}(m_\nu) > n^{1-\delta'_\ell}$ for some $\ell = 1,\ldots,k$ is less than $k\cdot 3^{-k}n < 2^{-k}n$.

Now consider the m_ν for which $i_1(m_\nu) = 1$. These are all divisible by $d(1)$. We have chosen δ_1 so that by Proposition 2, the number of $m \le n$ with $d_1(m) = 1$ is less than $3^{-k}n$. Thus assume $d(1) > 1$, so that $d(1) > n^{\delta_1}$. But then the number of $m \le n$ divisible by $d(1)$ is less than $n^{1-\delta_1} < 3^{-k}n$. We conclude that the number of m_ν for which $i_1(m_\nu) = 1$ is less than $3^{-k}n$.

We now consider the m_ν for which $i_\ell(m_\nu) = 1$ for some $\ell = 2,\ldots,k$ and $i_\lambda(m_\nu) > 1$ for some $\lambda < \ell$. Consider those values of ν where

$$D_{\ell-1}(m_\nu) \ne D_{\ell-1}(m_{\nu+1}) \ . \tag{3.1}$$

Say these are $\nu_1,\ldots,\nu_t$. Let μ_i be the least positive number such that

$$D_\ell(m_{\nu_i+\mu_i}) \ne D_\ell(m_{\nu_i+\mu_i+1})$$

for $i = 1,\ldots,t$. So the sets $\{m_{\nu_i+1},\ldots,m_{\nu_i+\mu_i}\}$ are precisely the blocks of m_ν where $i_\ell(m_\nu) = 1$ and $i_\lambda(m_\nu) > 1$ for some $\lambda < \ell$. Thus the number of such m_ν is exactly $\sum_{i=1}^{t} \mu_i$.

To estimate this sum, we first get an upper bound on t. From (3.1), there is a prime power $p^a | m_\nu$ with $p > n^{\delta_{\ell-1}}$ and $p^a \nmid m_{\nu+1}$ or there is a prime power $q^b | m_{\nu+1}$ with $q > n^{\delta_{\ell-1}}$ and $q^b \nmid m$. Thus, either

$$p\, m_{\nu+1} \,|\, [m_\nu, m_{\nu+1}] \quad \text{or} \quad q\, m_\nu \,|\, [m_\nu, m_{\nu+1}] \ .$$

Since $[m_\nu, m_{\nu+1}] \le n$, we have

$$\min\{m_\nu, m_{\nu+1}\} < n^{1-\delta_{\ell-1}} .$$

Therefore $t < 2n^{1-\delta_{\ell-1}}$.

Suppose $1 = b_1 < b_2 < \ldots$ are the integers all of whose primes exceed n^{δ_ℓ}. Then

$$\sum_{i=1}^{t} \mu_i \le \sum_{b \in \{D_\ell(m_{\nu_i+1}):\ i=1,\ldots,t\}} \psi(\tfrac{n}{b}, n^{\delta_\ell})$$

$$\le \sum_{i=1}^{t} \psi(\tfrac{n}{b_i}, n^{\delta_\ell}) \le \sum_{i<2n^{1-\delta_{\ell-1}}} \psi(\tfrac{n}{b_i}, n^{\delta_\ell})$$

$$< 3^{-k} n$$

by Proposition 2. Therefore, the number of m_ν with $i_\ell(m_\nu) = 1$ for some $\ell = 1,\ldots,k$ is less than $k \cdot 3^{-k} n \le 2^{-k} n$.

Let $\mathscr{L}(n)$ denote the set of $\nu = 1,\ldots,g(n)$ with $D_{\ell-1}(m_\nu) \le n^{1-\delta'_\ell}$ and $i_\ell(m_\nu) > 1$ for each $\ell = 1,\ldots,k$. Our goal now is to estimate $\#\mathscr{L}(n)$. Given a number D all of whose primes exceed $n^{\delta_{\ell-1}}$, let

$$U_\ell(D) = \{d_\ell(m_\nu):\ \nu \in \mathscr{L}(n),\ D_{\ell-1}(m_\nu) = D\}$$

$$u_\ell(D) = \#\, U_\ell(D) .$$

We now obtain an upper bound for $u_\ell(D)$. Say $\nu \in \mathscr{L}(n)$, $D_{\ell-1}(m_\nu) = D$ and $d_\ell(m_\nu) = d(i_1,\ldots,i_\ell)$. We may assume $D_\ell(m_{\nu-1}) \ne D_\ell(m_\nu)$. Since $i_\ell > 1$, we have $D_{\ell-1}(m_{\nu-1}) = D$ and $d_\ell(m_{\nu-1}) = d(i_1,\ldots,i_{\ell-1},i_\ell - 1)$, so that $d_\ell(m_{\nu-1}) \ne d_\ell(m_\nu)$. Since $[m_{\nu-1}, m_\nu] \le n$, we have

$$\min\{m_{\nu-1}, m_\nu\} \le n^{1-\delta_\ell} .$$

Now the number of $m \le n^{1-\delta_\ell}$ with $D_{\ell-1}(m) = D$ is $\psi(\frac{n^{1-\delta_\ell}}{D}, n^{\delta_{\ell-1}})$, which is at most $n^{1-\delta_\ell}/D$. Therefore,

$$u_\ell(D) \le 2n^{1-\delta_\ell}/D . \tag{3.2}$$

Note that if $D_{\ell-1}(m_\nu) = D$ for some $\nu \in \mathscr{L}(n)$, then $D \le n^{1-\delta'_\ell}$, so that $n/D \ge n^{\delta'_\ell}$. We clearly have

$$\#\mathscr{L}(n) \le \sum_{D \in \{D_k(m_\nu):\nu\in\mathscr{L}(n)\}} \psi(\tfrac{n}{D}, n^{\delta_k}) .$$

We now show that for each $\ell = 0,1,\ldots,k$, we have

(3.3) $$\#\mathscr{L}(n) \le 2^{\ell-k} \sum_{D \in \{D_\ell(m_\nu):\nu\in\mathscr{L}(n)\}} \psi(\tfrac{n}{D}, n^{\delta_\ell}) .$$

To show (3.3) we use induction on ℓ. We have already seen (3.3) for $\ell = k$. Suppose now $1 \le \ell \le k$ and (3.3) is true for ℓ. Then

$$\#\mathscr{L}(n) \le 2^{\ell-k} \sum_{D \in \{D_\ell(m_\nu):\nu\in\mathscr{L}(n)\}} \psi(\tfrac{n}{D}, n^{\delta_\ell})$$

(3.4) $$= 2^{\ell-k} \sum_{D \in \{D_{\ell-1}(m_\nu):\nu\in\mathscr{L}(n)\}} \ \sum_{d \in U_\ell(D)} \psi(\tfrac{n}{Dd}, n^{\delta_\ell}) .$$

Let $1 = a_1 < a_2 < \ldots$ denote the integers all of whose primes come from $(n^{\delta_\ell}, n^{\delta_{\ell-1}}]$. Thus the inner sum in (3.4) is at most

$$\sum_{i=1}^{u_\ell(D)} \psi(\tfrac{n}{Da_i}, n^{\delta_\ell}) = \sum_{i=1}^{\infty} \psi(\tfrac{n}{Da_i}, n^{\delta_\ell}) - \sum_{i>u_\ell(D)} \psi(\tfrac{n}{Da_i}, n^{\delta_\ell})$$

$$= \psi(\tfrac{n}{D}, n^{\delta_{\ell-1}}) - \sum_{i>u_\ell(D)} \psi(\tfrac{n}{Da_i}, n^{\delta_\ell})$$

$$\le \psi(\tfrac{n}{D}, n^{\delta_{\ell-1}}) - \sum_{i>2n^{1-\delta_\ell}/D} \psi(\tfrac{n}{Da_i}, n^{\delta_\ell})$$

$$= \psi(\tfrac{n}{D}, n^{\delta_{\ell-1}}) - \sum_{i>2n^{1-\delta_\ell}/D} [\tfrac{n}{Da_i}]$$

$$< \tfrac{1}{2}\, \psi(\tfrac{n}{D}, n^{\delta_{\ell-1}})$$

by (3.2) and Proposition 1. Thus from (3.4) we have

$$\# \mathscr{B}(n) \le 2^{\ell-1-k} \sum_{D \in \{D_{\ell-1}(m_\nu) : \nu \in \mathscr{L}(n)\}} \psi(\frac{n}{D}, n^{\delta_{\ell-1}}) .$$

This then establishes (3.3) for each $\ell = 0,1,\ldots,k$.

Using (3.3) with $\ell = 0$, we have

$$\# \mathscr{B}(n) \le 2^{-k} \sum_{D \in \{D_0(m_\nu) : \nu \in \mathscr{L}(n)\}} \psi(\frac{n}{D}, n^{\delta_0}) \le 2^{-k} n ,$$

since $D_0(m) = 1$ for any m and $\delta_0 = 1$. We thus have for all large n that

$$g(n) \le 2 \cdot 2^{-k} n + \# \mathscr{B}(n) \le 3 \cdot 2^{-k} n .$$

Since k is arbitrary, this shows that $g(n) = o(n)$, completing the proof of the theorem.

Remark. An examination of the proof reveals that we have actually shown a bit more. Namely, we have shown that for each $\varepsilon > 0$, there is a $\delta > 0$ and an n_o such that if $n \ge n_o$ and $m_1, m_2, \ldots, m_{h(n)}$ is the longest sequence of distinct numbers from $\{1,2,\ldots,n\}$ such that each $[m_\nu, m_{\nu+1}] \le n^{1+\delta}$, then $h(n) < \varepsilon n$.

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