AN INEQUALITY RELATED TO THE SIEVE OF ERATOSTHENES

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ABSTRACT. Let $\Phi(x,y)$ denote the number of integers $n \in [1,x]$ free of prime factors $\le y$. We show that but for a few small cases, $\Phi(x,y) < .6x/\log y$ when $y \le \sqrt{x}$.

1. INTRODUCTION

The sieve of Eratosthenes removes the multiples of the primes $p \le y$ from the set of positive integers $n \le x$. Let $\Phi(x,y)$ denote the number of integers remaining. Answering a question of Ford, the first-named author [7] recently proved the following theorem.

Theorem A. *When $2 \le y \le x$, $\Phi(x,y) < x/\log y$.*

If $y > \sqrt{x}$, then $\Phi(x,y) = \pi(x) - \pi(y) + 1$ (where $\pi(t)$ is the number of primes in $[1,t]$), and so by the prime number theorem, Theorem A is essentially best possible when $x^{1-\epsilon} < y < \epsilon x$. When $y \le \sqrt{x}$, there is a long history in estimating $\Phi(x,y)$, and in particular, we have the following theorem, essentially due to Buchstab (see [13, Theorem III.6.4]).

Theorem B. *For $\omega(u)$ the Buchstab function and $u = \log x/\log y \ge 2$ and $y \ge 2$,*

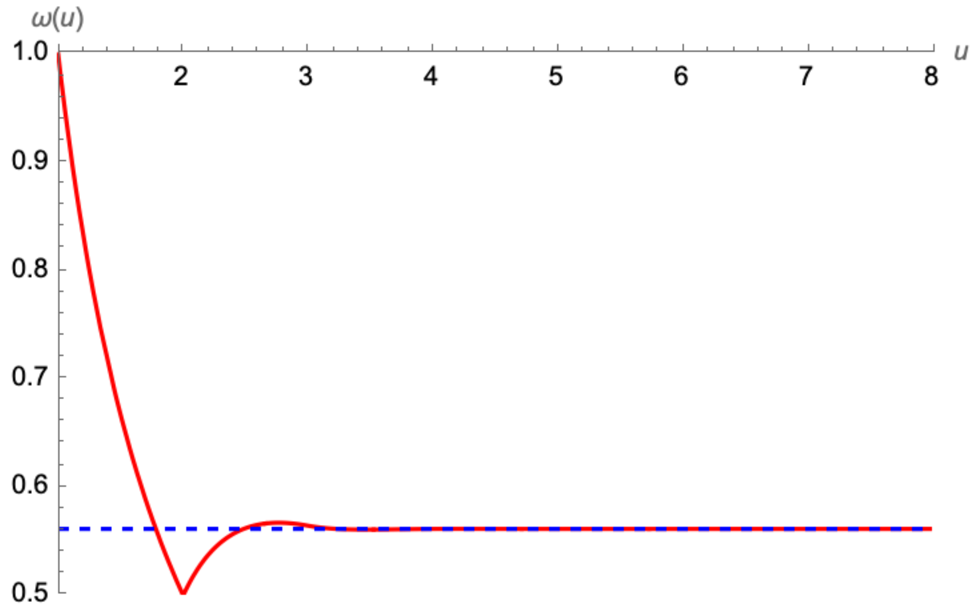
$$\Phi(x,y) = \frac{x}{\log y}\left(\omega(u) + O\left(\frac{1}{\log y}\right)\right).$$

The Buchstab function $\omega(u)$ is defined as the unique continuous function on $[1,\infty)$ such that

$$u\omega(u) = 1 \text{ on } [1,2], \quad (u\omega(u))' = \omega(u-1) \text{ on } (2,\infty).$$

Below is a graph of $\omega(u)$ for $u \in [1,8]$ generated by Mathematica.

Date: August 9, 2023.
2010 *Mathematics Subject Classification.* 11N25.
Key words and phrases. Buchstab's function, Selberg's sieve.



It is known that $\lim_{u\to\infty}\omega(u) = e^{-\gamma} = 0.561459483566885\ldots$ and that $\omega(u)$ oscillates above and below its limiting value infinitely often. The minimum value of $\omega(u)$ on $[2,\infty)$ is $1/2$ at $u = 2$ and the maximum value M_0 is $0.567143290409783\ldots$, occurring at $u = 2.763222834171 62\ldots$. In particular, it follows from Theorem B that if $c > M_0$ and $y \le \sqrt{x}$ with y sufficiently large depending on the choice of c, that $\Phi(x, y) < cx/\log y$. In addition, using an inclusion–exclusion argument plus the fact that the Mertens product $\prod_{p\le y}(1 - 1/p) < M_0/\log y$ for all $y \ge 2$, the inequality $\Phi(x, y) < cx/\log y$ can be extended to all $2 \le y \le \sqrt{x}$, but now with x sufficiently large depending on c.

In light of Theorem A and given that $\Phi(x, y)$ is a fundamental (and ancient) function, it seems interesting to try and make these consequences of Theorem B numerically explicit. We prove the following theorem.

Theorem 1. *For $3 \le y \le \sqrt{x}$, we have $\Phi(x, y) < .6x/\log y$. The same inequality holds when $2 \le y \le \sqrt{x}$ and $x \ge 10$.*

To prove this we use some numerically explicit estimates of primes due to Rosser–Schoenfeld, Büthe, and others. In addition we use a numerically explicit version of the upper bound in Selberg's sieve.

Theorem B itself is also appealing. It provides a simple asymptotic formula for $\Phi(x, y)$ as $y \to \infty$ which is applicable in a wide range.

Writing

$$\Phi(x, y) = \frac{x}{\log y}\left(\omega(u) + \frac{\Delta(x, y)}{\log y}\right),$$

one may attempt to establish numerically explicit lower and upper bounds for $\Delta(x, y)$ in the range $y \le \sqrt{x}$ for suitably large $y \ge y_0$, where $y_0 \ge 2$ is some numerically computable constant. More precisely, de Bruijn [3] essentially showed that for any given $\epsilon > 0$, one has

$$\Phi(x, y) = \mu_y(u)e^{\gamma} x \log y \prod_{p\le y}\left(1 - \frac{1}{p}\right) + O(x \exp(-(\log y)^{3/5-\epsilon}))$$

for all $x \ge y \ge 2$, where

$$\mu_y(u) := \int_0^{u-1} \omega(u - v)y^{-v}\, dv.$$

Recently, the first-named author [8] proved numerically explicit versions of this result applicable for y in wide ranges.

2. A prime lemma

Let $\pi(x)$ denote, as usual, the number of primes $p \le x$. Let

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t},$$

where the principal value is taken for the singularity at $t = 1$. There is a long history in trying to find the first point when $\pi(x) \ge \mathrm{li}(x)$, which we now know is beyond 10^{19}. We prove a lemma based on this research.

Lemma 1. *Let* $\beta_0 = 2.3 \times 10^{-8}$. *For* $x \ge 2$, *we have* $\pi(x) < (1 + \beta_0)\mathrm{li}(x)$.

Proof. The result is true for $x \le 10$, so assume $x \ge 10$. Consider the Chebyshev function

$$\theta(x) = \sum_{p\le x} \log p.$$

We use [10, Prop. 2.1], which depends strongly on extensive calculations of Büthe [4, 5] and Platt [11]. This result asserts in part that $\theta(x) \le x - .05\sqrt{x}$ for $1427 \le x \le 10^{19}$ and for larger x, $\theta(x) < (1+\beta_0)x$. One easily checks that $\theta(x) < x$ for $x < 1427$, so we have

$$\theta(x) < (1 + \beta_0)x, \quad x > 0.$$

By partial summation, we have

$$\begin{aligned}\pi(x) &= \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t(\log t)^2}\,dt \\ &< \frac{(1+\beta_0)x}{\log x} + \int_2^{10} \frac{\theta(t)}{t(\log t)^2}\,dt + (1+\beta_0)\int_{10}^x \frac{dt}{(\log t)^2}.\end{aligned}$$

Since $\int dt/(\log t)^2 = -t/\log t + \mathrm{li}(t)$, we have

$$\pi(x) < (1+\beta_0)\mathrm{li}(x) + \int_2^{10} \frac{\theta(t)}{t(\log t)^2}\,dt + (1+\beta_0)(10/\log 10 - \mathrm{li}(10))$$

$$< (1+\beta_0)\mathrm{li}(x) - .144. \tag{1}$$

This gives the lemma. □

After checking for $x \le 10$, we remark that an immediate corollary of (1) is the inequality

$$\pi(x) - k < (1+\beta_0)(\mathrm{li}(x) - k), \quad 2 \le k \le \pi(x),\ k \le 10^7. \tag{2}$$

3. Inclusion–exclusion

For small values of $y \ge 2$ we can do a complete inclusion–exclusion to compute $\Phi(x,y)$. Let $P(y)$ denote the product of the primes $p \le y$. We have

$$\Phi(x,y) = \sum_{d \mid P(y)} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor. \tag{3}$$

As a consequence, we have

$$\Phi(x,y) \le \sum_{d \mid P(y)} \mu(d)\frac{x}{d} + \sum_{\substack{d \mid P(y)\\ \mu(d)=1}} 1 = x\prod_{p \le y}\left(1 - \frac{1}{p}\right) + 2^{\pi(y)-1}. \tag{4}$$

We illustrate how this elementary inequality can be used in the case when $\pi(y) = 5$, that is, $11 \le y < 13$. Then the product in (4) is $16/77 < .207793$. The remainder term in (4) is 16. And we have

$$\Phi(x,y) < .207793x + 16 < .6x/\log 13$$

when $x \ge 613$. There remains the problem of dealing with smaller values of x, which we address momentarily. We apply this method for $y < 71$.

Pertaining to Table 1, for x beyond the "x bound" and y in the given interval, we have $\Phi(x,y) < .6x/\log y$. The column "max" in Table 1 is the supremum of $\Phi(x,y)/(x/\log y)$ for y in the given interval and $x \ge y^2$ with x below the x bound. The max statistic was computed by creating a table of the integers up to the x bound with a prime

TABLE 1. Small y.

y interval	x bound	max
$[2,3)$	22	.61035
$[3,5)$	51	.57940
$[5,7)$	96	.55598
$[7,11)$	370	.56634
$[11,13)$	613	.55424
$[13,17)$	1603	.56085
$[17,19)$	2753	.54854
$[19,23)$	6296	.55124
$[23,29)$	17539	.55806
$[29,31)$	30519	.55253
$[31,37)$	76932	.55707
$[37,41)$	1.6×10^5	.55955
$[41,43)$	2.9×10^5	.55648
$[43,47)$	5.9×10^5	.55369
$[47,53)$	1.4×10^6	.55972
$[53,59)$	3.0×10^6	.55650
$[59,61)$	5.4×10^6	.55743
$[61,67)$	1.2×10^7	.55685
$[67,71)$	2.4×10^7	.55641

factor $\leq y$, taking the complement of this set in the set of all integers up to the x bound, and then computing $(j \log p)/n$ where n is the jth member of the set and p is the upper bound of the y interval. The max of these numbers is recorded as the max statistic.

As one can see, for $y \geq 3$ the max statistic in Table 1 is below .6. However, for the interval $[2,3)$ it is above .6. One can compute that it is $< .6$ once $x \geq 10$.

This method can be extended to larger values of y, but the x bound becomes prohibitively large. With a goal of keeping the x bound smaller than 3×10^7, we can extend a version of inclusion-exclusion to $y < 241$ as follows.

First, we "pre-sieve" with the primes 2, 3, and 5. For any $x \geq 0$ the number of integers $n \leq x$ with $\gcd(n, 30) = 1$ is $(4/15)x + r$, where $|r| \leq 14/15$, as can be easily verified by looking at values of $x \in [0, 30]$. We change the definition of $P(y)$ to be the product of the primes in $(5, y]$. Then for $y \geq 5$, we have

$$\Phi(x,y) \leq \frac{4}{15} \sum_{d|P(y)} \mu(d) \frac{x}{d} + \frac{14}{15} 2^{\pi(y)-3}.$$

However, it is better to use the Bonferroni inequalities in the form

$$\Phi(x,y) \le \frac{4}{15}\sum_{j\le 4}\sum_{\substack{d\mid P(y)\\ \nu(d)=j}}(-1)^j\frac{x}{d}+\sum_{i=0}^{4}\binom{\pi(y)-3}{i} = xs(y)+b(y),$$

say, where $\nu(d)$ is the number of distinct prime factors of d. (We remark that the expression $b(y)$ could be replaced with $\frac{14}{15}b(y)$.) The inner sums in $s(y)$ can be computed easily using Newton's identities, and we see that

$$\Phi(x,y) \le .6x/\log y \text{ for } x > b(y)/(.6/\log y - s(y)).$$

We have verified that this x bound is smaller than 30,000,000 for $y < 241$ and we have verified that $\Phi(x,y) < .6x/\log y$ for x up to this bound and $y < 241$.

This completes the proof of Theorem 1 for $y < 241$.

4. When u is large: Selberg's sieve

In this section we prove Theorem 1 in the case that $u = \log x/\log y \ge 7.5$ and $y \ge 241$. Our principal tool is a numerically explicit form of Selberg's sieve.

Let $\mathcal{A}$ be a set of positive integers $a \le x$ and with $|\mathcal{A}| \approx X$. Let $\mathcal{P} = \mathcal{P}(y)$ be a set of primes $p \le y$. For each $p \in \mathcal{P}$ we have a collection of $\alpha(p)$ residue classes mod p, where $\alpha(p) < p$. Let $P = P(y)$ denote the product of the members of $\mathcal{P}$. Let g be the multiplicative function defined for numbers $d \mid P$ where $g(p) = \alpha(p)/p$ when $p \in \mathcal{P}$. We let

$$V := \prod_{p\in\mathcal{P}}(1-g(p)) = \prod_{p\in\mathcal{P}}\left(1-\frac{\alpha(p)}{p}\right).$$

We define $r_d(\mathcal{A})$ via the equation

$$\sum_{\substack{a\in\mathcal{A}\\ d\mid a}} 1 = g(d)X + r_d(\mathcal{A}).$$

The thought is that $r_d(\mathcal{A})$ should be small. We are interested in $S(\mathcal{A},\mathcal{P})$, the number of those $a \in \mathcal{A}$ such that a is coprime to P.

We will use Selberg's sieve as given in [9, Theorem 7.1]. This involves an auxiliary parameter $D < X$ which can be freely chosen. Let h be the multiplicative function supported on divisors of P such that $h(p) = g(p)/(1-g(p))$. In particular if each $\alpha(p) = 1$, then each $g(p) = 1/p$ and $h(p) = 1/(p-1)$, so $h(d) = 1/\varphi(d)$ for $d \mid P$, where

φ is Euler's function. Henceforth we will make this assumption (that each $\alpha(p) = 1$). Let

$$J = J_D = \sum_{\substack{d|P \\ d<\sqrt{D}}} h(d), \quad R = R_D = \sum_{\substack{d|P \\ d<D}} \tau_3(d)|r_d(\mathcal{A})|,$$

where $\tau_3(d)$ is the number of ordered factorizations $d = abc$, where a, b, c are positive integers. Selberg's sieve gives in this situation that

$$S(\mathcal{A}, \mathcal{P}) \leq X/J + R. \tag{5}$$

Note that if $D \geq P^2$, then

$$J = \sum_{d|P} h(d) = \prod_{p\in\mathcal{P}} (1 + h(p)) = \prod_{p\in\mathcal{P}} (1 - g(p))^{-1} = V^{-1},$$

so that $X/J = XV$. This is terrific, but if D is so large, the remainder term R in (5) is also large, making the estimate useless. So, the trick is to choose D judiciously so that R is under control with J being near to V^{-1}.

Consider the case when each $|r_d(\mathcal{A})| \leq r$, for a constant r. In this situation the following lemma is useful.

Lemma 2. *For $y \geq 241$, we have*

$$R \leq r \sum_{\substack{d<D \\ d|P(y)}} \tau_3(d) \leq rD(\log y)^2 \prod_{\substack{p\leq y \\ p\notin\mathcal{P}}} \left(1 + \frac{2}{p}\right)^{-1}.$$

Proof. Let $\tau(d) = \tau_2(d)$ denote the number of positive divisors of d. Note that

$$\sum_{d|P(y)} \frac{\tau(d)}{d} = \prod_{p\in\mathcal{P}} \left(1 + \frac{2}{p}\right) = \prod_{p\leq y} \left(1 + \frac{2}{p}\right) \prod_{\substack{p\leq y \\ p\notin\mathcal{P}}} \left(1 + \frac{2}{p}\right)^{-1}.$$

One can show that for $y \geq 241$ the first product on the right is smaller than $.95(\log y)^2$, but we will only use the "cleaner" bound $(\log y)^2$ (which holds when $y \geq 53$). Thus,

$$\sum_{\substack{d<D \\ d|P(y)}} \tau_3(d) = \sum_{\substack{d<D \\ d|P(y)}} \sum_{j|d} \tau(j) \leq \sum_{\substack{j<D \\ j|P(y)}} \tau(j) \sum_{\substack{d<D/j \\ d|P(y)}} 1$$

$$< D \sum_{\substack{j<D \\ j|P(y)}} \frac{\tau(j)}{j} < D(\log y)^2 \prod_{\substack{p\leq y \\ p\notin\mathcal{P}}} \left(1 + \frac{2}{p}\right)^{-1}.$$

This completes the proof. □

To get a lower bound for J in (5) we proceed as in [9, Section 7.4]. Recall that we are assuming each $\alpha(p)=1$ and so $h(d)=1/\varphi(d)$ for $d\mid P$.

Let

$$I=\sum_{\substack{d\ge\sqrt{D}\\ d\mid P}}\frac{1}{\varphi(d)},$$

so that $I+J=V^{-1}$. Hence

$$J=V^{-1}-I=V^{-1}(1-IV), \tag{6}$$

so we want an upper bound for IV. Let ε be arbitrary with $\varepsilon>0$. We have

$$I<D^{-\varepsilon}\sum_{d\mid P}\frac{d^{2\varepsilon}}{\varphi(d)}=D^{-\varepsilon}\prod_{p\le y}\left(1+\frac{p^{2\varepsilon}}{p-1}\right),$$

and so, assuming each $\alpha(p)=1$,

$$IV<D^{-\varepsilon}\prod_{p\in\mathcal{P}}\left(1+\frac{p^{2\varepsilon}-1}{p}\right)=:f(D,\mathcal{P},\varepsilon). \tag{7}$$

In particular, if $y\ge 241$ and each $r_d(\mathcal{A})\le r$, then

$$S(\mathcal{A},\mathcal{P})\le XV\big(1-f(D,\mathcal{P},\varepsilon)\big)^{-1}+rD(\log y)^2\prod_{\substack{p\le y\\ p\notin\mathcal{P}}}\left(1+\frac{2}{p}\right)^{-1}. \tag{8}$$

We shall choose D so that the remainder term is small in comparison to XV, and once D is chosen, we shall choose ε so as to minimize $f(D,\mathcal{P},\varepsilon)$.

4.1. The case when $y\le 500{,}000$ and $u\ge 7.5$.

We wish to apply (8) to estimate $\Phi(x,y)$ when $u\ge 7.5$, that is, when $x\ge y^{7.5}$. We have a few choices for $\mathcal{A}$ and $\mathcal{P}$. The most natural choice is that $\mathcal{A}$ is the set of all integers $\le x$, $X=x$, and $\mathcal{P}$ is the set of all primes $\le y$. In this case, each $r_d(\mathcal{A})\le 1$, so that we can take $r=1$ in (8) (since $r_d(\mathcal{A})\ge 0$ in this case). Instead we choose (as in the last section) $\mathcal{A}$ as the set of all integers $\le x$ that are coprime to 30 and we choose $\mathcal{P}$ as the set of primes p with $7\le p\le y$. Then $X=4x/15$ and one can check that each $|r_d(\mathcal{A})|\le 14/15$, so we can take $r=14/15$ in (8). Also,

$$\prod_{\substack{p\le y\\ p\notin\mathcal{P}}}\left(1+\frac{2}{p}\right)^{-1}=\frac{3}{14},$$

when $y \geq 5$. With this choice of $\mathcal{A}$ and $\mathcal{P}$, (8) becomes

$$\Phi(x,y) \leq XV\left(1 - D^{-\varepsilon}\prod_{7\leq p\leq y}\left(1+\frac{p^{2\varepsilon}-1}{p}\right)\right)^{-1} + \frac{1}{5}D(\log y)^2, \tag{9}$$

when $y \geq 241$.

Our "target" for $\Phi(x,y)$ is $.6x/\log y$. We choose D here so that our estimate for the remainder term is 1% of the target, namely $.006x/\log y$. Thus, in light of Lemma 2, we choose

$$D = .03x/(\log y)^3.$$

We have verified that for every value of $y \leq 500{,}000$ and $x \geq y^{7.5}$ that the right side of (9) is smaller than $.6x/\log y$. Note that to verify this, if p, q are consecutive primes with $241 \leq p < q$, then $S(\mathcal{A},\mathcal{P})$ is constant for $p \leq y < q$, and so it suffices to show the right side of (9) is smaller than $.6x/\log q$. Further, it suffices to take $x = p^{7.5}$, since as x increases beyond this point with $\mathcal{P}$ and ε fixed, the expression $f(D,\mathcal{P},\varepsilon)$ decreases. For smaller values of y in the range, we used Mathematica to choose the optimal choice of ε. For larger values, we let ε be a judicious constant over a long interval. As an example, we chose $\varepsilon = .085$ in the top half of the range.

4.2. When $y \geq 500{,}000$ and $u \geq 7.5$.

As in the discussion above we have a few choices to make, namely for the quantities D and ε. First, we choose $x = y^{7.5}$, since the case $x \geq y^{7.5}$ follows from the proof of the case of equality. We choose D as before, namely $.03x/(\log y)^3$. We also choose

$$\varepsilon = 1/\log y.$$

Our goal is to prove a small upper bound for $f(D,\mathcal{P},\varepsilon)$ given in (7). We have

$$f(D,\mathcal{P},\varepsilon) < D^{-\varepsilon}\exp\left(\sum_{7\leq p\leq y}\frac{p^{2\varepsilon}-1}{p}\right).$$

We treat the two sums separately. First, by Rosser–Schoenfeld [12, Theorems 9, 20], one can show that

$$-\sum_{p\leq y}\frac{1}{p} < -\log\log y - .26$$

for all $y \geq 2$, so that

$$-\sum_{7\leq p\leq y}\frac{1}{p} < -\log\log y - .26 + 31/30 \tag{10}$$

for $y \ge 7$. For the second sum we have

$$\sum_{7\le p\le y} p^{2\varepsilon-1} = 7^{2\varepsilon-1} + (\pi(y)-4)y^{2\varepsilon-1} + \int_{11}^{y}(1-2\varepsilon)(\pi(t)-4)t^{2\varepsilon-2}\,dt.$$

At this point we use (2) , so that

$$\begin{aligned}\frac{1}{1+\beta_0}\sum_{11\le p\le y} p^{2\varepsilon-1} &< (\mathrm{li}(y)-4)y^{2\varepsilon-1} + \int_{11}^{y}(1-2\varepsilon)(\mathrm{li}(t)-4)t^{2\varepsilon-2}\,dt\\ &= (\mathrm{li}(y)-4)y^{2\varepsilon-1} - (\mathrm{li}(t)-4)t^{2\varepsilon-1}\Big|_{11}^{y} + \int_{11}^{y}\frac{t^{2\varepsilon-1}}{\log t}\,dt\\ &= (\mathrm{li}(11)-4)11^{2\varepsilon-1} + \mathrm{li}(t^{2\varepsilon})\Big|_{11}^{y}\\ &= (\mathrm{li}(11)-4)11^{2\varepsilon-1} + \mathrm{li}(y^{2\varepsilon}) - \mathrm{li}(11^{2\varepsilon}),\end{aligned}$$

and so

(11)
$$\frac{1}{1+\beta_0}\sum_{7\le p\le y} p^{2\varepsilon-1} < 7^{2\varepsilon-1} + (\mathrm{li}(11)-4)11^{2\varepsilon-1} + \mathrm{li}(y^{2\varepsilon}) - \mathrm{li}(11^{2\varepsilon}).$$

There are a few things to notice, but we will not need them. For example, $\mathrm{li}(y^{2\varepsilon}) = \mathrm{li}(e^2)$ and $\mathrm{li}(11^{2\varepsilon}) \approx \log(11^{2\varepsilon}-1)+\gamma$.

Let $S(y)$ be the sum of the right side of (10) and $1+\beta_0$ times the right side of (11). Then

$$f(D,\mathcal{P},\varepsilon) < D^{-\varepsilon}e^{S(y)}.$$

The expression XV in (9) is

$$x\prod_{p\le y}\left(1-\frac{1}{p}\right).$$

We know from [10] that this product is $< e^{-\gamma}/\log y$ for $y \le 2\times 10^9$, and for larger values of y, it follows from [6, Theorem 5.9] (which proof follows from [6, Theorem 4.2] or [2, Corollary 11.2]) that it is $< (1+2.1\times 10^{-5})e^{-\gamma}/\log y$. We have

$$\begin{aligned}(12)\qquad \Phi(x,y) &\le XV\big(1-f(D,\mathcal{P},\varepsilon)\big)^{-1} + \frac{1}{5}D(\log y)^2\\ &< (1+2.1\times 10^{-5})\frac{x}{e^{\gamma}\log y}\big(1-D^{-\varepsilon}e^{S(y)}\big)^{-1} + \frac{.006x}{\log y}.\end{aligned}$$

We have verified that $(1-D^{-\varepsilon}e^{S(y)})^{-1}$ is decreasing in y, and that at $y = 500{,}000$ it is smaller than 1.057. Thus, (12) implies that

$$\Phi(x,y) < (1+2.1\times 10^{-5})\frac{1.057x}{e^{\gamma}\log y} + \frac{.006x}{\log y} < \frac{.5995x}{\log y}.$$

This concludes the case of $u \ge 7.5$.

5. Small u

In this section we prove that $\Phi(x,y) < .57163x/\log y$ when $u \in [2,3)$, that is, when $y^2 \le x < y^3$.

For small values of y, we calculate the maximum of $\Phi(x,y)/(x/\log y)$ for $y^2 \le x < y^3$ directly, as we did in Section 3 when we checked below the x bounds in Table 1 and the bound 3×10^7. We have done this for $241 \le y \le 1100$, and in this range we have

$$\Phi(x,y) < .56404\frac{x}{\log y}, \quad y^2 \le x < y^3, \quad 241 \le y \le 1100.$$

Suppose now that $y > 1100$ and $y^2 \le x < y^3$. We have

$$\Phi(x,y) = \pi(x) - \pi(y) + 1 + \sum_{y<p\le x^{1/2}} (\pi(x/p) - \pi(p) + 1). \tag{13}$$

Indeed, if n is counted by $\Phi(x,y)$, then n has at most 2 prime factors (counted with multiplicity), so $n = 1$, n is a prime in $(y,x]$ or $n = pq$, where p, q are primes with $y < p \le q \le x/p$.

Let p_j denote the jth prime. Note that

$$\sum_{p\le t} \pi(p) = \sum_{j\le\pi(t)} j = \frac{1}{2}\pi(t)^2 + \frac{1}{2}\pi(t).$$

Thus,

$$\sum_{y<p\le x^{1/2}} (\pi(p) - 1) = \frac{1}{2}\pi(x^{1/2})^2 - \frac{1}{2}\pi(x^{1/2}) - \frac{1}{2}\pi(y)^2 + \frac{1}{2}\pi(y),$$

and so

$$\Phi(x,y) = \pi(x) - M(x,y) + \sum_{y<p\le x^{1/2}} \pi(x/p), \tag{14}$$

where

$$M(x,y) = \frac{1}{2}\pi(x^{1/2})^2 - \frac{1}{2}\pi(x^{1/2}) - \frac{1}{2}\pi(y)^2 + \frac{3}{2}\pi(y) - 1.$$

We use Lemma 1 on various terms in (14). In particular, we have (assuming $y \ge 5$)

$$\Phi(x,y) < (1+\beta_0)\mathrm{li}(x) + \sum_{y<p\le x^{1/2}} (1+\beta_0)\mathrm{li}(x/p) - M(x,y). \tag{15}$$

Via partial summation, we have

$$\sum_{y<p\le x^{1/2}} \mathrm{li}(x/p) = x^{1/2}\mathrm{li}(x^{1/2}) \sum_{y<p\le x^{1/2}} \frac{1}{p} - \int_y^{x^{1/2}} \left(\mathrm{li}(x/t) - \frac{x/t}{\log(x/t)}\right) \sum_{y<p\le t} \frac{1}{p}\, dt. \tag{16}$$

For $1100 \le t \le 10^4$ we have checked numerically that

$$0 < \sum_{p\le t} \frac{1}{p} - \log\log t - B < .00624,$$

where $B = .261497\ldots$ is the Meissel–Mertens constant. Further, for $10^4 \le t \le 10^6$,

$$0 < \sum_{p\le t} \frac{1}{p} - \log\log t - B < .00161.$$

(The lower bounds here follow as well from [12, Theorem 20].) It thus follows for $1100 \le y \le 10^4$ that

$$\sum_{y<p\le x^{1/2}} \frac{1}{p} < \log \frac{\log(x^{1/2})}{\log y} + \beta_1, \quad \sum_{y<p\le t} \frac{1}{p} > \log \frac{\log t}{\log y} - \beta_1, \tag{17}$$

where $\beta_1 = .00624$. Now suppose that $y \ge 10^4$. Using [6, Eq. (5.7)] and the value 4.4916 for "η_3" from [2, Table 15], we have that

$$\left|\sum_{p\le t} \frac{1}{p} - \log\log t - B\right| < 1.9036/(\log t)^3, \ \ t \ge 10^6.$$

Thus, (17) continues to hold for $y \ge 10^4$ with .00624 improved to .00322. We thus have from (16)

$$\sum_{y<p\le x^{1/2}} \mathrm{li}(x/p) < x^{1/2}\mathrm{li}(x^{1/2}) \left(\log \frac{\log(x^{1/2})}{\log y} + \beta_1\right) - \int_y^{x^{1/2}} \left(\mathrm{li}(x/t) - \frac{x/t}{\log(x/t)}\right) \left(\log \frac{\log t}{\log y} - \beta_1\right) dt. \tag{18}$$

Let $R(t) = (1+\beta_0)\mathrm{li}(t)/(t/\log t)$, so that $R(t) \to 1+\beta_0$ as $t \to \infty$. We write the first term on the right side of (15) as

$$\frac{x}{u\log y} R(x) = \frac{R(y^u)}{u} \frac{x}{\log y},$$

and note that the first term on the right of (18) is less than

$$R(y^{u/2})\frac{2}{u}(\log(u/2)+\beta_1)\frac{x}{\log y}.$$

For the expression $\frac{1}{2}\pi(x^{1/2})^2 - \frac{1}{2}\pi(x^{1/2})$ in $M(x,y)$ we use the inequality $\pi(t) > t/\log t + t/(\log t)^2$ when $t \geq 599$, which follows from [1, Lemma 3.4] and a calculation (also see [6, Corollary 5.2]). Further, we use $\pi(y) \leq R(y)y/\log y$ for the rest of $M(x,y)$.

Using these estimates and numerical integration for the integral in (18) we find that

$$\Phi(x,y) < .57163\frac{x}{\log y}, \quad y \geq 1100, \quad y^2 \leq x < y^3.$$

6. Iteration

Suppose k is a positive integer and we have shown that

$$\Phi(x,y) \leq c_k\frac{x}{\log y} \tag{19}$$

for all $y \geq 241$ and $u = \log x/\log y \in [2,k)$. We can try to find some c_{k+1} not much larger than c_k such that

$$\Phi(x,y) \leq c_{k+1}\frac{x}{\log y}$$

for $y \geq 241$ and $u < k+1$. We start with c_3, which by the results of the previous section we can take as .57163. In this section we attempt to find c_k for $k \leq 8$ such that $c_8 < .6$. It would then follow from Section 4 that $\Phi(x,y) < .6x/\log y$ for all $u \geq 2$ and $y \geq 241$.

Suppose that (19) holds and that y is such that $x^{1/(k+1)} < y \leq x^{1/k}$. We have

$$\Phi(x,y) = \Phi(x,x^{1/k}) + \sum_{y<p\leq x^{1/k}} \Phi(x/p,p^-). \tag{20}$$

Indeed the sum counts all $n \leq x$ with least prime factor $p \in (y,x^{1/k}]$, and $\Phi(x,x^{1/k})$ counts all $n \leq x$ with least prime factor $> x^{1/k}$. As we have seen, it suffices to deal with the case when $y = q_0^-$ for some prime q_0.

Note that if (19) holds, then it also holds for $y = x^{1/k}$. Indeed, if y is a prime, then $\Phi(x,y) = \Phi(x,y+\epsilon)$ for all $0 < \epsilon < 1$, and in this case $\Phi(x,y) \leq c_k x/\log(y+\epsilon)$, by hypothesis. Letting $\epsilon \to 0$ shows we have $\Phi(x,y) \leq c_k x/\log y$ as well. If y is not prime, then for all sufficiently small $\epsilon > 0$, we again have $\Phi(x,y) = \Phi(x,y+\epsilon)$ and the same proof works.

Thus, we have (19) holding for all of the terms on the right side of (20). This implies that

$$\Phi(x,q_0^-)\le c_k x\left(\frac{1}{\log(x^{1/k})}+\sum_{q_0\le p\le x^{1/k}}\frac{1}{p\log p}\right). \tag{21}$$

We expect that the parenthetical expression here is about the same as $1/\log q_0$, so let us try to quantify this. Let

$$\epsilon_k(q_0)=\max\left\{\frac{-1}{\log q_0}+\frac{1}{\log(x^{1/k})}+\sum_{q_0\le p\le x^{1/k}}\frac{1}{p\log p}: y^k<x\le y^{k+1}\right\}.$$

Let q_1 be the largest prime $\le x^{1/k}$, so that

$$\epsilon_k(q_0)=\max\left\{\frac{-1}{\log q_0}+\frac{1}{\log q_1}+\sum_{q_0\le p\le q_1}\frac{1}{p\log p}: q_0<q_1\le q_0^{1+1/k}\right\}.$$

It follows from (21) that

$$\Phi(x,y)=\Phi(x,q_0^-)\le c_k x\left(\frac{1}{\log q_0}+\epsilon_k(q_0)\right)=\frac{c_k x}{\log y}(1+\epsilon_k(q_0)\log q_0).$$

Note that as k grows, $\epsilon_k(q_0)$ is non-increasing since the max is over a smaller set of primes q_1. Thus, we have the inequality

$$\Phi(x,q_0^-)\le c_3(1+\epsilon_3(q_0)\log q_0)^j\frac{x}{\log y},\quad x^{1/3}<q_0\le x^{1/(3+j)}. \tag{22}$$

Thus, we would like

$$c_3(1+\epsilon_3(q_0)\log q_0)^5<.6 \tag{23}$$

We have checked (23) numerically for primes $q_0<1000$ and it holds for $q_0\ge 241$.

This leaves the case of primes >1000. We have the identity

$$\sum_{q_0\le p\le q_1}\frac{1}{p\log p}$$
$$=\frac{-\theta(q_0^-)}{q_0(\log q_0)^2}+\frac{\theta(q_1)}{q_1(\log q_1)^2}+\int_{q_0}^{q_1}\theta(t)\left(\frac{1}{t^2(\log t)^2}+\frac{2}{t^2(\log t)^3}\right)dt,$$

via partial summation, where θ is again Chebyshev's function. First assume that $q_1<10^{19}$. Then, using [4], [5], we have $\theta(t)\le t$, so that

$$\sum_{q_0\le p\le q_1}\frac{1}{p\log p}<\frac{q_0-\theta(q_0^-)}{q_0(\log q_0)^2}+\frac{1}{\log q_0}-\frac{1}{\log q_1}.$$

We also have [4], [5] that $q_0 - \theta(q_0^-) < 1.95\sqrt{q_0}$, so that one can verify that

$$\epsilon_3(q_0) < \frac{1.95}{\sqrt{q_0}(\log q_0)^2}$$

and so (23) holds for $q_0 > 1000$. It remains to consider the cases when $q_1 > 10^{19}$, which implies $q_0 > 10^{14}$. Here we use $|\theta(t) - t| < 3.965t/(\log t)^2$, which is from [6, Theorem 4.2] or [2, Corollary 11.2]. This shows that (23) holds here as well, completing the proof of Theorem 1.

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