A 1935 Erdős paper on prime numbers and Euler's function

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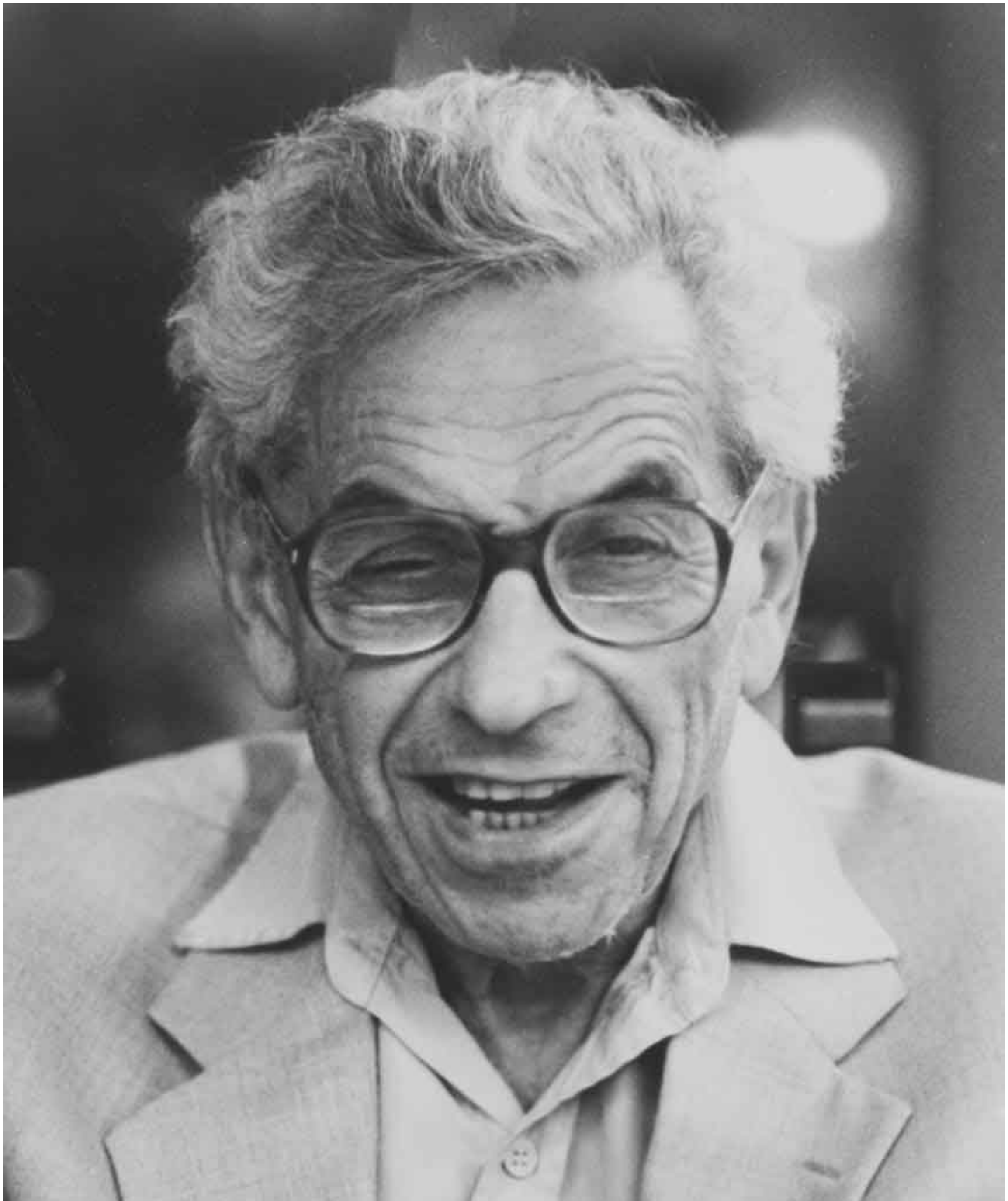

ON THE NORMAL NUMBER OF PRIME FACTORS OF $p-1$ AND SOME RELATED PROBLEMS CONCERNING EULER'S ϕ-FUNCTION

By PAUL ERDŐS (*Manchester*)

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This paper is concerned with some problems considered by Hardy and Ramanujan, Titchmarsh, and Pillai. Suppose we are given a set M of positive integers m. Let $N(n)$ denote the number of m in the interval $(0, n)$. By saying that the normal number of prime factors of a number m is $B(n)$, we mean that, as $n \rightarrow \infty$, there are only $o[N(n)]$ of the m $(\leq n)$ for which the number of prime factors does

Hardy & Ramanujan, **1917**: *The normal number of prime divisors of* n *is* $\log\log n$*.*

That is, for each fixed $\epsilon > 0$, the set of n with

$$|\omega(n) - \log\log n| > \epsilon \log\log n$$

has asymptotic density 0. Here, $\omega(n)$ is the number of prime divisors of n. The same is true for $\Omega(n) - \log\log n$, where $\Omega(n)$ is the number of prime power divisors of n.

Turán, **1934**: A beautiful “probabilistic” proof of the Hardy–Ramanujan theorem.

Erdős, **1935**: *The normal number of prime divisors of $p-1$, where p is prime, is* $\log\log p$*.*

Erdős could not adapt the slick Turán proof; rather he used the older Hardy–Ramanujan proof together with Brun’s (sieve) method.

As an application:

$$\sum_{p\le x}\tau(p-1)\ge x/(\log x)^{1-\log 2+o(1)},$$

where τ is the divisor function. Titchmarsh, in 1930, had exponent $1/2$ in the denominator.

That was a straightforward application. Next came a typically Erdős application. *What can one say about the range of Euler's function φ?* If $V(x)$ denotes the number of Euler values in $[1, x]$, then since φ is 1-to-1 on the primes, we have $V(x) \geq \pi(x) \sim x/\log x$.

Pillai, **1929**: $V(x) \ll x/(\log x)^{(\log 2)/e}$.

As the principal application of the normal order of $\omega(p-1)$:

Erdős, **1935**: $V(x) = x/(\log x)^{1+o(1)}$.

Using either the result of Pillai or Erdős one has that there are values of φ with arbitrarily many preimages. In particular, there is some $c > 0$ such that for all large x, below x there is a number with more than $(\log x)^c$ preimages.

Thus, the following seems completely unexpected!

Erdős, **1935**: *There is some $c > 0$ such that for all large x, below x there is a number with more than x^c preimages under φ.*

What have we learned since 1935?

One of the first applications of the Bombieri–Vinogradov inequality was a proof that

$$\sum_{p\le x} \tau(p-1) \sim Cx,$$

for a certain positive constant C, which thus solved the Titchmarsh divisor problem. (Solved earlier by Linnik using his "dispersion method".)

I believe we still don't know the aysmptotic order of $\sum_{p\le x} \tau_3(p-1)$, where $\tau_3(n)$ is the number of ordered factorizations of n into 3 factors.

Concerning $\omega(p-1)$, we know after Barban, Vinogradov, & Levin that we have an Erdős–Kac-type theorem. Namely the relative density of those primes p with

$$\omega(p-1) \le \log\log p + u(\log\log p)^{1/2}$$

is $G(u)$ (the Gaussian distribution).

For $V(x)$, the number of Euler values in $[1,x]$, we now know after papers of Erdős & Hall, Maier & Pomerance, and Ford, the true order of magnitude of $V(x)$. It is

$$\frac{x}{\log x}\exp\Big(c_1(\log_3 x - \log_4 x)^2 + c_2 \log_3 x + c_3 \log_4 x\Big)$$

for certain explicit constants c_1, c_2, c_3. We still do not have an asymptotic formula for $V(x)$, nor do we know that the number of Euler values in $[1,x]$ is asymptotically equal to the number of them in $[x,2x]$.

For popular values, after work of Wooldridge, Pomerance, Fouvry & Grupp, Balog, Friedlander, Baker & Harman, we now know that there are numbers below x with more than $x^{0.7067}$ Euler preimages.

This problem is connected to the distribution of Carmichael numbers in that improvements in the popular-value result are likely to lead to improvements in the lower bound in the distribution of Carmichael numbers.

Sketch of the Erdős proof on the range of φ:

- If $\varphi(n) \le x$, then $n \le X := cx \log\log x$.

- For K large enough, we may assume $\omega(n) \ge (1/K) \log\log x$.

- Primes p with $\omega(p-1) \le 40K$ are rare, so n may be assumed to be divisible by at least $(1/(2K)) \log\log x$ primes q with $\omega(q-1) > 40K$.

- Thus, but for $O(x/(\log x)^{1-\epsilon})$ values $\varphi(n) \le x$, we have $\Omega(\varphi(n)) > 20 \log\log x$. But there are very few such integers.

Let φ_k be the k-fold iterate of φ. What can one say about the range of φ_k?

The function φ_k was studied by Pillai: how many iterations to get to 1? For example, 31, 32, 33, 34, 35, 36, and 37 each take 5 iterations, but 38 takes only 4. Also studied by Shapiro and Erdős, Granville, Pomerance, & Spiro.

Using the Bateman–Horn conjecture, one can show that

$$V_k(x) \gg_k x/(\log x)^k$$

for each k, where $V_k(x)$ denotes the number of values of φ_k in $[1, x]$.

Indeed, consider primes p where $p - 1 = 2q$ with q prime, $q - 1 = 2r$, with r prime, etc.

Erdős & Hall, **1977**:

$$V_2(x) \ll \frac{x}{(\log x)^2} \exp\left(\frac{c \log_2 x \log_4 x}{\log_3 x}\right).$$

In addition they claimed they were able to prove that $V_2(x) \gg x/(\log x)^\alpha$ for some $\alpha > 2$.

Luca & Pomerance, **2009**:

$$\frac{x}{(\log x)^2} \ll V_2(x) \ll \frac{x}{(\log x)^2} \exp\left(37(\log_2 x \log_3 x)^{1/2}\right).$$

The upper bound generalizes:

$$V_k(x) \ll \frac{x}{(\log x)^k} \exp\left(13k^{3/2}(\log_2 x \log_3 x)^{1/2}\right)$$

uniformly for each positive integer k.

Sketch of proof:

The lower bound for $V_2(x)$ uses Chen's theorem and Brun's method.

For the upper bound on $V_k(x)$:

• Write $n = pm$ with p the largest prime factor of n, assume $n \le X := x(c \log\log x)^k$.

• We can assume $\Omega(\varphi_k(n)) \le 2.9k \log\log x$, so $\Omega(\varphi_k(p)) \le 2.9k \log\log x$ and $\Omega(\varphi_k(m)) \le 2.9k \log\log x$. Thus, $\Omega(\varphi(m)) \le 3k \log\log x$.

• Use the large sieve to get an upper bound for the number of such primes $p \le X/m$. Then use the old Erdős strategy to get an upper bound on $\sum 1/m$ over $m \le X$ with $\Omega(\varphi(m)) \le 3k \log\log x$.

The range of the Carmichael function λ

First discussed by Gauss, $\lambda(n)$ is the *exponent* of the multiplicative group $(\mathbb{Z}/n\mathbb{Z})^\times$. It is the smallest positive integer such that $a^{\lambda(n)} \equiv 1 \pmod{n}$ for all a coprime to n.

So, for coprime m, n, $\lambda(mn) = \mathrm{lcm}[\lambda(m), \lambda(n)]$. And $\lambda(p^j) = \varphi(p^j)$ for prime powers p^j, except when $p = 2, j \geq 3$ in which case $\lambda(2^j) = 2^{j-2}$.

Being so similar to φ, one might expect similar results about the range of λ. But one big headache appears: while $\varphi(n) \leq x$ implies $n \ll x \log\log x$ (in fact, the number of n with $\varphi(n) \leq x$ is $\sim cx$, a result of Bateman), there can be extraordinarily huge numbers n with $\lambda(n) \leq x$.

How huge? Try $\exp(x^{c/\log\log x})$.

In addition, there are $\gg x^{(\log x)^2}$ numbers n with $\lambda(n) \le x$.

So, with so many chances to hit numbers in $[1, x]$, it is not even clear that $V_\lambda(x)$, the number of λ-values in $[1, x]$, is $o(x)$.

But it is true; it follows from a lemma in

Erdős & Wagstaff, **1980**: *Let d_B be the upper density of those numbers n divisible by some $p - 1$ with $p > B$ prime. Then $\lim_{B\to\infty} d_B = 0$.*

Note that if $\lambda(n)$ is not divisible by any $p-1$ with $p>B$ prime, then n is not divisible by any prime $p>B$, so $\lambda(n)$ has no prime factors $>B$. So, there are few such integers.

This argument was first outlined by Erdős, Pomerance, & Schmutz (1991) claiming that $V_\lambda(x) \ll x/(\log x)^c$ for some $c>0$.

Friedlander & Luca, (2007):

$$V_\lambda(x) \le x/(\log x)^{1-(e/2)\log 2+o(1)},$$

where $1-(e/2)\log 2 = 0.05791\ldots$.

Luca & Pomerance, (2009?):

$$V_\lambda(x) \le x/(\log x)^{1-(1+\log\log 2)/\log 2+o(1)},$$

where $1-(1+\log\log 2)/\log 2 = 0.08607\ldots$.

Towards a lower bound:

Clearly $V_\lambda(x) \geq \pi(x)$ (since as with φ, λ is 1-to-1 on the primes), so $V_\lambda(x) \gg x/\log x$.

Can we do better?

Banks, Friedlander, Luca, Pappalardi, & Shparlinski, (2006):

$$V_\lambda(x) \gg \frac{x}{\log x} \exp\left(c(\log\log\log x)^2\right).$$

So, what is your instinct? Is 1 the "correct" exponent on $\log x$, or is it some number smaller than 1?

Luca & Pomerance, (2009?):

$$V_\lambda(x) \gg \frac{x}{(\log x)^{3/5}}.$$

In fact, we show this for numbers $\lambda(n)$, where $n = pq$ with p, q primes. This seems counter-intuitive, since the number of integers $n = pq \le x$ is $\sim x \log\log x/\log x$. But recall, it is not $n \le x$ that we need, but $\lambda(n) \le x$.

So, let $R(x)$ be the number of triples a, b, d where $abd \leq x$, $\gcd(a, b) = 1$, and $p = ad + 1, q = bd + 1$ are both prime. Then $\lambda(pq) = abd$, and except for possible overcounting, we have $V_\lambda(x) \geq R(x)$. But overcounting needs to be considered!

Assume $p \leq \exp((\log x)^c)$ with c chosen appropriately. Say $\lambda(pq) = \lambda(p'q')$. If $R_1(x)$ is the number of times this happens with $q = q'$ and $R_2(x)$ is the number of times this happens with $q \neq q'$, then by Cauchy–Schwarz,

$$V_\lambda(x) \geq R(x)^2/(R_1(x) + R_2(x)).$$

In getting upper bounds for $R_1(x), R_2(x)$ we assume further that parameters a, b, d have close to a set number of prime divisors, where the settings are chosen to optimize the final estimate.

A very new development

Results on the range of φ may be carried over with little difficulty to the range of σ, the sum-of-divisors function. Fifty years ago, Erdős proposed the problem of showing there are infinitely many integers that are simultaneously values of both functions.

It is obviously true! For example, it would follow from the conjecture that there are infinitely many Mersenne primes (primes of the form $2^p - 1$), since σ of one of these is a power of 2, and every power of 2 is in the range of φ.

It also follows if there are infinitely many twin primes, since if $p, p+2$ are both primes, then $\sigma(p) = \varphi(p+2)$.

It is easy to see that every factorial number $k!$ is a value of φ, and presumably each of these is also a value of σ (except for $k = 2$). For these reasons, Erdős wrote it was a "very annoying" problem.

Some years ago I came up with a conditional proof on the ERH (Extended Riemann Hypothesis). The idea was to take σ of the product of those primes $p \le x$ where the greatest prime factor of $p - 1$ is below $x^{1/2-\varepsilon}$, and then show using the ERH that this was indeed a φ value. To do this, one need only show that for each prime q dividing the candidate number, we also have $q - 1$ dividing the number.

I used to mention in talks that it would be much more profitable to prove that the intersection of the ranges of φ and σ is *finite*, since corollaries would be

1. There are only finitely many Mersenne primes.

2. There are only finitely many twin primes.

3. The ERH is false.

Now in a joint papeer with Ford and Luca, we have found a proof of the Erdős conjecture, so unfortunately we have not proved any of those corollaries of the negation!

There are two key thoughts that go into the proof, which is modeled after the ERH argument. First, Heath-Brown had shown in 1983 that either there are infinitely many twin primes or there are no Siegel zeros (loosely speaking). So, if there are are only finitely many twin primes, we get to rigorously assume that a weak form of the ERH holds. Since it is only a weak form, there may be some exceptional primes that need to be dealt with, and for this we use a second idea: a new paper of Ford, Konyagin, and Luca on the distribution of primes p for which a given prime divides a given iterate of φ at p.