Banff Conference in honor of Cam Stewart at 60

Fibonacci integers

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Leonardo of Pisa (Fibonacci) (c. 1170 – c. 1250)

We all know the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 510, \ldots .$$

The n-th one is given by Binet's formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta},$$

where $\alpha = (1 + \sqrt{5})/2$, $\beta = (1 - \sqrt{5})/2$ are the roots of $x^2 - x - 1$.

Thus, $F_n \sim \alpha^n/\sqrt{5}$ and the number of Fibonacci numbers in $[1, x]$ is $\log x/\log \alpha + O(1)$.

Thus, it is quite special for a natural number to be in the Fibonacci sequence, it is a rare event.

Say we try to "spread the wealth" by also including integers we can build up from the Fibonacci numbers using multiplication and division. Some examples that are not themselves Fibonacci numbers:

$$4 = 2^2,\ 6 = 2 \cdot 3,\ 7 = \frac{21}{3},\ 8 = 2^3,\ 9 = 3^2, \ldots .$$

Call such numbers Fibonacci integers. These are the numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, \ldots .$$

Perhaps we have spread the wealth too far?

Well, not every natural number is a Fibonacci integer, the first one missing is 37. To see that it must be missing, note that

$$F_{19} = 4181 = 37 \cdot 113,$$

so that the *rank of appearance* of 37 (and 113) is 19. Say

$$37 = \frac{F_{n_1} \dots F_{n_k}}{F_{m_1} \dots F_{m_l}},$$

where $n_1 \le \cdots \le n_k$ and $m_1 \le \cdots \le m_l$, $n_k \ne m_l$. Then $n_k \ge 19$. Carmichael showed that each F_n has a primitive prime factor (i.e., not dividing a smaller F_k) when $n \ne 1, 2, 6, 12$. Thus, F_{n_k} has a primitive prime factor $p \ne 37$ (if $n_k = 19$, then $p = 113$). So p must appear in the denominator, so that $m_l \ge n_k$ and indeed $m_l > n_k$. Then repeat with a primitive prime factor q of m_l, getting $n_k > m_l$.

Robert D. Carmichael (1879–1967)

Let $N(x)$ denote the number of Fibonacci integers in $[1, x]$. We have

$N(10) = 10,\ N(100) = 88,\ N(1000) = 534,\ N(10{,}000) = 2681.$

So, what do you think?

$$N(x) \approx x/(\log x)^c \ ?$$
$$N(x) \approx x/\exp((\log x)^c) \ ?$$
$$N(x) \approx x^c$$
$$N(x) \approx \exp((\log x)^c) \ ?$$
$$N(x) \approx (\log x)^c \ ?$$

Luca, Porubský (2003). *With $N(x)$ the number of Fibonacci integers in $[1, x]$, we have*

$$N(x) = O_c\left(x/(\log x)^c\right)$$

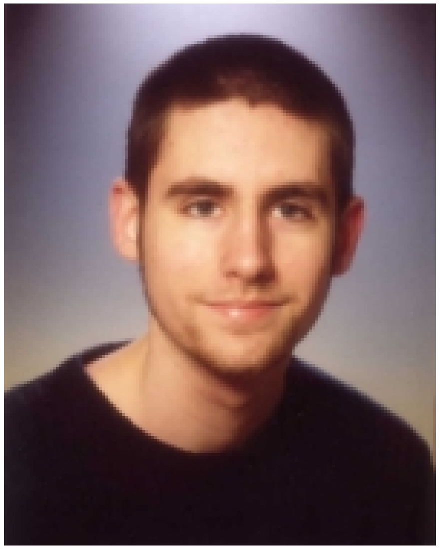
for every positive number c.

Štefan Porubský

Luca, Pomerance, Wagner (2010). *With $N(x)$ the number of Fibonacci integers in $[1, x]$, for each $\epsilon > 0$,*

$$\exp\left(C(\log x)^{1/2} - (\log x)^{\epsilon}\right) < N(x) < \exp\left(C(\log x)^{1/2} + (\log x)^{1/6+\epsilon}\right)$$

for x sufficiently large, where $C = 2\zeta(2)\sqrt{\zeta(3)/(\zeta(6)\log\alpha)}$.



Stephan Wagner

The problem of counting Fibonacci integers is made more difficult because of allowing denominators. That is, if we just looked at the *semigroup* generated by the Fibonacci numbers, rather than integers in the group that they generate, life would be simpler.

In fact, because of Carmichael's primitive prime factors, if we throw out F_n for $n = 1, 2, 6, 12$, then an integer represented as a product of Fibonacci numbers is uniquely so up to order; the semigroup is free.

And we can now begin to see the shape of the counting function (for this restricted problem). We're essentially taking partitions of integers up to $\log x / \log \alpha$, and the total number of them should be of the approximate shape $\exp((\log x)^{1/2})$.

Ban denominators? The cyclotomic factorization:

Let $\Phi_n(x)$ denote the n-th cyclotomic polynomial, so that

$$x^n - 1 = \prod_{d|n} \Phi_d(x), \quad \Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}.$$

Let $\Phi_n(x,y) = y^{\varphi(n)}\Phi(x/y)$ be the homogenization of $\Phi_n(x)$. Then for $n > 1$,

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \prod_{d|n,\ d>1} \Phi_d(\alpha,\beta), \quad \Phi_n(\alpha,\beta) = \prod_{d|n} F_d^{\mu(n/d)}.$$

Abbreviate $\Phi_n(\alpha,\beta)$ as Φ_n. For $n > 1$, Φ_n is a natural number, and in fact it is a Fibonacci integer.

Thus, the Fibonacci integers are also generated by the cyclotomic numbers $\Phi_n = \Phi_n(\alpha, \beta)$ for $n > 1$. The number $\Phi_n = \Phi_n(\alpha, \beta)$ divides F_n, and it has all of the primitive prime factors of F_n (with the same exponents as in F_n). So they too (for $n \neq 1, 2, 6, 12$) freely generate a semigroup that now contains many more Fibonacci integers than the semigroup generated by just the Fibonacci numbers themeselves.

But we do not get all of them, unfortunately.

Consider the Fibonacci integer 23. We can see that it is one, since

$$F_{24} = 2^5 \cdot 3^2 \cdot 7 \cdot 23, \text{ so that } 23 = \frac{F_{24}}{F_3^5 F_4 F_8}.$$

The primitive part of F_{24} is 23, so this will appear in Φ_{24}. However $\Phi_{24} = 46$, and we have

$$23 = \frac{\Phi_{24}}{\Phi_3}.$$

Thus, denominators are still necessary.

Let $z(p)$ be the rank of appearance of the prime p; that is, the least n with $p \mid F_n$. Then for any positive integer k,

$$\Phi_{p^k z(p)} = p \times (\text{the primitive part of } F_{p^k z(p)}).$$

And if n is not in the form $p^k z(p)$, then Φ_n is exactly the primitive part of F_n.

For example, $\Phi_{19} = F_{19} = 37 \cdot 113$, as we've seen. Thus,

$$\frac{\Phi_{37^k \cdot 19} \Phi_{113^l \cdot 19}}{\Phi_{19}}$$

is a Fibonacci integer for any choice of positive integers k, l.

It is possible to figure out the *atoms* for the Fibonacci integers, namely those Fibonacci integers exceeding 1 that cannot be factored into smaller Fibonacci integers. And with these atoms, we would not need denominators; that is, the Fibonacci integers would indeed be the semigroup generated by the atoms.

However, we do not have unique factorization into atoms. Call the above example $n(k, l)$. It is easy to see that they are atoms, but $n(1, 1)n(2, 2) = n(1, 2)n(2, 1)$.

Our strategy: ignore the difficulties and plow forward.

First, let $N_\Phi(x)$ be the number of integers in $[1,x]$ representable as a product of Φ_n's (for $n \neq 1, 2, 6, 12$). Clearly $N(x) \geq N_\Phi(x)$.

Since different words in these factors (order of factors not counting) give different Fibonacci integers and

$$\Phi_n \approx \alpha^{\varphi(n)} \ \ \text{(in fact, } \alpha^{\varphi(n)-1} \leq \Phi_n \leq \alpha^{\varphi(n)+1}\text{)},$$

we can tap into the partition philosophy mentioned earlier. Following the analytic methods originally laid out by Hardy and Ramanujan, we can show that

$$\exp\left(C(\log x)^{1/2} - (\log x)^{\epsilon}\right) \leq N_\Phi(x) \leq \exp\left(C(\log x)^{1/2} + (\log x)^{\epsilon}\right).$$

G. H. Hardy (1877–1947)

S. Ramanujan (1887–1920)

The basic plan is to consider the generating function

$$D(z) = \sum_{n \in \Phi} n^{-z} = \prod_{n \neq 1,2,6,12} (1 - \Phi_n^{-z})^{-1},$$

where Φ is the multiplicative semigroup generated by the Φ_n's. By a standard argument, the Mellin transform of $\log D(z)$ is $\Gamma(s)\zeta(s+1)C(s)$, where

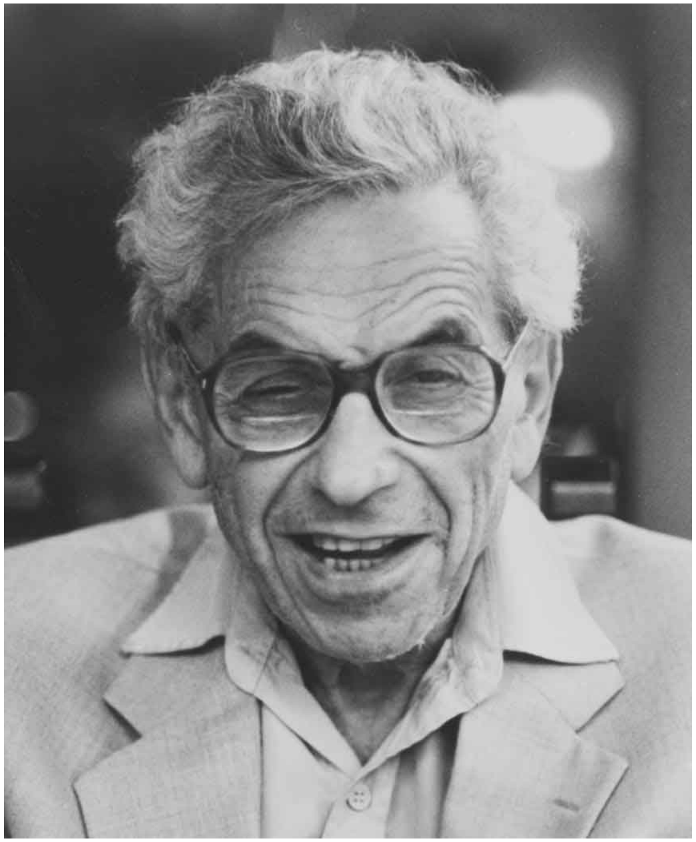
$$C(s) = \sum_{n \neq 1,2,6,12} (\log \Phi_n)^{-s}.$$

Then $C(s)$ differs from $(\log \alpha)^{-s} \sum_n \varphi(n)^{-s}$ by a function analytic in $\Re(s) > 0$ with nice growth behavior in the vertical aspect.

Then comes the saddle point method, and so on.

And with more work we believe we can attain an asymptotic formula for $N_\Phi(x)$.

Could there be an elementary approach for this part of the proof? There very well might be, since Erdős showed in 1942 that by elementary methods one can get an asymptotic formula for $p(n)$ (but not a determination of the constant outside of the exponential).



Paul Erdős (1913–1996)

Our next step in the plowing-ahead program is to estimate the number of extra Fibonacci integers that are not words in the Φ_n's. We show, via a fairly delicate combinatorial argument, that these extra Fibonacci integers introduce a factor of at most

$$\exp\left((\log x)^{1/6+\epsilon}\right).$$

Further, we can show that if Φ_n has a prime factor larger than n^K for each fixed K and all sufficiently large n, depending on K, then "1/6" may be replaced with 0.

So, what is stopping us from doing this?

Or might I ask, *who* is stopping us from doing this?

Cameron L. Stewart

Let P_n denote the largest prime factor of Φ_n.

Exhibit A:

Stewart (1977): *For numbers n with $\tau(n) \le (\log n)^{1-\epsilon}$, $P_n > C_\epsilon \varphi(n)(\log n)/\tau(n)$.*

Exhibit B:

Stewart (1977): *For most numbers n, $P_n > \epsilon(n) n (\log n)^2 / \log\log n$, where $\epsilon(n) \to 0$ is arbitrary.*

Note that these results do not even show $P_n > n^{1+\epsilon}$ for most n, let alone for all large n.

For the defense: Obviously this is a very difficult problem; it was hard making even this meager progress.

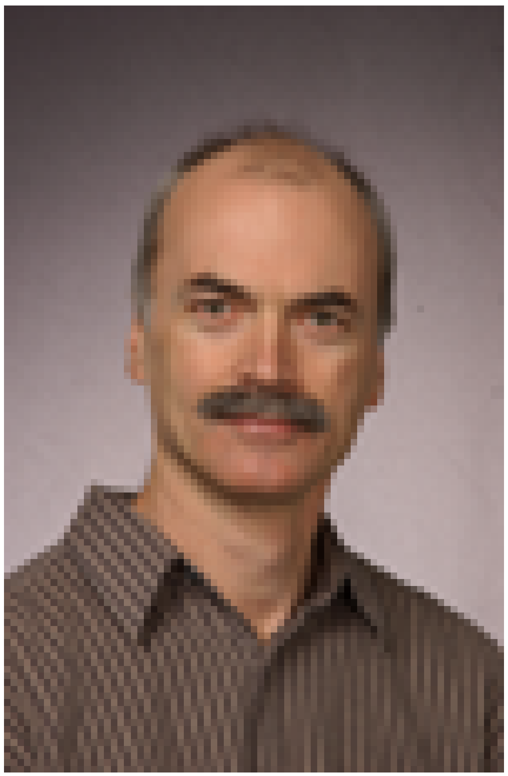
It is not known that $P_n/n \to \infty$. The best result for all large n is that the largest prime factor exceeds $2n$ for all $n > 12$, a result of Schinzel.

Assuming a strong form of the ABC conjecture, due to Stewart and Tenenbaum, we can at least get that $P_n > n^{2-\epsilon}$ for all large n. This then allows an improvement of "1/6" in the theorem to "1/8".

In our proof, we did not use very much, for example, we did not use that for most primes p, we have $z(p) > p^{1/2}$, so it is conceivable that some improvements can be made.

Finally: Our result for Fibonacci integers carries over too to "Mersenne integers" (integers in the multiplicative group generated by the Mersenne numbers $2^n - 1$) and other similar constructs created from binary recurrent sequences. Only the constant coefficient of $(\log x)^{1/2}$ in the exponent changes.

These slides and a draft of our paper can be found at
www.math.dartmouth.edu/~carlp/


Happy Birthday Cam!