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The first function and the Guy–Selfridge conjecture

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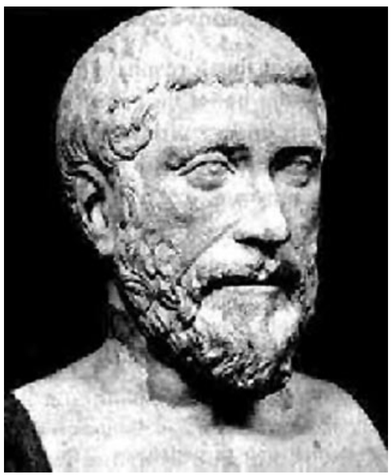


As we all know, functions in mathematics are ubiquitous and indispensable.

But what was the very first function mathematicians studied?

I would submit as a candidate, the function $s(n)$ of **Pythagoras**.

The sum-of-proper-divisors function

Let $s(n)$ be the sum of the *proper* divisors of n:

For example:

$$s(10) = 1 + 2 + 5 = 8,$$
$$s(11) = 1,$$
$$s(12) = 1 + 2 + 3 + 4 + 6 = 16.$$

In modern notation: $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum of all of n's natural divisors.

Pythagoras noticed that $s(6) = 1 + 2 + 3 = 6$.
If $s(n) = n$, we say n is *perfect*.

And he noticed that

$$s(220) = 284, \quad s(284) = 220.$$

If $s(n) = m$, $s(m) = n$, and $m \neq n$, we say n, m are an *amicable pair* and that they are *amicable* numbers.

So 220 and 284 are amicable numbers.

So, not only did Pythagoras give us the first function, he suggested iterating it, giving us the first dynamical system.

Let's take a look.

$$1 \to 0$$
$$2 \to 1 \to 0$$
$$3 \to 1 \to 0$$
$$4 \to 3 \to 1 \to 0$$
$$5 \to 1 \to 0$$
$$6 \to 6 \to 6 \ldots$$

$10 \to 8 \to 7 \to 1 \to 0$

$12 \to 16 \to 15 \to 9 \to 4 \to 3 \to 1 \to 0$

$14 \to 10 \ldots$

$18 \to 21 \to 11 \to 1 \to 0$

$20 \to 22 \to 14 \ldots$

$24 \to 36 \to 55 \to 17 \to 1 \to 0$

$25 \to 6 \to 6 \ldots$

$26 \to 16 \ldots$

$28 \to 28$

$30 \to 42 \to 54 \to 66 \to 78 \to 90 \to 144 \to 259 \to 45 \to 33 \to 15 \ldots$

$\vdots$

Some questions:

- Are there infinitely many perfect numbers? (There are 49 known, all of them in the Euclid–Euler form: $2^{p-1}(2^p - 1)$.)

- Are there infinitely many amicable pairs? (There are over 10^9 known.)

- Are there any 3-cycles?

- Can cycles be arbitrarily long? (The longest cycle known has length 28.)

- Are there infinitely many *sociable* numbers (i.e., numbers involved in a cycle)?

- Do the sociable numbers have asymptotic density 0?

- Is every orbit bounded?

- Is the orbit starting with 276 bounded?

In 1888, **Catalan**, inspired by a question raised by **Oltramare** the previous year, proposed the following "empirical theorem": *Every orbit either terminates at 0 or reaches a perfect number.*

In 1913, **Dickson** corrected this to: *Every orbit is bounded.*

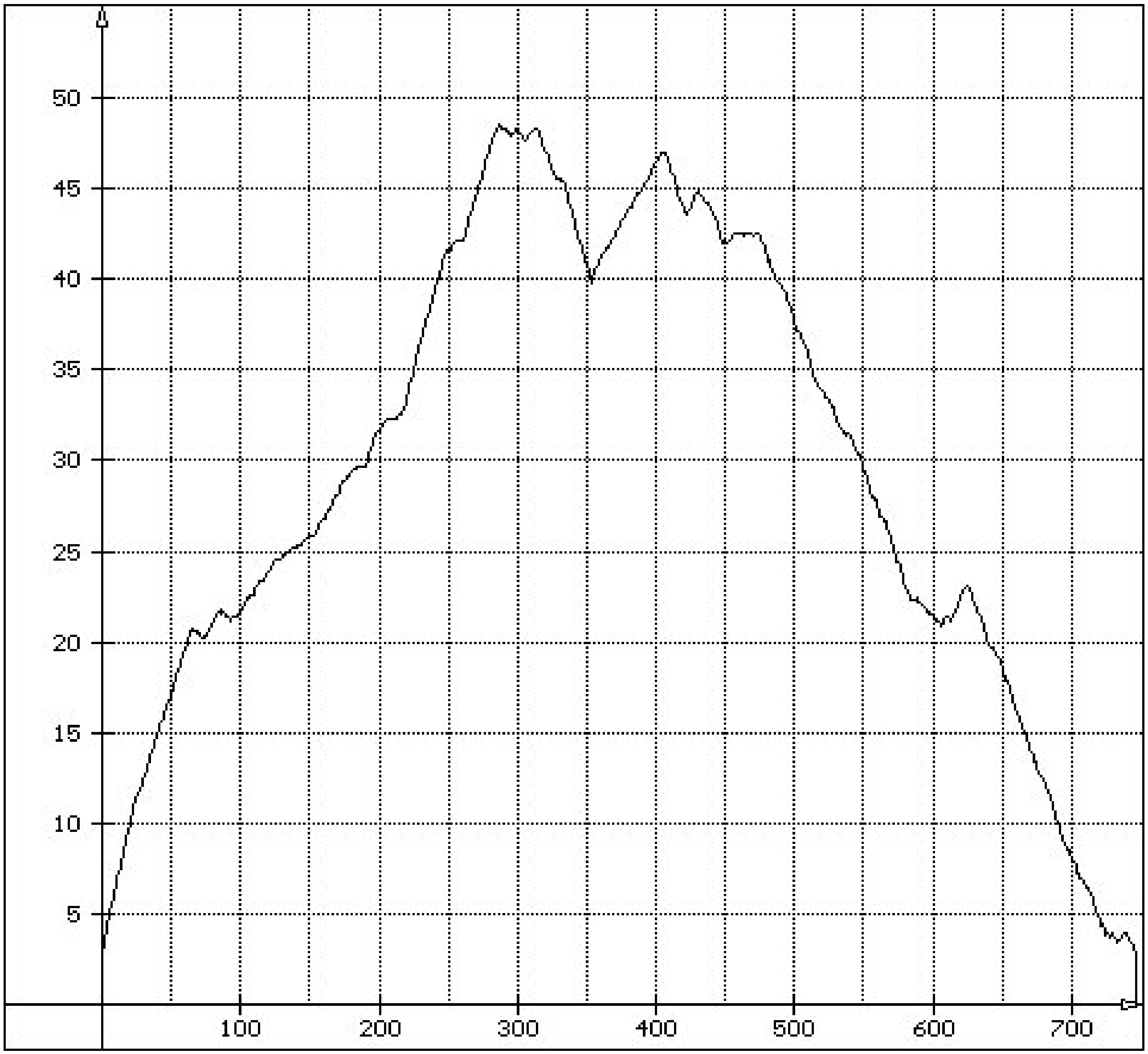
This is now known as the **Catalan–Dickson conjecture**.

But there is also the **Guy–Selfridge counter conjecture** (1975): *Discarding a set of asymptotic density* 0, *orbits starting from an odd number are bounded, while orbits starting from an even number are unbounded.*

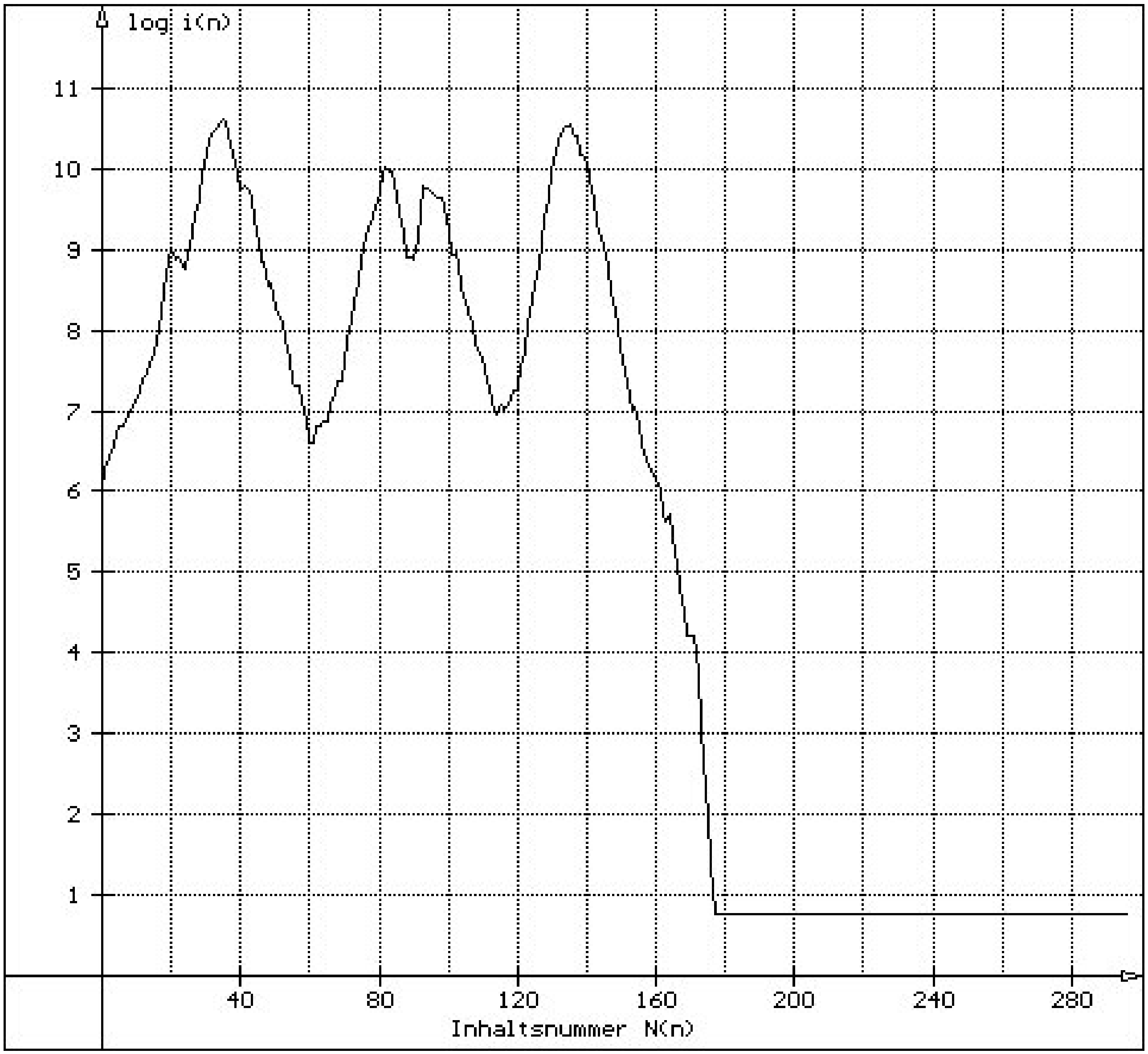
The reasoning behind Catalan–Dickson: It just takes a single prime to kill off the sequence, or a single sociable number. If a sequence should diverge to infinity, it would have to dodge these stoppers infinitely often.

The reasoning behind Guy–Selfridge: The average value for $s(n)/n$ for n even is larger than 1. (It is about 1.056.) So, for n even, there is an average tendency to grow. In addition, it should be very unusual for a sequence to switch parity when one is at high levels. This occurs only when one hits a square or the double of a square.

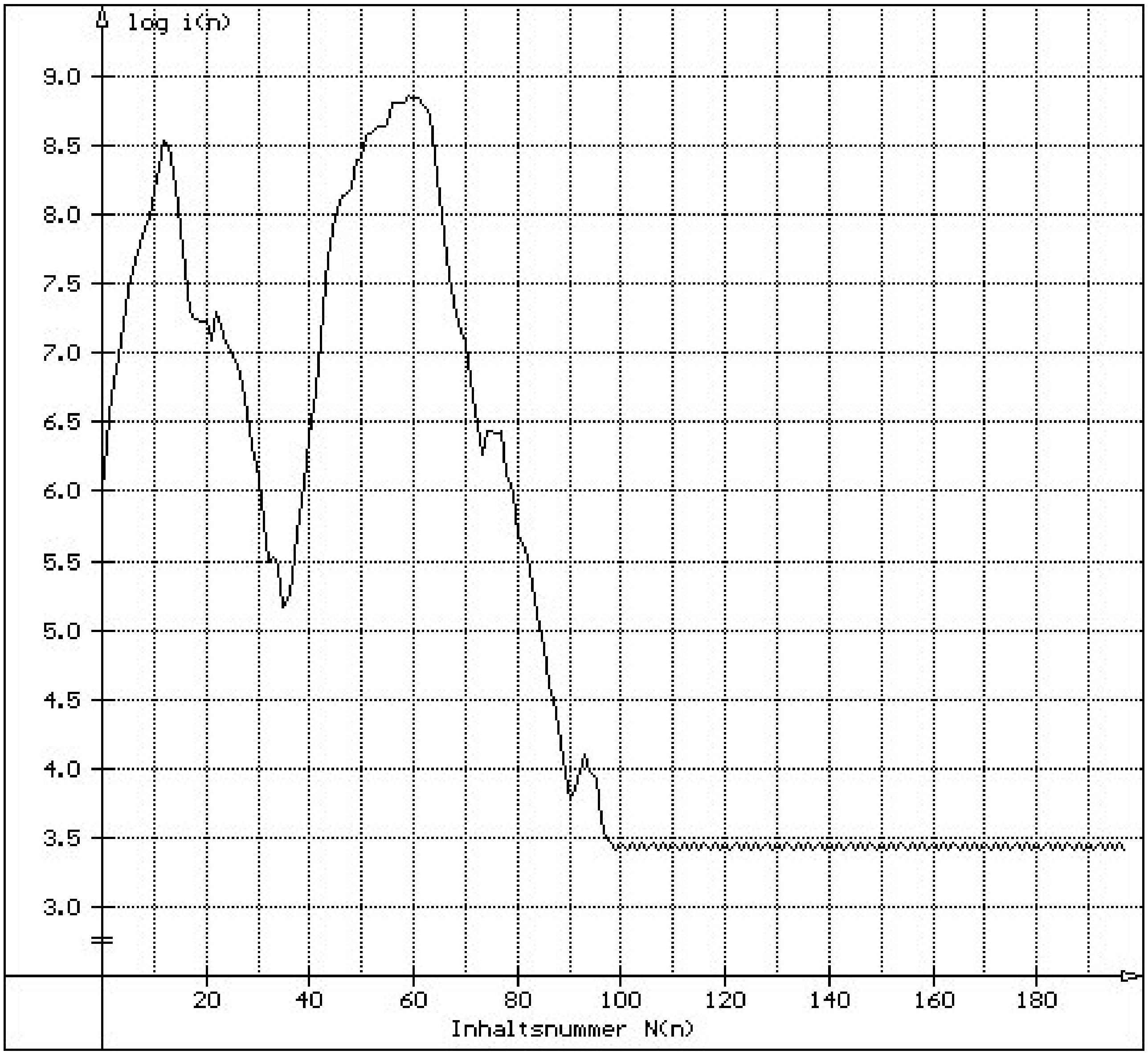
Some of the extensive calculations in computing orbits are summed up in these graphs taken from `aliquot.de` maintained by **Wolfgang Crayaufmüller**.



840


log i(n)
11
10
9
8
7
6
5
4
3
2
1
40
80
120
160
200
240
280
Inhaltsnummer N(n)

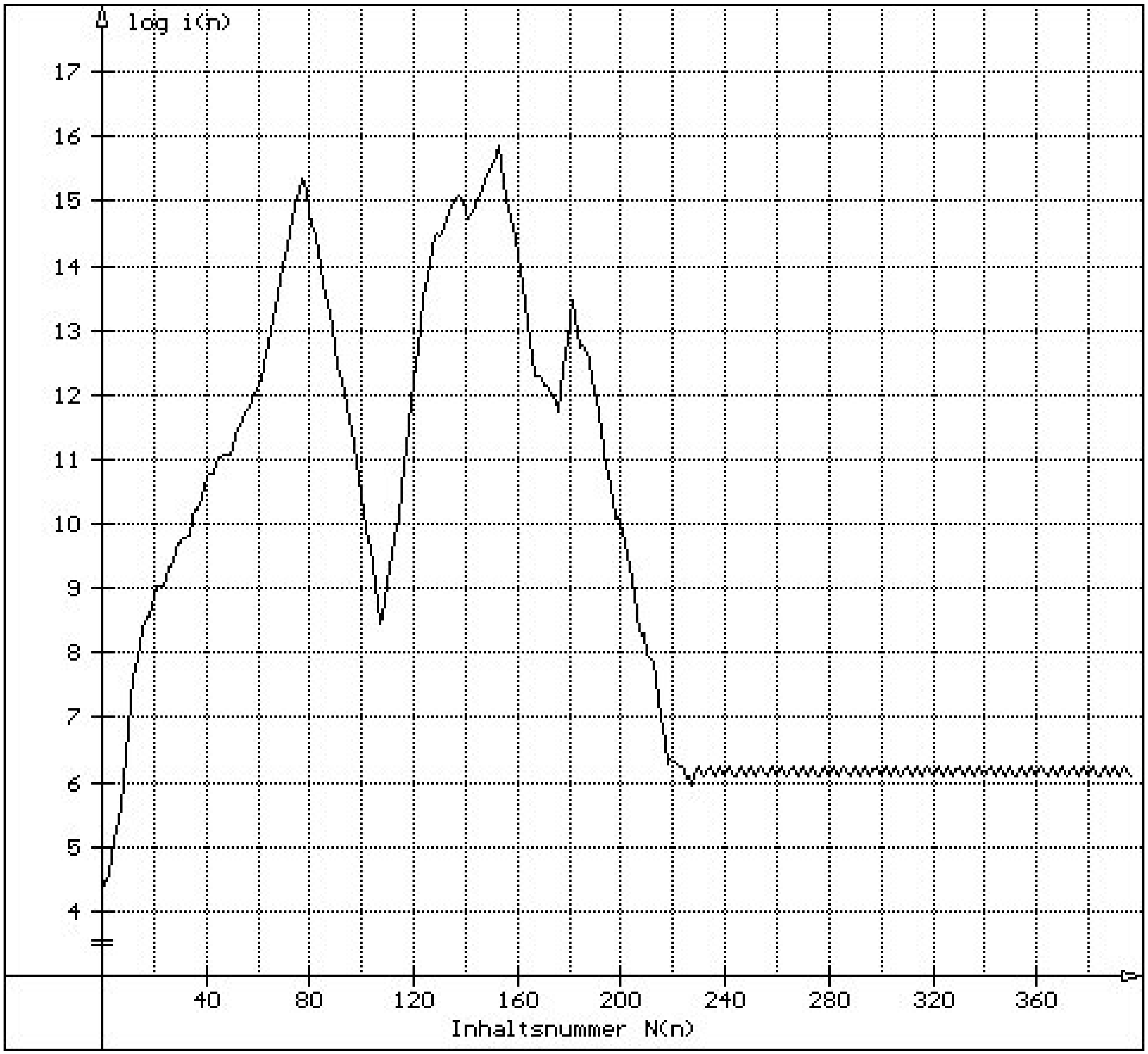
976950



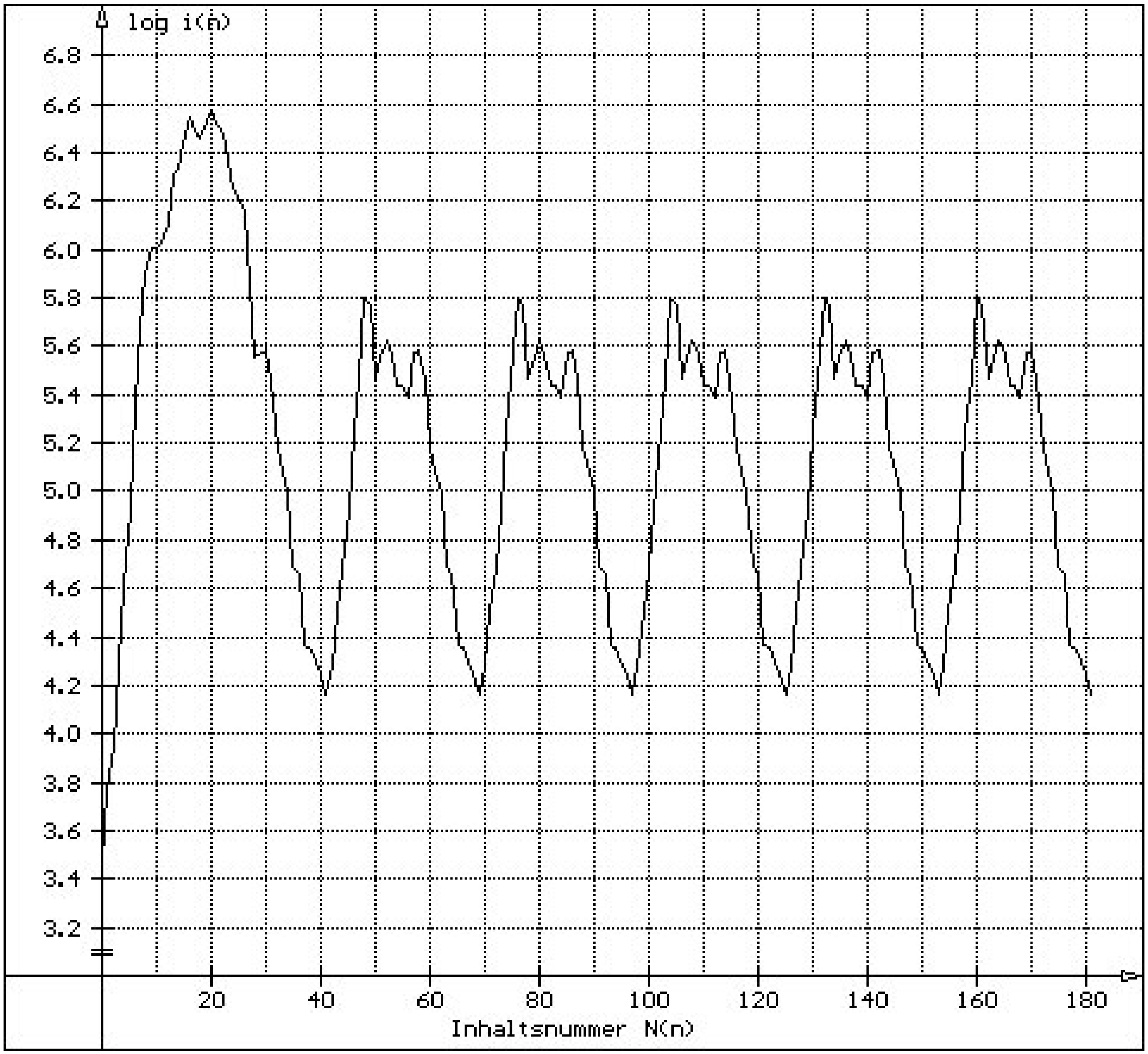
log i(n)
9.0
8.5
8.0
7.5
7.0
6.5
6.0
5.5
5.0
4.5
4.0
3.5
3.0
20
40
60
80
100
120
140
160
180
Inhaltsnummer N(n)

980460

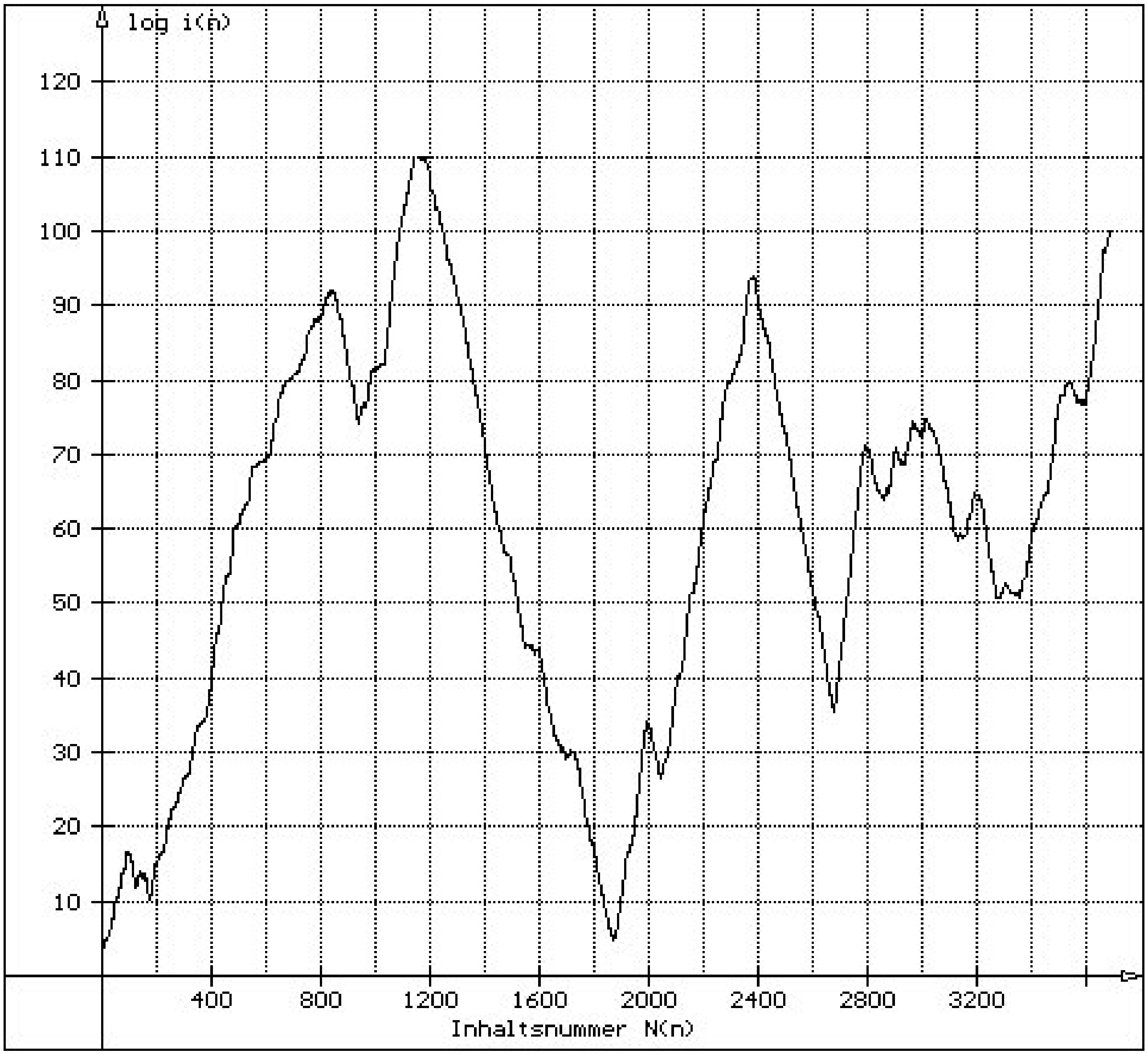
17490



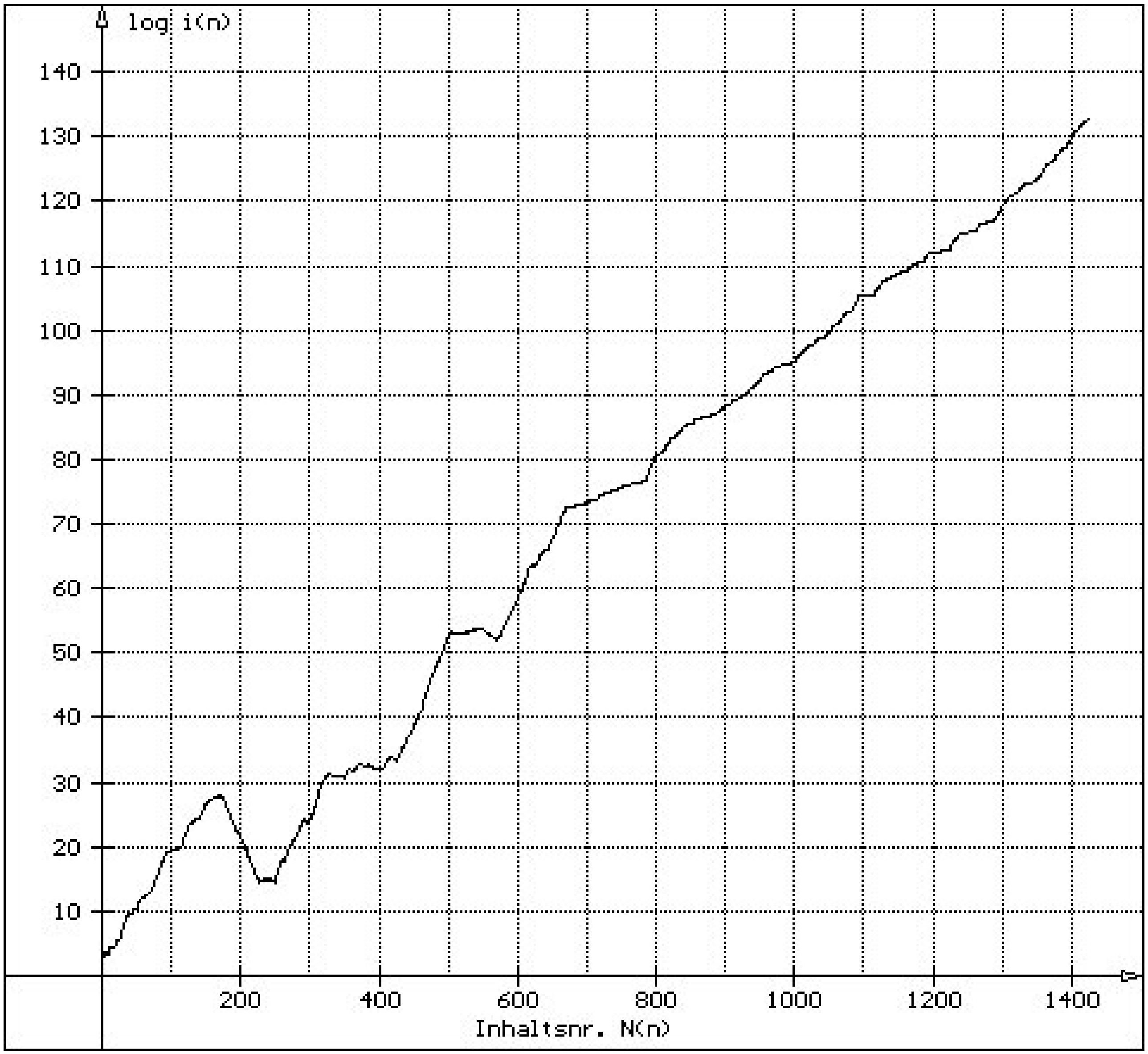
log i(n)
17
16
15
14
13
12
11
10
9
8
7
6
5
4
40
80
120
160
200
240
280
320
360
Inhaltsnummer N(n)


log i(n)
6.8
6.6
6.4
6.2
6.0
5.8
5.6
5.4
5.2
5.0
4.8
4.6
4.4
4.2
4.0
3.8
3.6
3.4
3.2
20
40
60
80
100
120
140
160
180
Inhaltsnummer N(n)

2856


log i(n)
120
110
100
90
80
70
60
50
40
30
20
10
400
800
1200
1600
2000
2400
2800
3200
Inhaltsnummer N(n)

1578



log i(n)
140
130
120
110
100
90
80
70
60
50
40
30
20
10
200
400
600
800
1000
1200
1400
Inhaltsnr. N(n)

276

Before discussing further numerical studies, it is fair to ask if there are any relevant theorems.

Lenstra (1975):
There are arbitrarily long increasing "aliquot" sequences

$$n < s(n) < s(s(n)) < \cdots < s_k(n).$$

Erdős (1976): *In fact, for each fixed k, if $n < s(n)$, then almost surely the sequence continues to increase for $k - 1$ more steps.*

Erdős claimed his proof would go through for decreasing chains (if it decreases at the first step, almost always it will continue to decrease for $k-1$ more steps), but this claim was retracted in a later paper with **Granville, P, & Spiro**. (We were able to prove it for $k=2$.)

We also showed the k-steps decreasing assertion would follow from the following conjecture: *If A is a set of integers of asymptotic density 0, then $s^{-1}(A)$ also has density 0.*

The set of perfect numbers have asymptotic density 0, a result essentially due to **Euler**. The best we know is that the number of them up to x is at most $x^{o(1)}$ as $x \to \infty$ (**Hornfeck & Wirsing** 1957).

Erdős (1955): *The amicable numbers have asymptotic density zero.*

P (2015): *The number of amicable numbers below x is at most $x/e^{\sqrt{\log x}}$, when x is large.*

Kobayashi, Pollack, & P (2009): *The even sociable numbers have asymptotic density* 0. *The odd ones have density at most about* 0.002.

When iterating s, already at the second level we are not looking at all numbers, just numbers that are in the range of s.

What can we say about the set $s(\mathbb{N})$? Again there is a bifurcation between odd and even:

Asymptotically all odd numbers are in $s(\mathbb{N})$, *while a positive proportion of even numbers are missing.*

These results are due to **Erdős** (1973).

The case of odd numbers goes as follows: If p, q are different primes, then $s(pq) = p + q + 1$. A slightly stronger form of Goldbach's conjecture asserts that every even number $n \geq 8$ is the sum of two different primes, and so a corollary would be that every odd number ≥ 9 is in $s(\mathbb{N})$. (In addition, $s(2^k) = 2^k - 1$, so 1, 3, and 7 are also in $s(\mathbb{N})$.) Goldbach's conjecture is still unproved, but we do know that those even numbers not the sum of two different primes has asymptotic density 0.

The Erdős proof that a positive proportion of evens is missing is a bit trickier. It can be shown that but for a set of n of asymptotic density 0, we have $\sigma(n)$ divisible by every prime power up to $(\log\log n)^{1-\epsilon}$. In particular, $12 \mid \sigma(n)$ almost always. So $s(n) \equiv -n \pmod{12}$ almost always. If we wish to look for s-values that are multiples of 12, almost all of them come from numbers n that are also multiples of 12. But $s(n) \geq \frac{4}{3}n$ when $12 \mid n$, so at least $\frac{1}{4}$ of the multiples of 12 are not in $s(\mathbb{N})$.

This gives a set of at least density 1/48 missing from $s(\mathbb{N})$ (**De Koninck & Luca** 2007). **Chen & Zhao** (2011) achieved at least density 0.06 missing.

Luca & P (2015): *The set* $s(\mathbb{N})$ *contains a positive proportion of even numbers. In fact, it contains a positive proportion of any fixed residue class.*

Does the set of numbers missing from $s(\mathbb{N})$ have an asymptotic density?

Pollack & P (2016): Heuristically, yes, and this density is

$$\lim_{y\to\infty} \frac{1}{\log y} \sum_{a\le y,\, 2|a} \frac{1}{a} e^{-a/s(a)} \approx 0.1718.$$

P & Yang (2014): To 10^9 the density is 0.1658.
Pollack & P (2016): To 10^{10} the density is 0.1682.
Mosunov (2016): To 10^{12} the density is 0.1712.

Recently **Bosma** did a statistical study of aliquot sequences with starting numbers below 10^6. About one-third of the even starters are still open and running beyond 10^{99}. Evidence for **Guy–Selfridge**?

But: he and **Kane** (2010) found the *geometric* mean of the ratios $s(2n)/2n$ asymptotically; it is slightly below 0.97. Evidence for **Catalan–Dickson**?

P (2016): *The asymptotic geometric mean of the ratios $s(s(2n))/s(2n)$ is the same as for $s(2n)/2n$. Assuming the conjecture of Erdős, Granville, P, & Spiro, for each fixed k, there is a set A_k of asymptotic density 1 such that the asymptotic geometric mean of $s_k(2n)/s_{k-1}(2n)$ on A_k is the same as for $s(2n)/2n$ on all n.*

One more new result: How large is the set $s^{-1}(n)$?

P (2016): *For n odd, $n > 1$, the number of $m \neq pq$ with $s(m) = n$ is $O(n^{3/4} \log n)$.*
For n even, $\# s^{-1}(n) \leq n^{2/3+o(1)}$ as $n \to \infty$.

Last week **Richard Guy** wrote me about some calculations he's been doing. He is looking at a few even numbers close to 2^{256} which might be a struggle for the Guy–Selfridge conjecture. He is choosing only those starters that do not have an "updriver" (a divisor that predicts the sequence will contnue to increase for a few or more than a few steps). He chose 27 numbers and began iterating. Of these, one of them collapsed to 1. The remaining 26 are still chugging along well beyond 100 digits.

It seems fitting, don't you think!

Happy Birthday Richard!