Letters from the master: My correspondence with Paul Erdős

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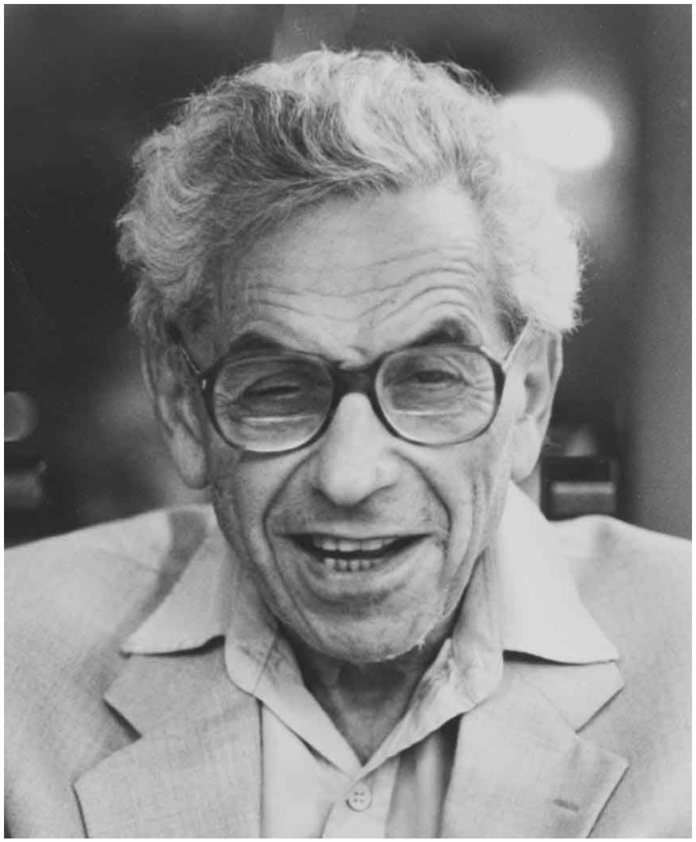
and . . .



Paul Erdős, 1913–1996

Located online at
http://www.cah.utexas.edu/collections/math_erdos.php
one can find 973 pages of my correspondence with Paul Erdős over 20+ years starting in 1974.

In this talk I'd like to describe how this correspondence began, some vignettes, and why it may be mathematically useful.

I first heard about Paul Erdős from a postdoc at Harvard who was on my dissertation committee: John Coates. My thesis was very un-Harvard-like, and he sensed that my true mathematical center was more in the problem-solving arena championed by Erdős. Coates specifically told me about the Erdős covering congruence conjecture: *For every B there is a finite set of pairs (a, n) where the integers n are distinct and larger than B, and every integer can be found in one of the residue classes $a \pmod{n}$.*

Coates told me that Erdős was offering \$100 for a proof or disproof.

(A few years ago I wrote a paper with Filaseta, Ford, Konyagin, and Yu that made some progress. Just recently Bob Hough settled the conjecture in the negative.)

I began my career at the University of Georgia in 1972. In the spring of my second year there a fortuitous event happened: On April 8, 1974, Hank Aaron of the Atlanta Braves hit his 715th career homerun, thus finally eclipsing the supposedly unbeatable total of 714 set some 40 years earlier by Babe Ruth.

I was watching the game on television, and I noticed that

$$714 \times 715 = 2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17,$$

so we have two consecutive integers whose product is also the product of the first k primes for some k. It seemed to me that this was not likely to occur ever again.

The next day I challendged by colleague David Penney to find an interesting property of 714 and 715, and he quickly saw the same thing. He asked a numerical analysis class he was teaching, and one of the students came up with another property:

$$\sum_{p|714} p = 2+3+7+17 = 29, \qquad \sum_{p|715} p = 5+11+13 = 29.$$

So, here we have two consecutive integers where the sum of the primes in one is equal to the sum of the primes in the other. We found a heuristic argument that this should occur infinitely often, but that it should be rare.

With Penney and a student, Carol Nelson, I quickly wrote a paper for the Journal of Recreational Mathematics, which was accepted by return mail and was published that same spring.

J. RECREATIONAL MATHEMATICS, Vol. 7, No. 2, Spring, 1974

714 AND 715

Carol Nelson
David E. Penney
Carl Pomerance
Department of Mathematics
University of Georgia

On April 8, 1974, in Atlanta, Georgia, Henry Aaron hit his 715th major league homerun, thus eclipsing the previous mark of 714 long held by Babe Ruth. This event received so much advance publicity that the numbers 714 and 715 were on millions of lips. Questions like "When do you think he'll get 715?" were perfectly understood, even with no mention made of Aaron, Ruth, or homerun.

In all of the hub-bub it appears certain interesting properties of 714 and 715 were overlooked. Indeed we first note that

$$714 \cdot 715 = 510510 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = P_7$$

This bit of frivolity began the correspondence!

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1974 X 31

COMPUTER SCIENCE DEPARTMENT

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Dear Professor Pomerance,

I just read your interesting paper on 714, 715. Denote by $P(m)$ the greatest prime factor of m, I expect that $P(m(m+1))/\log m \to \infty$ is true and this would imply your conjecture, but I expect that this is hopelessly difficult to prove.

If I remember right ~~2-3~~ a few years ago in Math of Computation there was a paper by Mohan Lal which considered the function $S(m) = \sum \alpha_i p_i$ where $m = \prod p_i^{\alpha_i}$. $S(m) = S(m+1)$ for inf many m seems hopeless to me but I can prove that the density of integers m for which $S(m) = S(m+1)$ is 0 the proof is not quite simple. It is easy to see that for almost all m $S(m)/P(m) \to 1$. I believe that if we neglect a sequence of density 0 then $\frac{P(m)}{P(m+1)}$ tends to 0 or ∞.

The density of integers with $P(n) > P(n+1)$ is $\frac{1}{2}$?
this last question seems hopeless. (I proved long ago [Proc Cambridge Phil Soc 1936] that the density of integers for which $d(n) > d(n+1)$ is $\frac{1}{2}$).

My adress for the rest of this year is Univ. of Cambridge Math. Dept. Mill Lane Cambridge England.

Sincerely yours

P. Erdős

I wrote back sending him some preprints of some less frivolous papers of mine, and inquired on how he would prove that the set of n with $S(n) = S(n+1)$ has density 0. He wrote again with some insightful comments on my papers and some more details on what he had in mind about the density 0 problem.

This led to his visiting me at U. Georgia and our first joint paper.

Aequationes Mathematicae **17** (1978) 311–321
University of Waterloo
Birkhäuser Verlag, Basel

On the largest prime factors of n and $n+1$

PAUL ERDÖS AND CARL POMERANCE

§1. Introduction

If $n \geq 2$ is an integer, let $P(n)$ denote the largest prime factor of n. For every $x > 0$ and every t, $0 \leq t \leq 1$, let $A(x, t)$ denote the number of $n \leq x$ with $P(n) \geq x^t$. A well-known result due to Dickman [4] and others is

THEOREM A. *The function*

$$a(t) = \lim_{x \to \infty} x^{-1} A(x, t)$$

is defined and continuous on $[0, 1]$.

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In fact it is even shown that $a(t)$ is strictly decreasing and differentiable. Note that $a(0) = 1$ and $a(1) = 0$.

If $0 \leq t$, $s \leq 1$, denote by $B(x, t, s)$ the number of $n \leq x$ with $P(n) \geq x^t$ and $P(n+1) \geq x^s$. One might guess that

This paper followed the more usual glacial path to publication, not appearing till 1978. By then we had exchanged many more letters and met up many more times.

About 6 months after his first visit I wrote him with some math and the personal tidbit that my wife had just given birth to a baby boy. This was his reply.

1975 XI 26

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Dear Pomerance,

First of all congratulation for your ε may he be a bigger prodigy than Galois and a bigger dotigy than Weizmann and never know the terrors of fascism and war.

I leave England the day after tomorrow fly to Bordeaux and after Dec 10 my adress is Technion Math Dept Haifa Israel. From Israel I come to Texas, hope to see you at the San-Antonio meeting.

I can speak about application of combinatorics + graph theory

It turns out that the word "dotigy" was coined by Ulam, one of the principal collaborators on the *Scottish Book* of problems emanating from the Scottish Cafe in Lwow, Poland in the 1930s. He barely escaped the Nazi invasion of Poland and his parents died in the Holocaust. In the US he worked on the Manhattan project and later collaborated with Teller in the creation of thermonuclear weapons. In an obituary for Ulam in 1985, Erdős wrote:

Ulam was clearly both a prodigy and a "dotigy". The word dotigy *cannot be found in any dictionary and is due to Ulam himself. I gave a talk on child prodigies and Ulam remarked that we were both "dotigies", i.e., that we should be in our dotage but can still "prove and conjecture". Perhaps it is a sad commentary on human fate that the best wish we can make for a baby is "May you be a prodigy and then later a dotigy".*


Stanislaw Ulam, 1909–1984

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1980 I 14

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Dear Carl,

You speak here at 1:30 p.m. monday Jan 21 – see you sunday between 6 and 7 at the union. I will phone in any case later this week.

Kind regards to all

E. P.

Not fitting conveniently on these slides, lets look at some letters over a one month period in October/November 1980.

In 1995 Emory University awarded Paul Erdős an honorary degree at their spring commencement. There was a reception the evening before for all of the honorees to which Erdős invited the math professors at Emory who had nominated him and also my wife and me, since we were not very far away.

You could imagine my surprise when I spotted another of the honorees, Hank Aaron! I introduced myself to him and told him that his athletic feat was responsible for the start of my correspondence with Erdős, and in fact, had profoundly changed my professional life.

He looked at me like he had just met a very strange person. Nevertheless, he was gracious. I introduced him to Erdős, and the following photo was taken.

Ron Gould, seen standing in the photo, knew that Aaron would be there that evening, but not that there was a connection to Erdős. He had brought some baseballs for Aaron to sign, and he gave me one of them. I had Erdős sign it too.

So, Aaron deos not have a joint paper with Erdős, but he does have a joint baseball.

Thank you