Joint Mathematics Meetings, 2018

Session on Computational Combinatorics and Number Theory

New results on an ancient function

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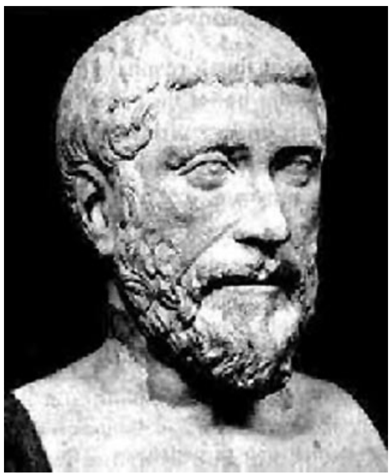

As we all know, functions in mathematics are ubiquitous and indispensable.

But what was the very first function mathematicians studied?

I would submit as a candidate, the function $s(n)$ of **Pythagoras**.

The sum-of-proper-divisors function

Let $s(n)$ be the sum of the *proper* divisors of n:

For example:

$$s(10) = 1 + 2 + 5 = 8,$$
$$s(11) = 1,$$
$$s(12) = 1 + 2 + 3 + 4 + 6 = 16.$$

In modern notation: $s(n) = \sigma(n) - n$, where $\sigma(n)$ is the sum of all of n's natural divisors.

Pythagoras noticed that $s(6) = 1 + 2 + 3 = 6$
If $s(n) = n$, we say n is *perfect*.

And amazingly, he noticed that

$$s(220) = 284, \quad s(284) = 220.$$

By iterating s, Pythagoras was looking at the first dynamical system!

If $s(n) = m$, $s(m) = n$, and $m \neq n$, we say n, m are an *amicable pair* and that they are *amicable* numbers.

So 220 and 284 are amicable numbers, forming a 2-cycle in the s-dynamical system.

Some problems

- Are there infinitely many perfect numbers?, amicable pairs? What can we say about their distribution?

- What can we say about the s-dynamical system?

- What numbers are of the form $s(n)$?

- Can a set of positive density be mapped by s to a set of density 0?

Euclid came up with a formula for perfect numbers 2300 years ago:
If $2^p - 1$ is prime, then $2^{p-1}(2^p - 1)$ is perfect.

Euler proved that all even perfect numbers are given by **Euclid**'s formula.

We suspect that $2^p - 1$ is prime infinitely often; so far we have discovered 50 of them, the largest being $2^{77,232,917} - 1$. (This last one was found a few weeks ago by a FedEx employee in Tennessee.)

What about odd perfect numbers? None are known and it's expected there are none.

Hornfeck & Wirsing (1957) The number of perfect numbers $\leq x$ is $O(x^\epsilon)$.

For amicable numbers We now know about 12,000,000 amicable pairs and suspect there are infinitely many.

Let $\mathcal{A}(x)$ denote the number of integers in $[1, x]$ that belong to an amicable pair. We have no good lower bounds for $\mathcal{A}(x)$ as $x \to \infty$, but what about an upper bound?

Erdős (1955) was the first to show $\mathcal{A}(x) = o(x)$, that is, the amicable numbers have asymptotic density 0.

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P (2014): $\mathcal{A}(x) \le x/\exp((\log x)^{1/2})$, x large.

Note that the last two results imply by a simple calculus argument that the reciprocal sum of the amicable numbers is finite.

So, what is this sum of reciprocals? Using a complete roster of all amicables to 10^{14} we can show the reciprocal sum A satisfies

$$A > 0.0119841556\ldots.$$

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Nguyen, P (2017): $A < 222$.

Another new result: **Lichtman** (2018): The reciprocal sum of those n with $s(n) \geq n$ and $s(d) < d$ for all $d \mid n, d < n$ is < 14. (**Erdős** (1934) had proved it is finite.)

Let's take a look at the s-dynamical system. A sequence under s-iteration is known as an *aliquot* sequence:

$10 \to 8 \to 7 \to 1$

$12 \to 16 \to 15 \to 9 \to 4 \to 3 \to 1$

$14 \to 10 \ldots$

$18 \to 21 \to 11 \to 1$

$20 \to 22 \to 14 \ldots$

$24 \to 36 \to 55 \to 17 \to 1$

$25 \to 6 \to 6$

$26 \to 16 \ldots$

$28 \to 28$

$30 \to 42 \to 54 \to 66 \to 78 \to 90 \to 144 \to 259 \to 45 \to 33 \to 15 \ldots$

The **Catalan–Dickson** conjecture: Every aliquot sequence is bounded.

The **Guy–Selfridge** counter conjecture: Most aliquot sequences starting from an even number are unbounded.

No unbounded aliquot sequence is known, the least starter in doubt is 276, having been pursued for over two thousand iterations. Computations have bogged down where the numbers involved have about 210 digits.

If p, q are different primes and $n = p + q + 1$, then $n = s(pq)$ is a value of s. A slightly stronger form of **Goldbach**'s conjecture implies that every even number starting with 8 is the sum of two different odd primes p, q, so this conjecture implies that starting from any odd number $n \geq 9$ there is an infinite sequence $\cdots > n_2 > n_1 > n_0 = n$, where $s(n_i) = n_{i-1}$.

In 1990, **Erdős, Granville, P, Spiro** showed that this argument works for "almost all" odd numbers n. In particular there are arbitrarily long decreasing aliquot sequences.

Lenstra (1975):
There are arbitrarily long increasing aliquot sequences

$$n < s(n) < s(s(n)) < \cdots < s_k(n).$$

Erdős (1976): *In fact, for each fixed k, if $n < s(n)$, then almost surely the sequence continues to increase for $k-1$ more steps.*

(A corollary: The amicable numbers have asymptotic density 0, since if n is the smaller member of a pair, we have $s(s(n)) = n < s(n)$.)

Recently **Bosma** did a statistical study of aliquot sequences with starting numbers below 10^6. About one-third of the even starters are still open and running beyond 10^{99}. Evidence for **Guy–Selfridge**? But: he and **Kane** (2012) found the geometric mean of the ratios $s(2n)/2n$ asymptotically, finding it slightly below 1. Evidence for **Catalan–Dickson**?

They showed that

$$\frac{2}{x}\sum_{\substack{n\le x\\ n\ \text{even}}}\log\frac{s(n)}{n}\sim\lambda<-0.03.$$

P (2017): *This holds with a power-saving error estimate and*

$$\lambda=-0.03325\,94807\,800\dots.$$

P (2016):

• *The asymptotic geometric mean of the ratios $s(s(n))/s(n)$ for n even is also* e^λ.

• *Assuming a conjecture of* **Erdős, Granville, P, & Spiro**, *for each fixed k, there is a set A_k of asymptotic density 1 such that the asymptotic geometric mean of $s_k(2n)/s_{k-1}(2n)$ on A_k is also* e^λ.

The conjecture mentioned:

If A has positive density, then $s(A)$ cannot have density 0.

Pollack, P, Thompson (2017): This conjecture holds in the case that E is very sparse, with counting function $O(x^{1/2+\epsilon})$.

Very recent: **Chum, Guy, Jacobson, Mosunov** have numerical experiments suggesting that $s_k(n)/s_{k-1}(n)$ perhaps behaves on average like $s(n)/n$, for $k \le 10$.

One can also ask about cycles in the s-dynamical system beyond the fixed points (perfect numbers) and 2-cycles (amicable pairs). There are about 12 million cycles known, with all but a few being 2-cycles, and most of the rest being 1-cycles and 4-cycles. There are no known 3-cycles, and the longest known cycle has length 28.

Say a number is *sociable* if it is in some cycle. Do the sociable numbers have density 0? The **Erdős** result on increasing aliquot sequences shows this if one restricts to cycles of bounded length. **Kobayashi, Pollack, & P** (2009) showed that apart possibly from odd sociable numbers n with $n < s(n)$, they have density 0. Further, we computed that the density of odd n with $n < s(n)$, whether or not sociable, is about 0.002.

In 1973, **Erdős** considered the range of $s(n)$: Which integers m are in the form $s(n)$? He showed that

- Almost all odd numbers are of the form $s(n)$. (As mentioned, in 1990 Erdős, et al. showed that almost all odd numbers are values of every iterate of s.)

- There is a positive proportion of even numbers not in the range.

In 2014, **Luca** and **P** showed that a positive proportion of even numbers *are* in the range, and the same goes for any residue class.

Pollack and **P** (2016) gave a heuristic argument for the density of the range of s. The heuristic is based on the theorem that for a given positive integer a, we have, apart from a set of density 0, that $a \mid n$ if and only $a \mid s(n)$. Further, the ratio $s(n)/n$ is usually closely determined by the small prime factors of n. Assuming randomness otherwise, we came up with the expression

$$\lim_{y\to\infty} \frac{1}{\log y} \sum_{\substack{a \le y \\ 2 \mid a}} \frac{1}{a\,\mathrm{e}^{a/s(a)}}$$

for the density of integers not in the range of s. This limit is not so easy to compute, but the value of the expression at $y = 10^{12}$ is about 0.1728, while the frequency of numbers not in the range to 10^{12} is about 0.1711. (**Anton Mosunov** 2017)

Thank You!