PERMUTATIONS WITH ARITHMETIC CONSTRAINTS

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ABSTRACT. Let $S_{\rm lcm}(n)$ denote the set of permutations π of $[n] = \{1, 2, \dots, n\}$ such that $\operatorname{lcm}[j, \pi(j)] \le n$ for each $j \in [n]$. Further, let $S_{\rm div}(n)$ denote the number of permutations π of $[n]$ such that $j \mid \pi(j)$ or $\pi(j) \mid j$ for each $j \in [n]$. Clearly $S_{\rm div}(n) \subset S_{\rm lcm}(n)$. We get upper and lower bounds for the counts of these sets, showing they grow geometrically. We also prove a conjecture from a recent paper on the number of "anti-coprime" permutations of $[n]$, meaning that each $\gcd(j, \pi(j)) > 1$ except when $j = 1$.

In memory of Eduard Wirsing (1931–2022)

1. Introduction

Recently, in [7] some permutation enumeration problems with an arithmetic flavor were considered. In particular, one might count permutations π of $[n] = \{1, 2, \dots, n\}$ where each $\gcd(j, \pi(j)) = 1$ and also permutations π where each $\gcd(j, \pi(j)) > 1$ except for $j = 1$. It was shown in [7] that the coprime count is between $n!/c_1^n$ and $n!/c_2^n$ for all large n, where $c_1 = 3.73$ and $c_2 = 2.5$. Shortly after, Sah and Sawhney [9] showed that there is an explicit constant $c_0 = 2.65044\dots$ with the count of the shape $n!/(c_0 + o(1))^n$ as $n \to \infty$. The "anti-coprime" count was shown in [7] to exceed $n!/(\log n)^{(\alpha+o(1))n}$ as $n \to \infty$, where $\alpha = e^{-\gamma}$, with γ Euler's constant. It was conjectured in [7] that this lower bound is sharp, which we will prove here.

There are several papers in the literature that have considered the divisibility graph on $[n]$ where $i \ne j$ are connected by an edge if i divides j or vice versa, and the closely related lcm graph, where edges correspond to $\operatorname{lcm}[i, j] \le n$. In particular, it was shown in [6] that the length of the longest simple path in such graphs is $o(n)$, and this has been improved to order-of-magnitude $n/\log n$, see Saias [10] for a recent paper on the topic. One might also consider permutations of $[n]$ compatible with these graphs. Let $S_{\rm div}(n)$ denote the set of permutations π of n such that for each $j \in [n]$, either $j \mid \pi(j)$ or $\pi(j) \mid j$.

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TABLE 1. Counts for $S_{\mathrm{div}}(n)$ and $S_{\mathrm{lcm}}(n)$ and their nth roots.

n	$\#S_{\mathrm{div}}(n)$	$(\#S_{\mathrm{div}}(n))^{1/n}$	$\#S_{\mathrm{lcm}}(n)$	$(\#S_{\mathrm{lcm}}(n))^{1/n}$
1	1	1.0000	1	1.0000
2	2	1.4142	2	1.4142
3	3	1.4422	3	1.4422
4	8	1.6818	8	1.6818
5	10	1.5849	10	1.5849
6	36	2.8272	56	1.9560
7	41	1.6998	64	1.8114
8	132	1.8411	192	1.9294
9	250	1.8469	332	1.9060
10	700	1.9254	1,184	2.0292
11	750	1.8254	1,264	1.9142
12	4,010	1.9965	12,192	2.1903
13	4,237	1.9011	12,872	2.0708
14	10,680	1.9398	37,568	2.1221
15	24,679	1.9626	100,836	2.1556
16	87,328	2.0362	311,760	2.2048
17	90,478	1.9569	322,320	2.1087
18	435,812	2.0573	2,338,368	2.2585
19	449,586	1.9839	2,408,848	2.1671
20	1,939,684	2.0625	14,433,408	2.2802
21	3,853,278	2.0588	32,058,912	2.2773
22	8,650,900	2.0669	76,931,008	2.2828
23	8,840,110	2.0046	78,528,704	2.2043
24	60,035,322	2.1091	919,469,408	2.3631
25	80,605,209	2.0714	1,158,792,224	2.3044
26	177,211,024	2.0761	2,689,828,672	2.3051
27	368,759,752	2.0757	4,675,217,824	2.2811
28	1,380,348,224	2.1205	21,679,173,184	2.3396
29	1,401,414,640	2.0673	21,984,820,864	2.2731
30	8,892,787,136	2.1460	381,078,324,992	2.4324
31	9,014,369,784	2.0947	386,159,441,600	2.3646
32	33,923,638,848	2.1334	1,202,247,415,040	2.3851
33	59,455,553,072	2.1208		
34	126,536,289,568	2.1210		
35	207,587,882,368	2.1055		

Further, let $S_{\mathrm{lcm}}(n)$ denote the set of permutations π of $[n]$ such that for each $j \in [n]$, $\mathrm{lcm}[j, \pi(j)] \le n$. Clearly, $S_{\mathrm{div}}(n) \subset S_{\mathrm{lcm}}(n)$. There is a small literature on these topics. In particular, counts for $\#S_{\mathrm{div}}(n)$ are on OEIS [5] (due to Heinz and Farrokhi), which we reproduce here, together with new counts for $\#S_{\mathrm{lcm}}(n)$.

The table suggests that $\#S_{\mathrm{lcm}}(n) > \#S_{\mathrm{div}}(n) > 2^n$ for n large, and that there may be a similar upper bound. In this note we will prove that $(\#S_{\mathrm{div}}(n))^{1/n}$ is bounded above 1 and $(\#S_{\mathrm{lcm}}(n))^{1/n}$ is bounded below infinity. We conjecture they tend to limits, but we lack the numerical evidence or heuristics to suggest values for these limits.[1] We will also show that $\#S_{\rm lcm}(n)/\#S_{\rm div}(n)$ tends to infinity geometrically.

One might also ask for the length of the longest cycle among permutations in $S_{\rm div}(n)$ or in $S_{\rm lcm}(n)$. This seems to be only slightly less (if at all) than the length of the longest simple chain in the divisor graph or lcm graph on $[n]$ mentioned above. Other papers have looked at tilings of $[n]$ with divisor chains, for example see [4]. This could correspond to asking about the cycle decomposition for permutations in $S_{\rm div}(n)$ or in $S_{\rm lcm}(n)$.

We mention the paper [2] of Erdős, Freud, and Hegyvári where some other arithmetic problems connected with integer permutations are discussed. Finally, we note the recent paper [1] which also has a similar flavor.

2. An upper bound for $\#S_{\rm lcm}(n)$

Theorem 1. *We have $\#S_{\rm lcm}(n) \le e^{2.61n}$ for all large n.*

Proof. Let n be large. For $j \in [n]$, let $N(j)$ denote the number of $j' \in [n]$ with $\operatorname{lcm}[j, j'] \le n$. This condition can be broken down as follows: $\operatorname{lcm}[j, j'] \le n$ if and only if there are integers a, b, c with

$$j = ab, \quad j' = bc, \quad \gcd(a, c) = 1, \quad abc \le n. \tag{1}$$

That is, $N(j)$ is the number of triples a, b, c with $j = ab$ satisfying (1). Since $a \mid j$, $b = j/a$, and $c \le n/j$, we have $N(j) \le \tau(j)n/j$, where τ is the divisor function (which counts the number of positive divisors of its argument). For $\pi \in S_{\rm lcm}(n)$, the number of possible values for $\pi(j)$ is at most $N(j)$, so we have

$$\#S_{\rm lcm}(n) \le \prod_{j\in[n]} N(j) \le \prod_{j\in[n]} \tau(j)n/j. \tag{2}$$

This quickly leads to an estimate for $\#S_{\rm lcm}(n)$ that is of the form $n!^{o(1)}$ as $n \to \infty$, but to do better we will need to work harder. In particular we use a seemingly trivial property of permutations: they are one-to-one. In particular, there are not many values of j with $\pi(j)$ small, since there are not many small numbers. This thought leads to versions of (2) where τ is replaced with a restricted divisor function that counts only small divisors.

Let $k = 30$. We partition the interval $(0, n]$ into subintervals as follows. Let $J_0 = (n/k, n]$. Let i_0 be the largest i such that $L :=$

[1] This conjecture was very recently proved by McNew, see arXiv:2207.09652 [math.NT].

$k^{2^i} \le \log n$, so that $(\log n)^{1/2} < L \le \log n$. For $i = 1, \dots, i_0$, let $J_i = (n/k^{2^i}, n/k^{2^{i-1}}]$, and let $J_{i_0+1} = (0, n/L]$.

For $\pi \in S_{\rm lcm}(n)$ we have sets X_i, Y_i as follows:

$$X_i := \{j \in J_i : \pi(j) > n/k^{2^i}\}, \quad 0 \le i \le i_0,$$
$$Y_i := \{j > n/k^{2^{i-1}} : \pi(j) \in J_i\}, \ 1 \le i \le i_0 + 1.$$

These sets depend on the choice of π, but the number of choices for the sets Y_i is not so large. We begin by counting the number of possibilities for the sequence of sets $Y_1, \dots, Y_{i_0+1}$.

Since π is a permutation it follows that $y_i := \#Y_i$ is at most the number of integers in J_i, so that $y_i \le n/k^{2^{i-1}}$. The number of subsets of $(n/k^{2^{i-1}}, n]$ of cardinality $\le y_i$ is less than

$$\sum_{u \le y_i} \binom{n}{u} \le 2\binom{n}{y_i} \le \frac{2n^{y_i}}{y_i!} \le \exp\left(\frac{n}{k^{2^{i-1}}}(2^{i-1}\log k + 1)\right),$$

for n sufficiently large, using the inequality $2/j! < (e/j)^j$ for $j \ge 3$. Multiplying these estimates we obtain that the number of choices for a sequence of sets $\{Y_i\}$ as described is

$$\le \exp(0.1554n) \tag{3}$$

for all sufficiently large n.

Fix now a specific sequence of sets $\{Y_i\}$, which then determines a complementary sequence of sets $\{X_i\}$ with $X_i = J_i \setminus Y_{i+1}$. We will give the set X_0 special treatment, so for now, assume that $1 \le i \le i_0$. For $j \in X_i$, the number of possible choices j' to which j may be mapped by a permutation in $S_{\rm lcm}(n)$ (with sequence of sets $\{Y_i\}$) is at most the number of choices for a, c as in (1). Here $c \le n/j$ and $a \mid j$ with $a \le n/j' < k^{2^i}$. Let $\tau_z(m)$ be the number of divisors of m that are $< z$. With this notation, the number of choices for $\pi \in S_{\rm lcm}(n)$ restricted to X_i is at most

$$\begin{aligned}\prod_{j \in X_i} \tau_{k^{2^i}}(j)\frac{n}{j} &\le \prod_{j \le n/k^{2^{i-1}}} \tau_{k^{2^i}}(j)\frac{n}{j} \\ &\le \left(\frac{1}{\lfloor n/k^{2^{i-1}} \rfloor} \sum_{j \le n/k^{2^{i-1}}} \tau_{k^{2^i}}(j)\right)^{n/k^{2^{i-1}}} \frac{n^{n/k^{2^{i-1}}}}{\lfloor n/k^{2^{i-1}} \rfloor !},\end{aligned} \tag{4}$$

by the AM-GM inequality (the arithmetic mean geometric mean inequality).

Since the harmonic sum $\sum_{d<z} 1/d$ is bounded above by $\log z + 1$, we have

$$\sum_{j\le x} \tau_z(j) = \sum_{d<z} \left\lfloor \frac{x}{d} \right\rfloor = \sum_{d<z} \left\lfloor \frac{\lfloor x\rfloor}{d} \right\rfloor < \lfloor x\rfloor(\log z + 1). \tag{5}$$

We apply this above getting that the number of assignments for the numbers $j \in X_i$ is at most

$$\begin{aligned}\big(\log(k^{2^i})+1\big)^{n/k^{2^{i-1}}} &\exp\Big(\frac{n}{k^{2^{i-1}}}(\log(k^{2^{i-1}})+1)\Big)\\ &= \exp\Big(\frac{n}{k^{2^{i-1}}}\big(\log(k^{2^{i-1}}) + \log(\log(k^{2^i})+1)+1\big)\Big).\end{aligned}$$

Thus, multiplying these estimates we have that the number of assignments for numbers j in the sets X_i, $1 \le i \le i_0$ is

$$\le \exp(0.2269n) \tag{6}$$

for n sufficiently large.

We next deal with the elements of the sets Y_i. Again referring to (1), for each $j \in Y_i$ we are to count pairs a, c with $a \mid j$ and $a \le n/j' < k^{2^i}$ and $c \le n/j < k^{2^{i-1}}$. Assuming Y_i is not empty, the number of assignments for elements of Y_i is at most

$$\begin{aligned}\prod_{j\in Y_i}\big(\tau_{k^{2^i}}(j)k^{2^{i-1}}\big) &\le \Bigg(\frac{1}{y_i}\sum_{j\in Y_i}\tau_{k^{2^i}}(j)k^{2^{i-1}}\Bigg)^{y_i}\\ &\le \Bigg(\frac{1}{y_i}\sum_{j\le n}\tau_{k^{2^i}}(j)k^{2^{i-1}}\Bigg)^{y_i}\\ &\le \Bigg(\frac{n}{y_i}(\log(k^{2^i})+1)k^{2^{i-1}}\Bigg)^{y_i},\end{aligned}$$

using (5). We have $y_i < n/k^{2^{i-1}}$ and in this range, the above estimate is increasing as y_i varies. So, the count is at most

$$\exp\Big(\frac{n}{k^{2^{i-1}}}\big(\log(k^{2^i}) + \log(\log(k^{2^i})+1)\big)\Big).$$

Multiplying these estimates we have that the number of assignments for numbers j in the sets Y_i is

$$\le \exp(0.3134n). \tag{7}$$

For X_0 we directly look at all pairs j, j' with $j, j' \in (n/k, n]$ with $\mathrm{lcm}[j, j'] \le n$. Take for example, the case $k = 3$. Then possibilities for (a, c) in (1) are $(1, 1)$, $(1, 2)$, and $(2, 1)$. For each j we can take $j' = j$, this corresponds to $(1, 1)$. For $j \in (n/3, n/2]$, we can also take $j' = 2j$, corresponding to $(1, 2)$. And for $j \in (2n/3, n]$ with j even, we can take

$j' = \frac{1}{2}j$, corresponding to $(2, 1)$. Letting $N_k(j)$ be the number of j' that can correspond to j, we thus have $N_k(j) = 2$ for $j \in (n/3, n/2]$ and for even j in $(2n/3, n]$, so that

$$\prod_{j \in (n/3, n]} N_3(j) \asymp 2^{n/3}.$$

(The symbol $\asymp$ indicates the two sides are of the same magnitude up to a bounded factor.) However, we are taking $k = 30$, and this simple argument becomes more complicated, but nevertheless can be estimated. We have the number of assignments for numbers j in X_0 is

$$\le \exp(1.9115n). \tag{8}$$

This estimate is arrived at as follows. We are interested in $\prod_{j \in J_0} N_k(j)$, which is equal to

$$\prod_{\substack{j \in J_0 \\ N_k(j) > 1}} N_k(j).$$

We have $N_k(j) = 1$ if and only if $j \in (n/2, n]$ and j is not divisible by any prime $< k$. Thus, up to an error of $O(1)$, the number of factors in the above product is νn, where

$$\nu = 1 - 1/k - (1/2) \prod_{p < k} (1 - 1/p).$$

By the AM-GM inequality,

$$\prod_{\substack{j \in J_0 \\ N_k(j) > 1}} N_k(j) \le \left(\frac{1}{\nu n} \sum_{\substack{j \in J_0 \\ N_k(j) > 1}} N_k(j) \right)^{\nu n + O(1)}.$$

To compute the sum we refer to (1). By reversing the order of summation, the sum is

$$n \sum_{\substack{a, c < k \\ \gcd(a,c) = 1}} \left(\frac{1}{ac} - \max\left\{ \frac{1}{ka}, \frac{1}{kc} \right\} \right) - \frac{n}{2} \prod_{p < k} \left(1 - \frac{1}{p} \right) + O(1). \tag{9}$$

Computing this when $k = 30$ we arrive at the estimate (8).

This leaves the contribution of numbers $j \in J_{i_0+1}$. Note that J_{i_0+1} is a subset of $[A]$, where $A = \lceil n/(\log n)^{1/2} \rceil$. As with (4), (5), this is at most

$$\begin{aligned}\prod_{j\le A}(\tau(j)n/j) &\le \left(\frac{1}{A}\sum_{j\le A}\tau(j)\right)^A \frac{n^A}{A!}\\ &\le (\log n)^A \exp\Big(A\log n - A\log A + A\Big)\\ &\le \exp\Big(\frac{3}{2}A\log\log n + A\Big).\end{aligned}$$

This last estimate is of the form $e^{o(n)}$, so it suffices to multiply the estimates in (3), (6), (7), and (8), getting that for all sufficiently large n, we have $\#S_{\rm lcm}(n) \le \exp(2.6071n)$. This completes the proof. □

Remark 1. This argument gives up a fair amount in computing the contribution for $j \in X_0$, which is the estimate (8) with $k = 30$. Another way of estimating this count is to take some large numbers n and directly compute the product of the numbers $N_k(j)$. It is seen that the nth root of this product hardly varies as n varies, and thus one can empirically arrive at a constant that is presumably more accurate than the one in (8). With $k = 30$, one gets in this way the number 1.5466, which leads to the estimate $\#S_{\rm lcm}(n) \le \exp(2.2423n)$. In fact, if one is prepared to reason in this way, then one can do a little better by taking $k = 100$. This improves the numbers in (3), (6), and (7) to 0.0571, 0.0807, and 0.1175, with the number in (8) moving to 1.8709, which would give the estimate $\#S_{\rm lcm}(n) \le \exp(2.1262n)$.

Remark 2. Since $S_{\rm div}(n) \subset S_{\rm lcm}(n)$, the upper bound counts of this section hold as well for $\#S_{\rm div}(n)$. However, we can do a little better than this. The savings comes from X_0, namely the estimate in (8). For $j \in J_0$, let $N'_k(j)$ denote the number of $j' \in J_0$ with either $j' \mid j$ or $j \mid j'$. We are interested in

$$\sum_{\substack{n/k<j\le n\\ N'_k(j)>1}} N'_k(j). \tag{10}$$

The number of $j' \mid j$ with $j' \in J_0$ is the number of $a \mid j$ with $j/a > n/k$, and the number of $j' \in J_0$ with $j \mid j'$ is the number of $c \le n/j$. We should subtract 1 since otherwise $j' = j$ would be counted twice. We have up to an error of $+O(1)$,

$$\sum_{n/k<j\le n}\ \sum_{\substack{a|j\\ a<kj/n}} 1 = \sum_{a<k}\Big(\frac{n}{a} - \frac{n}{k}\Big),$$

and

$$\sum_{n/k<j\le n}\sum_{c\le n/j}1=\sum_{c<k}\Big(\frac{n}{c}-\frac{n}{k}\Big).$$

Thus, the sum in (10), up to $+O(1)$, is

$$2\sum_{a<n}\frac{n}{a}-3n\Big(1-\frac{1}{k}\Big)-\frac{n}{2}\prod_{p<k}\Big(1-\frac{1}{p}\Big).$$

Using this when $k=100$ in place of (9) and the $k=100$ estimates for (3), (6), and (7) as mentioned in Remark 1, we obtain $\#S_{\rm div}(n)<\exp(2.1745n)$ for all large n. Doing the numerical experiment analogous to the one at the end of Remark 1, the number in (8) moves to 1.6161, giving $\#S_{\rm div}(n)<\exp(1.8714n)<6.5^n$.

3. Lower bounds

Let b denote a positive integer, and for $a\mid b$, let

$$s(a,b)=\{d\mid b: d\le a\}.$$

Further, let $P(a,b)$ denote the set of permutations π of $s(a,b)$ such that for each $d\in s(a,b)$, we have $\operatorname{lcm}[d,\pi(d)]\le a$. Write the divisors a of b in increasing order: $1=a_1<a_2<\cdots<a_k=b$, where $k=\tau(b)$. Let

$$c(b)=\frac{\log(\tau(b)!)}{b}+\sum_{i=1}^{\tau(b)-1}\Big(\frac{1}{a_i}-\frac{1}{a_{i+1}}\Big)\log(\#P(a_i,b)).$$

Theorem 2. *For any positive integer b we have*

$$\#S_{\rm lcm}(n)\ge\exp((c(b)\varphi(b)/b+o(1))n)$$

as $n\to\infty$.

We illustrate Theorem 2 in the first interesting case: $b=2$. Then $p(1,2)=1$ and $p(2,2)=2$, so that $c(2)=\log(2)/2$ and the theorem asserts that $\#S_{\rm lcm}(n)\ge\exp((\log(2)/4+o(1))n)$ as $n\to\infty$. To see why this is true, look at sets $\{j,2j\}$ where $j\le n/2$ and j is odd. There are $n/4+O(1)$ of these pairs and any permutation π of $[n]$ for which $\pi(\{j,2j\})=\{j,2j\}$ for each j, and π otherwise acts as the identity, is in $S_{\rm lcm}(n)$. Since the sets $\{j,2j\}$ are pairwise disjoint, this shows that $S_{\rm lcm}(n)$ contains at least $2^{n/4+O(1)}$ elements. The sets are pairwise disjoint since we are taking j odd. But a weaker condition also insures this. Let $v_p(j)$ be the exponent on p in the canonical prime factorization of j. Then we take sets $\{j,2j\}$ where $j\le n/2$ and $v_2(j)$ is even. This insures that the sets $\{j,2j\}$ are pairwise disjoint, and now there are

$n/3 + O(\log n)$ pairs, leading to $\#S_{\mathrm{lcm}}(n) \geq \exp((\log(2)/3 + o(1))n)$ as $n \to \infty$.

In fact, this improvement generalizes. For a prime power p^i let

$$\alpha(p^i) = \frac{p^{i+1} - p^i}{p^{i+1} - 1},$$

and extend α as a multiplicative function on the positive integers. Note that $\alpha(b)$ is the density of the set of integers j such that for all primes $p \mid b$, $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$. (Steve Fan pointed out to me that $\alpha(b) = b/\sigma(b)$, where σ is the sum-of-divisors function.)

Theorem 3. *For any positive integer b we have*

$$\#S_{\mathrm{lcm}}(n) \geq \exp((c(b)\alpha(b) + o(1))n)$$

as $n \to \infty$.

Proof. For $1 \leq i \leq \tau(b) - 1$ consider the intervals $I_i := (n/a_{i+1}, n/a_i]$ and $I_{\tau(b)} = (0, n/b]$. For $j \in I_i$ with $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$ for each prime $p \mid b$, we have the set $T(i,j) := \{dj : d \in s(a_i, b)\}$ as a subset of $[n]$. Moreover, the sets $T(i,j)$ are pairwise disjoint for all pairs i,j with $j \in I_i$ and $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$ for each prime $p \mid b$. For $j \in I_i$, we can view a permutation $\pi \in P(a_i, b)$ as a permutation on $T(i,j)$, where dj gets sent to $\pi(d)j$. So, consider a permutation π on $[n]$ such that for each i,j with $i \leq \tau(b)$, $j \in I_i$, and $v_p(j) \equiv 0 \pmod{v_p(b) + 1}$, it acts on $T(i,j)$ like a permutation in $P(a_i, j)$, and otherwise acts as the identity. Then $\pi \in S_{\mathrm{lcm}}(n)$.

For a given value of $i < \tau(b)$ there are $\sim (1/a_i - 1/a_{i+1})\alpha(b)n$ values of j, and for $i = \tau(b)$, there are $\sim n\alpha(b)/a_{\tau(b)}$ values of j. We conclude that $\#S_{\mathrm{lcm}}(n)$ is at least

$$(\#P(a_{\tau(b)}, b))^{(1+o(1))n\alpha(b)/a_{\tau(b)}} \prod_{i=1}^{\tau(b)-1} (\#P(a_i, b))^{(1+o(1))n\alpha(b)(1/a_i - 1/a_{i-1})}.$$

Since $a_{\tau(b)} = b$ and $p(b,b) = \tau(b)!$, the result follows. □

We have an analogous result for $S_{\mathrm{div}}(n)$. Let $p_{\mathrm{d}}(a,b)$ denote the number of permutations π of $s(a,b)$ such that for each $d \in s(a,b)$, we have $d \mid \pi(d)$ or $\pi(d) \mid d$. Let

$$c_{\mathrm{d}}(b) = \frac{\log(p_{\mathrm{d}}(b,b))}{b} + \sum_{i=1}^{\tau(b)-1} \Big(\frac{1}{a_i} - \frac{1}{a_{i+1}}\Big) \log(p_{\mathrm{d}}(a_i, b)).$$

Corollary 1. *For any positive integer b we have*

$$\#S_{\mathrm{div}}(n) \geq \exp((c_{\mathrm{d}}(b)\alpha(b) + o(1))n)$$

as $n \to \infty$.

TABLE 2. Some values of $c(b)\alpha(b)$ and $c_{\rm d}(b)\alpha(b)$ to 6 places with their exponentials rounded down to 4 places.

b	$c(b)\alpha(b)$	$e^{c(b)\alpha(b)}$	$c_{\rm d}(b)\alpha(b)$	$e^{c_{\rm d}(b)\alpha(b)}$
4	.354987	1.4261	.354987	1.4261
12	.536243	1.7095	.479872	1.6158
24	.602065	1.8258	.542689	1.7206
48	.638300	1.8932	.578122	1.7826
60	.646856	1.9095	.552061	1.7368
96	.658201	1.9313	.597849	1.8182
120	.707611	2.0291	.610358	1.8410
144	.704928	2.0237	.631752	1.8809
210	.600981	1.8239	.496559	1.6430
240	.740127	2.0962	.648821	1.9132
288	.723607	2.0618	.650371	1.9162
420	.716176	2.0465	.597383	1.8173
480	.757765	2.1335	.660864	1.9364

So, by Table 2 and Theorem 3 with $b = 480$, we have $\#S_{\rm lcm}(n) \geq 2.1335^n$ for all large values of n and by Corollary 1 we have $\#S_{\rm div}(n) \geq 1.9364^n$ for all large n.

4. COMPARING $\#S_{\rm div}(n)$ AND $\#S_{\rm lcm}(n)$

Let

$$R(n) = \#S_{\rm lcm}(n)/\#S_{\rm div}(n).$$

It appears from a glance at Table 1 that $R(n)$ grows at least geometrically. Here we prove this.

Theorem 4. *There is a constant $c > 1$ such that for all large values of n we have $R(n) > c^n$.*

Proof. Let A denote the set of integers a with $n/7 < a \leq n/6$ and with a not divisible by any prime $< 10^4$. Note that

$$\#A \geq \frac{n}{42} \prod_{p<10^4} \left(1 - \frac{1}{p}\right) + O(1),$$

so that $\#A > 14n/10^4$ for all large n. For $a \in A$, let

$$B_a = \{a, 2a, 3a, 4a, 5a, 6a\}.$$

Any divisor of a member of B_a that is not itself in B_a must be $< (n/6)/10^4$. Since clearly each member of B_a has no multiple in $[n]$ that is not in B_a, we have that each $\pi \in S_{\rm div}(n)$ has $\pi(B_a) \neq B_a$ for at most $n/10^4$ values of $a \in A$. We conclude that each $\pi \in S_{\rm div}(n)$ has $\pi(B_a) = B_a$ for at least $13n/10^4$ values of $a \in A$.

Let $\pi \in S_{\rm div}(n)$ with $\pi(B_a) = B_a$, and let π_0 be π restricted to $[n] \setminus B_a$. There are exactly 36 permutations $\pi \in S_{\rm div}(n)$ which give rise to the same π_0 corresponding to the 36 permutations σ of B_a with each $ja \mid \sigma(ja)$ or $\sigma(ja) \mid ja$ (since $\#S_{\rm div}(6) = 36$). However, for a given π_0 here, there are exactly 56 permutations $\pi \in S_{\rm lcm}(n)$ with π restricted to $[n] \setminus B_a$ equal to π_0 (since $\#S_{\rm lcm}(6) = 56$).

It thus follows that

$$R(n) > (56/36)^{13n/10^4},$$

so the theorem is proved with $c = (56/36)^{13/10^4} > 1.00057$. □

5. An upper bound for anti-coprime permutations

Let $A(n)$ denote the number of anti-coprime permutations π of $[n]$. We prove the following theorem.

Theorem 5. *We have $A(n) = n!/(\log n)^{(e^{-\gamma}+o(1))n}$ as $n \to \infty$.*

Proof. In light of the lower bound from [7], it suffices to prove that $A(n) \le n!/(\log n)^{(e^{-\gamma}+o(1))n}$ as $n \to \infty$.

Let $\psi(n) \to \infty$ arbitrarily slowly, but with $\psi(n) = o(\log\log n)$. Let $\alpha = 1 + 1/(\psi(n))^{1/2}$, so that $\alpha \to 1^+$ as $n \to \infty$. Let the integer variable i satisfy

$$\psi(n) < i < \log\log n/\log\alpha - \psi(n). \tag{11}$$

Thus, $e^{\alpha^i} \to \infty$ and $e^{\alpha^i} = n^{o(1)}$ as $n \to \infty$. For each i satisfying (11), let

$$I_i := (e^{\alpha^{i-1}}, e^{\alpha^i}], \quad J_i := \{j \in [n] : P^-(j) \in I_i\},$$

where $P^-(j)$ is the least prime factor of j. By the Fundamental Lemma of the Sieve (see [3, Theorem 6.12]) and Mertens' theorem, we have

$$\#J_i \sim \frac{n}{e^\gamma \alpha^{i-1}} - \frac{n}{e^\gamma \alpha^i} = \frac{n(\alpha-1)}{e^\gamma \alpha^i} \tag{12}$$

uniformly in i satisfying (11), as $n \to \infty$.

For $j \in [n]$ the number of $j' \in [n]$ with $\gcd(j', j) > 1$ is $\le \sum_{p|j} n/p$. (This is a poor bound for most integers j, but fairly accurate for most j's without small prime factors, as is the case for members of J_i.) Thus, the total number of assignments for the numbers $j \in J_i$ in an anti-coprime permutation of $[n]$ is

$$\le \prod_{j\in J_i}\left(n\sum_{p|j}\frac{1}{p}\right) \le \left(\frac{n}{\#J_i}\sum_{j\in J_i}\sum_{p|j}\frac{1}{p}\right)^{\#J_i},$$

by the AM-GM inequality. The double sum here is

$$\sum_{p>e^{\alpha^{i-1}}} \sum_{\substack{j\in J_i\\ p|j}} \frac{1}{p} \ll n \sum_{p>e^{\alpha^{i-1}}} \frac{1}{p^2\alpha^i} \ll \frac{n}{e^{\alpha^{i-1}}\alpha^{2i}},$$

uniformly, using an upper bound for the sieve. Thus, for n large,

$$\frac{n}{\#J_i}\sum_{j\in J_i}\sum_{p|j}\frac{1}{p} \le \frac{n}{(\alpha-1)e^{\alpha^{i-1}}} \le \frac{n}{e^{\alpha^{i-2}}}.$$

Hence, using (12),

$$\begin{aligned}\prod_{j\in J_i}\left(n\sum_{p|j}\frac{1}{p}\right) &\le \left(\frac{n}{e^{\alpha^{i-1}}}\right)^{\#J_i} = \exp(\#J_i\log n - \#J_i\alpha^{i-2})\\ &= \exp(\#J_i\log n - (1+o(1))n(\alpha-1)/e^\gamma)\end{aligned}$$

uniformly.

Let N be the number of $j\in[n]$ not in any J_i, so that $N = n-\sum_i \#J_i$. Further the number of values of i is at most $\log\log n/\log\alpha - 2\psi(n)$. After assignments have been made for the values of $j \in \cup J_i$, there are at most $N!$ remaining assignments for $j\notin \cup J_i$ and multiplying this by the product of the previous estimate for all i is at most

$$\exp\left(n\log n - (1+o(1))\frac{n(\alpha-1)}{e^\gamma}\Big(\frac{\log\log n}{\log\alpha} - 2\psi(n)\Big)\right).$$

It remains to note that $(\alpha-1)/\log\alpha \sim 1$ as $n\to\infty$, which completes the proof of the theorem. □

Dedication and Acknowledgments

Eduard Wirsing is not primarily known for his work in combinatorial number theory, yet one of his papers that influenced me a great deal is his joint work with Hornfeck, later improved on his own (see [11]), on the distribution of integers n with $\sigma(n)/n=\alpha$, for a fixed rational number α, where σ is the sum-of-divisors function. In a survey I wrote with Sárközy [8] on combinatorial number theory, we singled out this particular work for being a quintessential exemplar of the genre. It is in this spirit that I offer this note on combinatorial number theory in remembrance of Eduard Wirsing.

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References

[1] E. Berkove and M. Brilleslyper, Subgraphs of coprime graphs on sets of consecutive integers, Integers **22** (2022), #A47.

[2] P. Erdős, R. Freud, and N. Hegyvári, Arithmetical properties of permutations of integers, Acta Math. Hungar. **41** (1983), 169–176.

[3] J. Friedlander and H. Iwaniec, Opera de Cribro, Amer. Math. Soc., Providence, RI, 2010.

[4] N. McNew, Counting primitive subsets and other statistics of the divisor graph of $\{1, 2, \ldots, n\}$, European J. Combin. **92** (2021), Paper No. 103237, 20 pp.

[5] Online Encyclopedia of Integer Sequences, A320843.

[6] C. Pomerance, On the longest simple path in the divisor graph, Proceedings of the fourteenth Southeastern conference on combinatorics, graph theory and computing (Boca Raton, Fla., 1983). Congr. Numer. **40** (1983), 291–304.

[7] C. Pomerance, Coprime permutations, arXiv:2203.03085 [math.NT], 20 pp., Integers, to appear.

[8] C. Pomerance and A. Sárközy, Combinatorial number theory, in Handbook of Combinatorics, R. L. Graham, M. Grötschel, L. Lovász, eds., Elsevier Science B.V., 1995, pp. 967–1018.

[9] A. Sah and M. Sawhney, Enumerating coprime permutations, arXiv: 2203.06268 [math.NT], 11 pp.

[10] E. Saias, Etude du graphe divisoriel 5, arXiv:2107.03855 [math.NT], 39 pp.

[11] E. Wirsing, Bemerkung zu der Arbeit über volkommene Zahlen, Math. Ann. **137** (1959), 316–318.

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